Information-Based Complexity of Uncertainty Sets in Feedback Control

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Abstract—In this paper, a notion of information-based complexity is introduced to characterize complexities of plant uncertainty sets in feedback control settings, and to understand relationships between identification and feedback control in dealing with uncertainty. This new complexity measure extends the Kolmogorov entropy to problems involving information acquisition (identification) and processing (control), and provides a tangible measure of “difficulty” of an uncertainty set of plants.

In the special cases of robust stabilization for systems with either gain uncertainty or unstructured additive uncertainty, the complexity measures are explicitly derived.

Index Terms—Complexity, feedback, identification, information, metric robustness, uncertainty.

I. INTRODUCTION

IN THIS PAPER, a notion of information-based complexity, termed as control complexity, is introduced to characterize complexities of uncertainty sets in feedback control settings, and to understand complexity relationships between identification and feedback control.

Compared to tremendous progress in design methodologies of robust and adaptive control, the central issue of complexity in control systems remains largely unexplored. Essentially, study of system complexity intends to answer the question: What is a “difficult” plant to model, to identify, and to control? Traditionally, a common perception of complexity of a finite-dimensional linear time invariant (LTI) system is the order or dimension in its representation such as transfer functions or state space models. This complexity is closely related to many numerical issues involved in representing, identifying, designing, and simulating the system. Many aspects of model reduction have been explored based on this understanding.

Several new developments since the 1970s have provided an alternative paradigm of measures of plant complexity, which is related to feedback mechanism in a more fundamental manner. These include optimal robustness in the theories of $H^\infty$, $L^1$, $\mu$, gap, etc. from control points of view, and the Kolmogorov entropy and $\eta$-width, Gelfand $\eta$-width and identification $\eta$-width for system modeling and identification.

In this paper, we will provide a new perspective in characterizing plant complexity. The basic ideas and approaches we take in this paper to characterize complexity of an uncertainty set can be illustrated in the following example.

A. Illustrative Examples

Example 1: A device’s dynamic behavior varies significantly with the altitude $h$ where it is installed. The installer of the device is supposed to input the value $h$, and then a pre-designed controller suitable for that $h$ is automatically selected for implementation. Suppose that the total range of altitude is $\Omega_h = [h_m, h_M]$. $\Omega_h$ is equipped with a metric, namely, the absolute value $|\cdot|$. Assume that the best controllers can provide robust stability and performance only for a range of uncertainty on $h$ of length $r$. $r$ is a metric measure of optimal robustness of controllers. The question is: How many robust controllers must be predesigned and stored in the system computer? Let $n$ be the minimum number of such controllers.

1) Case 1—$h$ can be Accurately Measured: In this case, the output of measuring the altitude will belong to $\Omega_p = \{\hat{h} : \hat{h} \in [h_m, h_M]\}$.

It can be shown that the minimum number of controllers is

$$n = \left\lceil \frac{h_M - h_m}{r} \right\rceil$$

(1)

and one example of robust ranges of these controllers is

$$[h_m, h_m+\delta], [h_m+\delta, h_m+2\delta], \ldots, [h_m+(n-1)r, h_m+nr]$$

(2)

where, by (1), $h_m + nr \geq h_M$.

2) Case 2—$h$ can Only be Measured with an Accuracy of $\delta/2$, $\delta < h_M - h_m$. In this case, the output of measuring the altitude will belong to

$$\Omega_p = \left\{ \left[ \hat{h} - \frac{\delta}{2}, \hat{h} + \frac{\delta}{2} \right] : \hat{h} \in [h_m, h_M] \right\}.$$

To implement a robust controller based on the measured $\hat{h}$, it is necessary that $r > \delta$. Under this condition, we can show that the minimum number of controllers is

$$n = \left\lceil \frac{h_M - h_m}{r - \delta} \right\rceil$$

(3)

and one example of robust ranges of these controllers is

$$[h_m - \delta, h_m - \delta + \delta], [h_m - 2\delta + r, h_m - 2\delta + 2r], \ldots, [h_m - n\delta + (n-1)r, h_m - n\delta + nr]$$

(4)
Note that the overlapping of the robust ranges is necessary to cover, e.g., the possible measuring output $\hat{h} = h_m - (3\delta/2) + r + \varepsilon$, where $\varepsilon > 0$ is a very small number.

From (3), we can easily see that the better the metric robustness (larger $r$), the smaller the complexity (smaller $n$); the better the identification in reducing metric uncertainty (smaller $\delta$), the smaller the complexity.

In this example, the control scheme involves a finite set $S$ of controllers $F$ selected from a space of admissible controllers $C$. Let $s$ be the set of all such schemes. Since the individual controllers can be designed off-line, one relevant measure of complexity of the scheme $s$ is the number of controllers in the set, rather than the complexity of individual controllers. This leads to a complexity measure $m(s)$ on $s$. Let $s_n$ be the set of schemes in $s$ with complexity bounded by $n$. For a given uncertainty set $\Omega$ of plants and a performance index $f$

$$n_p(\Omega) = \inf \left\{ n : \inf_{s \in s_n} f(\Omega, s) \leq \rho \right\}$$

becomes a complexity measure of the uncertainty set $\Omega$. This is in the sense that $\Omega$ is considered as “difficult” if more complex controllers must be used. Viewed from the point of hybrid systems, the number of controllers is directly related to the size of discrete-event state space, while system orders are related to the size of analog state space.

In a sense, we are encountering adaptation in this example, at least a primitive one: selecting different controllers based on the outcome of an identification process. Apparently, the complexity measure in this example does not characterize time-varying aspects of the systems involved. As a result, it cannot be used directly to characterize complexity of adaptive or switched systems in which time-varying behavior is dominant. However, there are applications in which the size of discrete-event state space is a relevant complexity measure and reduction of it is desirable. Also, for slowly time-varying systems, the complexity measure will become relevant. In fact, the effect of time variation on complexity resembles identification errors, and hence it increases complexity, as shown in the following example from [17].

**Example 2:** Continuing Example 1, we now study gain scheduling when the altitude $h$ varies with time, slowly with rate $d$: $[h(t) - h(t - T)] \leq dT$, for all $t$ and $T > 0$. One may simply imagine that the device is now mounted on a mobile station and an altitude sensor is also installed. For simplicity, assume that the sensor can read altitude accurately. Adaptation is now done by the standard gain scheduling. Given $m + 1$ controllers $F_1, \ldots, F_{m+1}$, a set of up-switching points

$$\mathcal{H} = \{b_2, \ldots, b_{m+1}\}$$

and a set of down-switching points

$$\mathcal{H} = \{b_2, \ldots, b_{m+1}\}$$

are established with

$$b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_m \leq b_{m+1}.$$

From the current controller $F_i$, switching of controllers from $F_i$ to $F_{i+1}$ occurs when $h$ crosses the point $b_{i+1}$ in $\mathcal{H}$. For instance, if the controllers in (2) of Case 1 were used, the switching points, which will be shown shortly to be undesirable, would be

$$b_2 = h_m + r, \quad b_3 = h_m + 2r, \ldots$$

$$b_n = h_m + (n-1)r$$

and

$$b_{n+1} = h_m + nr.$$

For simplicity, assume that the plant and robust controllers are linear, and when $h$ is in the robust range of the implemented controller, the resultant closed-loop system is exponentially stable. It is well known that under this exponential stability, there exists a finite staying time $T_0$ such that if $h$ always stays in the robust range of the controller at least $T_0$ seconds, then the total adaptive system will remain stable.

It can be easily verified that if $l = \max_{i=1,2,\ldots,n} \frac{b_i - b_{i-1}}{l} > 0$ and the variation rate $d$ satisfies $d \leq l/T_0$ then the minimum staying time $T_0$ will be guaranteed. Controllers in (2) have $l = 0$, and, hence, not suitable for adaptation here.

Next, we modify (2) to increase the number of controllers from $n$ to $n_l$ to provide the robustness ranges of $h$ as follows:

$$[h_m, h_m + r], \quad [h_m - l + r, h_m - l + 2r], \ldots$$

$$[h_m - nl + (n_l - 1)r, h_m - nl + nr]$$

and

$$n_l = \left\lfloor \frac{h_m - h_m}{r - l} \right\rfloor.$$  \hspace{1cm} (5)

Unlike the controllers in (2), the robustness ranges in (5) have overlaps. Now, the up-switching points become

$$b_2 = h_m + r, b_3 = h_m - l + 2r, \ldots, b_{n_l} = h_m - (n_l - 1)(r - l)$$

and down-switching points are

$$b_1 = h_m - l + r, b_2 = h_m - 2l + 2r, \ldots, b_{n_l-1} = h_m - nl + (n_l - 1)r.$$  \hspace{1cm} (6)

The complexity $n_l$ is higher than $n$. By substituting $l = dT_0$ in (6), we have

$$n_d = \left\lfloor \frac{h_m - h_m}{r - dT_0} \right\rfloor.$$  \hspace{1cm} (7)

Compared to (3), this expression tells us that time variation increases complexity; and in this special case, the effect is similar to identification uncertainty $\delta$. In particular, $\lim_{d \to 0} n_d = n$

Hence, the complexity measure in (3) becomes a limiting complexity when variation rates approach zero.

Further discussions and simulation results will be presented in Section VI. However, rigorous analysis and development of complexity measures in time-varying systems remain an open research problem and are beyond the scope of this paper.

With intuitive understanding from these examples, we now proceed to a more rigorous description of the complexity measure introduced in this paper.
B. Metric Robustness

Classical control developed in the framework of Bode, Nyquist, and Nichols aims at designing feedback controllers under which the closed-loop systems are insensitive to modeling errors and disturbances, namely robust in today’s terms. It was observed that some plants are intrinsically difficult to control in terms of this sensitivity analysis, no matter how much effort was made in tuning controllers. This observation led to an intuitive perception of the “complexity” of controlling such plants.

The seminal work of George Zames and great research progress on robust control since the 1980s have ushered in a new era in our understanding and characterization of feedback robustness. Optimal robustness problems in metric spaces, formulated by using metrics such as the $H^\infty$ norm, $L^1$ norm, gap metric, $\mu$ metric, etc., have provided perhaps the most relevant measures of “difficulties” of a plant to be controlled robustly. For example, it is now clearly understood that plants with near unstable pole/zero cancellations are difficult plants, in the sense that they allow very small robustness radii (in the selected metric space) no matter how controllers are selected. In several important special cases, such as gain uncertainty, additive and multiplicative unstructured uncertainty, gap uncertainty and coprime-factor uncertainty, optimal robustness problems are solved and optimal controllers can now be either constructed in closed forms or obtained numerically. These results provide metric measures of complexity of a plant on the one hand, and characterize the ability of feedback to achieve robustness on the other. Hence, from this point of view, a plant which allows only small optimal robustness is more complex to control than those with larger optimal robustness radii. This idea of characterizing plant complexity is particularly useful in understanding the issues of modeling, uncertainty, and control from an information point of view: A plant of high complexity requires more effort in acquiring information during modeling and identification before a robust controller can be successfully applied.

C. Basic Ideas

The main ideas we employ in this paper can be briefly summarized as follows. Suppose that a performance index is given to describe the desired behavior of a feedback system.

1) Metric robustness, or equivalently robustness ranges, of controllers is expressed as balls in a metric space, such as the length $r$ in Example 1. Such a ball in the metric space is characterized by its center, norm radius, and, more importantly, by the robust performance level (say, $\rho$) achievable by applying a robust feedback. Such a ball will be called a $\rho$-level ball.

2) The ability of identification in reducing uncertainty is expressed in terms of metric errors in the same metric space, such as $\delta$ in Example 1.

3) For a required performance specification given by the level $\rho$, if the uncertainty set can be covered by a single $\rho$-level ball, then robust control is sufficient. For an uncertainty set which cannot be covered by a single $\rho$-level ball, a set of controllers will be used so that their robustness ranges can collectively cover the uncertainty set. In this case, the smallest covering of the uncertainty set by using $\rho$-level balls, which is also the size of control tables, is a measure of complexity of the plant uncertainty set, and is called control complexity of the uncertainty set.

This complexity measure is closely related to the Kolmogorov entropy [8]. For a given set $\Omega$ of functions and a set $M$ of modeling spaces in a metric space, the Kolmogorov $\varepsilon$-complexity of $\Omega$ is defined as the minimum number of functions from $M$ that can approximate the set $\Omega$ within the error bound $\varepsilon$. The main differences between our complexity measure and the Kolmogorov complexity are: 1) in Kolmogorov entropies balls of uniform norm radii are used to model an uncertainty set, whereas we are using balls of uniform performance levels; and 2) we take into consideration of identification errors.

In closing, we would like to comment that complexity measures have been explored extensively in the problems of communications, system modeling and identification. This is exemplified by system orders for numerical computations and control design; Kolmogorov $\eta$-width [8], [11], Kolmogorov $\varepsilon$-entropy [8], [21], and identification $\eta$-width [23] in deterministic identification problems; Akaïke’s information criterion [1] and Rissanen’s shortest description [13] in stochastic identification problems; and Shannon’s information in communication theory.

D. Organization of the Paper

The rest of the paper is organized as follows. The main assumptions on systems and uncertainty sets are discussed in Section II. The basic concepts of control complexity are introduced in Section III. Some generic properties of these complexity measures are provided (Proposition 1). To illustrate the potential utility and computation of these measures, the concepts are applied to robust stabilization problems for systems subject to gain or unstructured additive uncertainties in Sections IV and V. For systems with gain uncertainties, the exact control complexity is obtained (Theorem 1). For systems with unstructured additive uncertainties, their control complexities are related to the Kolmogorov $\varepsilon$-entropies for which many results are available for various classes of uncertainty sets (Theorem 2). Examples and simulation are provided in Section VI. Complexity relationships between identification and feedback are further discussed in Section VII. Finally, some conclusions are drawn in Section VIII for several open issues along the direction of this paper.

E. Related Literature

Complexity issues in modeling and identification have been pursued by many researchers. The concepts of $\varepsilon$-net and $\varepsilon$-dimension in the Kolmogorov sense [8] were first employed by Zames [21], [22] in studies of model complexity and system identification. For certain classes of continuous- and discrete-time systems, the $\eta$-widths and $\varepsilon$-dimensions in the $L^1$ kernel norm and the $H^\infty$ norm were obtained by Zames, Lin, and Wang [21], [18]. The notion of identification $\eta$-width was introduced in [23] to characterize intrinsic complexities in worst-case identification problems. Complexity issues in system identification were rigorously studied by Tse, Dahleh,
and Tsitsiklis [4]. Poolla and Tikku [12]. n-widths of many other classes of functions and operators were summarized in the books by Pinkus [11] and Vitushkin [16]. A general framework of information based complexity was comprehensively developed by Traub, Wasilkowski, and Wozniakowski [15]. The essential features and generic characterizations of topological approaches in robust and adaptive control are summarized and elaborated by Pait in [10].

In stochastic systems, complexity measures such as Shannon information and Kolmogorov probability entropies have been widely used in communication and control systems. For control applications, the reader is referred to the book by Saridis [14] for an exposure on entropy approaches, and to the book by Caines [2] for an encyclopedical treatment of linear stochastic systems including entropy methods. Recently, Xie and Guo [20] obtained a complete characterization of feedback robustness against uncertainties on growth rates of nonlinear time-varying first-order systems.

The concept of control complexity introduced in this paper is an extension of the Kolmogorov ε-entropy [8], [21] which characterizes the complexity of a set of functions by the minimal number of functions which can approximate the set to a precision level ε. While the Kolmogorov entropy has been extensively used in approximation theory, computational complexity theory, information theory, and operator theory, as well as system modeling and identification, it often fails to provide a relevant measure of complexity of uncertainty sets in feedback control problems. The main reason is that the ability of feedback to achieve robust stability and performance depends on the nominal system structure, not merely the size of uncertainty. The control complexities introduced in this paper incorporate the ability of feedback in achieving design specifications robustly. The Kolmogorov entropy was introduced in the 1950s [8]. The book by Carl and Stephani [3] contains many new developments on the theory and applications of the Kolmogorov entropy up to the late 1980s. Some terms used in this paper, such as covering numbers, are borrowed from [3].

A more general framework of uncertainty, information and complexity in information acquisition and processing is reported in [17], in which complexity problems beyond single-input–single-output (SISO) systems, metric spaces, or feedback are studied.

**Notation:** \(\mathbb{R}, \mathbb{C}, \mathbb{Z}\) denote the real numbers, complex numbers, and integers. The subscript + denotes restriction to non-negative values, as in \(\mathbb{R}_+\) and \(\mathbb{Z}_+\). The absolute value of \(x \in \mathbb{C}\) is \(|x|\). The spaces of vectors or matrices with elements in \(\mathbb{R}\) will be denoted by \(\mathbb{R}^n\) or \(\mathbb{R}^{n \times m}\). For a matrix \(M \in \mathbb{R}^{n \times m}\), \(M^T\) is its transpose.

Suppose \(L\) is a normed space with norm \(\|\cdot\|\). \(\text{Ball}(K, r) \subset L\) denotes the ball of center \(K\) and radius \(r\) in \(L\). For \(M_1, \ldots, M_n \subset L\), \(\bigcup_{i=1}^n M_i\) is the union of the subsets. For \(M \subset L\), \(d(M)\) is the diameter of \(M\)

\[
d(M) = \sup_{G_1, G_2 \in M} \|G_1 - G_2\|.
\]

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**II. UNCERTAINTY SETS AND FEEDBACK**

**A. Systems and Feedback Configurations**

\(\mathcal{B}\) denotes a normed algebra of SISO causal stable linear time invariant systems with norm \(\|\cdot\|\). Examples of \(\mathcal{B}\) include the following typical algebras of stable systems.

1) (Discrete-time stable systems) \(\mathcal{H}^\infty(D)\): The Hardy space of analytic functions \(K\) on the unit disk with \(\|K\|_\infty := \sup_{\theta \in [-\pi, \pi]} |K(e^{j\theta})| < \infty\).

2) (Continuous-time stable systems) \(\mathcal{H}^\infty(H)\): The Hardy space of analytic functions \(K\) on the right-half plane with \(\|K\|_\infty := \sup_{|\omega| < \infty} |K(i\omega)| < \infty\).

3) \(\mathcal{L}^2\): Discrete-time systems \(K\) with impulse responses \(k\) which satisfy \(\|k\|_1 := \sum_{n=-\infty}^{\infty} |k(n)| < \infty\).

4) \(\mathcal{L}^2\): Continuous-time systems \(K\) with impulse responses \(k\) which satisfy \(\|k\|_1 := \int_{-\infty}^{\infty} |k(t)| dt < \infty\).

Finite dimensional subspaces of the above systems are also such examples. Since we are dealing with SISO LTI systems, we will denote \(K\) to denote both the system and its transfer function, and \(k\) its impulse response.

Unstable systems will belong to \(\mathcal{B}_e\), the extended space of \(\mathcal{B}\) (see, e.g., [5] and [19] for detailed discussions on extended spaces). \(\mathcal{B}_e\) contains \(\mathcal{B}\) as a subspace.

Consider the feedback system in Fig. 1. \(r, d, e, y\) are reference input, disturbance, actuator signal, and plant output, respectively. The interconnection of a feedback \(F\) and a plant \(P\) in \(\mathcal{B}_e\) is well posed if all elements of the closed-loop system

\[
\mathcal{K}(P, F) = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} := \begin{pmatrix} (I + FP)^{-1} & P(I + FP)^{-1} \\ F(I + PF)^{-1} & (I + PF)^{-1} \end{pmatrix}
\]

are in \(\mathcal{B}_e\), and stable if they are in \(\mathcal{B}\). Suppose that the set of admissible controllers is \(F \subset \mathcal{B}_e\). \(F \in \mathcal{B}\) is said to robustly stabilize a subset \(\Omega \subset \mathcal{B}_e\) of plants if \(\mathcal{K}(F, P)\) is stable for all \(P \in \Omega\). We denote by \(\Pi(\Omega)\) the set of all feedback controllers \(F \in \mathcal{B}\) which can robustly stabilize \(\Omega\). \(\Pi(\Omega)\) may be empty, implying that \(\Omega\) is not robustly stabilizable by using controllers in \(\mathcal{B}\). In the special case where \(\Omega\) contains only one plant \(P\) with a coprime factorization representation and \(F = \mathcal{B}_e\), \(\Pi(\Omega)\) is given explicitly by the Youla parameterization of all stabilizing controllers for \(P\).
B. Uncertainty Sets

Uncertainty sets considered in this paper are limited to those parameterized by an uncertain variable $\Delta$ taking values in a subset $U$ of a normed space $\mathbb{L}$. Formally, let $\Omega \subseteq \mathbb{B}_e$ be an uncertainty set of plants represented by

$$\Omega = \mathcal{G}(U) = \{G(\Delta) : \Delta \in U \subseteq \mathbb{L}\}$$

where $\mathcal{G} : \mathbb{L} \to \mathbb{B}_e$ is a mapping relating the plant to its uncertainty variable $\Delta$. As a result, descriptions of a plant uncertainty set $\Omega$ will include the structure information $\mathcal{G}$ and the uncertainty set $U$. The structure information is assumed to be available from the outset and will be carried intact in the definitions of complexity measures.

Examples of uncertain systems with such parameterization include most cases commonly studied in the robust control literature.

Example 3:

1) Gain uncertainty: $\mathbb{L} = \mathbb{R}$ with norm $\| \cdot \|$ and $U = [a, b]$. $\Omega = \{kG_0 : k \in U = [a, b]\}$. Or phase uncertainty: $\mathbb{L} = \mathbb{R}$ with norm $\| \cdot \|$ and $U = [\theta_1, \theta_2]$. $\Omega = \{e^{j\theta}G_0 : \theta \in \mathbb{U} = [\theta_1, \theta_2]\}$, where $G_0 \in \mathbb{B}_e$ is a nominal plant.

2) Parametric uncertainty: $\mathbb{L} = \mathbb{R}^{n+m+1}$ with norm $\| \cdot \|_\infty = \max_{i=1, \ldots, n+m+1} |v_i|$ and $U$ is a compact set in $\mathbb{L}$.

$$\Omega = \left\{ \frac{b_m s^n + b_{m-1} s^{n-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}, \quad [a_0, \ldots, a_{n-1}, b_0, \ldots, b_m]^T \in U \subseteq \mathbb{R}^{n+m+1} \right\}.$$

3) Additive unstructured uncertainty: $\Omega = \{G_0 + W\Delta : \Delta \in U \subseteq \mathbb{B}\},$ or multiplicative unstructured uncertainty: $\Omega = \{G_0 (1 + W\Delta) : \Delta \in U \subseteq \mathbb{B}\}.$ Where $G_0 \in \mathbb{B}_e$ is a nominal plant and $W \in \mathbb{B}$ a weighting function, $L = \mathbb{B}$ may be $\mathbb{H}_\infty(D), \mathbb{H}_\infty(H), \mathbb{P}, \mathbb{L}_1,$ or other metric spaces of stable systems.

4) Linear fraction transformation (LFT) type: $\Omega = \{M_{22} + M_{21} \Delta (1 - M_{11} \Delta)^{-1} M_{12} : \Delta \in U \subseteq \mathbb{B}\}$, where $M_{ij} \in \mathbb{B}, i, j = 1, 2$ are nominal closed-loop systems. $L = \mathbb{B}$ may be $\mathbb{H}_\infty(D), \mathbb{H}_\infty(H), \mathbb{P}, \mathbb{L}_1,$ or other metric spaces of stable systems.

C. Metric Characterization of Feedback Robustness

Generally, the set of plants which can be stabilized by a given controller $F \in \mathbb{B}$ is expressed in the Youla parameterization. This is a complete characterization of feedback robustness. Such characterization is, however, far too complicated for information processing.

In this paper, we employ the following metric characterization of feedback robustness. Suppose that a plant $G$ has the structure $G = \mathcal{G}(\Delta)$ with $\Delta \in \mathbb{L}$, where $\mathcal{G}$ is equipped with norm $\| \cdot \|$, and controllers are selected from a set $F \subseteq \mathbb{B}_e$. The performance of a feedback system is measured by a certain performance index

$$f : \mathbb{B}_e \times \mathbb{B}_e \to \mathbb{R}_+$$

which assigns a finite nonnegative value $f(G, F)$ to a plant-controller pair $(G, F) \in \mathbb{B}_e \times \mathbb{B}_e$ if the feedback interconnection $\mathcal{K}(G, F)$ is stable, and $\infty$ otherwise.

We will use balls in the metric space $\mathbb{L}$ to characterize feedback robustness. For a given ball $\text{Ball}(\Delta_0, r_0) \subseteq \mathbb{L}$, the corresponding plant uncertainty set is $\Omega_0 = \mathcal{G}(\text{Ball}(\Delta_0, r_0))$. $\Pi(\Omega_0)$ denotes the set of controllers in $\mathbb{F}$ which robustly stabilize $\Omega_0$. We define the optimal robust performance by

$$\rho_0 = \Phi(\text{Ball}(\Delta_0, r_0))$$

$$= \begin{cases} \inf_{F \in \Pi(\Omega_0)} & f(G(\Delta), F), \quad \text{if } \Pi(\Omega_0) \text{ is not empty} \\ \infty, & \text{otherwise.} \end{cases}$$

Namely, $\rho_0$ is the best achievable robust performance for $\text{Ball}(\Delta_0, r_0)$ by using controllers in $\mathbb{F}$. By including the performance level, the ball is now characterized by its center $\Delta_0$, norm radius $r_0$, and the performance level $\rho_0$.

For a given performance level $\rho$, the set of all balls in $\mathbb{L}$ with performance levels bounded by $\rho$ is called the set of $\rho$-level balls $\mathcal{M}_\rho = \{\text{Ball}(\Delta_0, r_0) \subseteq \mathbb{L} : \Phi(\text{Ball}(\Delta_0, r_0)) \leq \rho\}$. (9)

One may interpret the set $\mathcal{M}_\rho$ from different angles. First, for a given $\Delta_0$ and a desired performance level $\rho$, one can define

$$r_{\max}(\Delta_0; \rho) = \max\{r_0 : \Phi(\text{Ball}(\Delta_0, r_0)) \leq \rho\}.$$
problems. For example, for a plant with the structure \( G = (s+a)/(s+b) + ((s+2)/(s+3))\Delta, \) \( a > 0, b > 0, \)
\( \Delta \in H^\infty(H), \) the balls \( Ball(\Delta_0, \varepsilon) \) can be robustly stabilized for any center \( \Delta_0 \) and finite \( \varepsilon. \) On the other hand, for \( G = (s-a)/(s-b) + ((s+2)/(s+3))\Delta, \) the ball \( Ball(0, \varepsilon) \) can be robustly stabilized only if \( \varepsilon < c, \) where \( c \) is a finite constant depending on \( a \) and \( b. \)

III. CONTROL COMPLEXITY OF UNCERTAINTY SETS

Consider an uncertainty set \( \Omega = \{G(\Delta): \Delta \in U \subseteq L\}, \) defined in (8). As in other applications, modeling of uncertainty sets aims at representing the uncertainty set by simple base uncertainty sets. The main approach we employ in this paper is based on the following two principles: 1) The structural information is kept intact in the modeling process. In other words, instead of modeling \( \Omega, \) we are modeling \( U. \) In Example 3, \( U \) are stable and bounded, and takes the simple forms of intervals or convex sets, whereas \( \Omega \) often contains unstable systems and is complicated. As a result, it is much easier to model \( U \) than \( \Omega. \) 2) The modeling space is \( M_p \) defined in (9). In approximation theory, one is seeking a finite set of functions which can approximate the uncertainty set to a precision level \( \varepsilon. \) In other words, the balls of norm radii \( \varepsilon \) are used as a modeling space. This leads to the concept of Kolmogorov entropy and dimension. By using the set \( M_p \) as our modeling space, we represent the uncertainty set \( U \) by balls of performance level \( \rho. \)

Definition: Suppose that for a given \( \rho > 0, V \) is an indexed class of sets in \( M_p. \)

\[ V = \{Ball(\Delta_j, r_j) \subseteq M_p: j = 1, \ldots, n\}. \]

\( V \) is said to be a \( \rho \)-covering of \( U \subseteq L \) if

\[ U \subseteq \bigcup_{j=1}^{n} V_j. \]

Interchangeably, we say that \( U \) is modeled by \( V. \)

A. Identification

Here, the term “identification” is used in the broad sense of information acquisition by some unspecified means. Possible examples of identification include direct measurement of parameters (possibly with sensor noises), estimation of parameters from corrupted input–output data, data transfer through a network with limited channel capacity, or parameters obtained from model reduction, etc. Rather than dealing with specific details of characterizing and computing identification errors, in this paper we focus on the generic impact of infidelity of information acquisition on control complexity. The treatment reported in this paper is limited to metric information. For more general discussions, see [17].

Consider a plant that belongs to a given prior uncertainty set \( \Omega = \{G(\Delta): \Delta \in U \subseteq L\}. \) Identification experiments provide additional information about the plant and reduce the uncertainty on the plant uncertain variable \( \Delta \) from \( U \) to a posterior uncertainty set \( U(j). \) Denote by \( U_p \) the class of all possible posterior uncertainty sets \( U(j). \)

\[ U_p = \{U(j): U(j) \subseteq L \text{ is a possible posterior uncertainty set}\}. \]

The variable \( j \) parameterizes all possible outcomes of an identification experiment. Since every \( \Delta \in U \) is a possible true variable, \( U_p \) covers \( U, \) namely,

\[ U \subseteq \bigcup_{U(j) \subseteq U_p} U(j). \]

We will denote the set of all possible posterior information on the plant by

\[ \Omega_p = \{G(U(j)): U(j) \in U_p\}. \]

B. Complexity

If the prior uncertainty set \( U \) is not contained in the robustness range \( RF \) of any single controller \( F \in F, \) we are seeking multiple controllers \( F_1, \ldots, F_r, \) whose robustness ranges \( RF_1, \ldots, RF_r \) will collectively contain or cover \( U. \) The main question is: How many controllers are needed? The situation becomes further compounded by the fact that identification is usually imperfect.

Feedback control which follows identification is implemented based on the posterior uncertainty sets \( \Omega_p. \) We note that since all \( \Delta \in U \) is a possible true variable, the robustness ranges of the set of controllers must be at least a \( \rho \)-covering of \( U. \) However, we will demonstrate by an example that this covering may not be sufficient to accommodate identification errors.

Example 4: Consider a plant with gain uncertainty \( \Omega = G(U) = \{kG_0: k \in [1, 4]\}, \) where \( G_0 \) is the LTI nominal plant. Suppose \( \Omega \) is not robustly stabilizable by a single LTI controller, but for a covering \( \{V_1 = [1, 2], V_2 = [2, 4]\} \) of \( U, \) \( \Omega(1) = G(V_1) \) and \( \Omega(2) = G(V_2) \) are robustly stabilizable separately by controllers \( F_1 \) and \( F_2. \) A control table which provides such robustness ranges for \( U \) is given by Fig. 2.

However, if an identification scheme can only locate \( k \) to, say, an accuracy of 0.2, that is, the posterior information takes the form of

\[ \Omega_p = \{G(U(k_0)): U(k_0) \in [1, 4]\} \]

then \( \{V_1, V_2\} \) will not be an appropriate covering. For instance, when the posterior uncertainty set produced by the identification scheme is \( G(U(2)) = \{kG_0: k \in (1.8, 2.2)\}, \) one cannot determine a robust controller from the control table in Fig. 2 since

<table>
<thead>
<tr>
<th>Gain Robustness Ranges</th>
<th>Controllers</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1, 2]</td>
<td>F_1</td>
</tr>
<tr>
<td>[2, 4]</td>
<td>F_2</td>
</tr>
</tbody>
</table>

Fig. 2. Control table for the example.
is not contained in any robustness ranges of the controllers in the table.

In fact, for an appropriate selection of controllers based on posterior information $U_{p}$, $\rho$-level coverings must satisfy the additional inclusion property: for any uncertainty set $U(j) \in U_{p}$, there exists $V_{i} \in V$ such that $U(j) \subseteq V_{i}$.

Intuitively, this means that for any possible outcome $U(j)$ of the identification scheme, there should be a controller in the table which can achieve robust performance level $\rho$ for $U(j)$. This motivates the following notion of control complexity.

First, we recall the mathematical notion of refiners from set theory. Suppose $M$ and $N$ are two classes of subsets of $L$.

$$M = \{M_{i} \subseteq L\}; \quad N = \{N_{j} \subseteq L\}.$$  

$M$ is said to be a refiner of $N$, denoted by $M \preceq N$, if for every $M_{i} \in M$, there exists $N_{j} \in N$ such that $M_{i} \subseteq N_{j}$.

Let $U_{p}$ and $\Omega_{p}$ be defined as in (10) and (12). Let $V = \{V_{j}, j = 1, \ldots, n\}$ be an indexed set of balls in $L$, where

$$V_{j} = Ball(\Delta_{j}; \gamma_{j}) \subseteq L.$$  \hspace{1cm} (13)

$V$ is said to be a covering of $U_{p}$ if $U_{p}$ is a refiner of $V$

$$U_{p} \preceq V$$

and a $\rho$-covering of $U_{p}$ if, in addition, for all $j = 1, \ldots, n$, $d(Ball(\Delta_{j}; \gamma_{j})) \leq \rho$.

Now, we introduce the notion of control complexity.

**Definition 1:**

1. The control complexity $N_{G}(U_{p}, \rho)$ of $U_{p}$ is the minimum size of all $\rho$-coverings of $U_{p}$. If there exists no $\rho$-covering of $U_{p}$, then $N_{G}(U_{p}, \rho) = \infty$. $U_{p}$ is said to be control compact if $N_{G}(U_{p}, \rho) < \infty$.

2. When the underlying identification is accurate, namely, $U(j)$ is reduced to a singleton, and $U_{p} = \{K\}; K \in U_{p}$, we will use the symbol $n_{G}(U, \rho)$ to denote the corresponding control complexity. The following properties of $N_{G}(U_{p}, \rho)$ can be derived from their definitions.

**Proposition 1:** $N_{G}(U_{p}, \rho)$ has the following properties.

1. **Monotonicity**

   a. $N_{G}(U_{p}, \rho) \geq n_{G}(U, \rho)$.

   b. For $0 \leq \rho_{1} \leq \rho_{2} \leq \infty$, $N_{G}(U_{p}, \rho_{1}) \geq N_{G}(U_{p}, \rho_{2})$.

   c. If $U_{p} \preceq U_{p}$, then $N_{G}(U_{p}, \rho) \leq N_{G}(U_{p}, \rho)$.

   d. Make explicit the dependence of the complexity on the underlying set $F$ (for admissible controllers) by the notation $N_{G}(U_{p}, \rho; F)$. If $F_{1} \subseteq F_{2}$, then $N_{G}(U_{p}, \rho; F_{1}) \geq N_{G}(U_{p}, \rho; F_{2})$.

2. **Convergence**

   $$N_{G}(U_{p}^{n+1}; \rho) \leq N_{G}(U_{p}^{n}; \rho) + N_{G}(U_{p}^{n}; \rho).$$

3. **Subadditivity**

   $$N_{G}(U_{p}^{n} \bigcup U_{p}^{m}; \rho) \leq N_{G}(U_{p}^{n}; \rho) + N_{G}(U_{p}^{m}; \rho).$$

Remarks:

1. The monotonicity states that control based on perfect identification schemes can always be less complex than that based on imperfect identification [see a)]. It claims also that the tighter the performance specifications, the more complex the control must be [see b)]; the better the identification schemes, the less complex the control can be [see c]); the stronger the ability of feedback to achieve
performance specifications, the less complex the control is [see d)].

2) Under the stated conditions, $n_C(U, \rho)$ is the limiting complexity when identification becomes accurate.

3) The feedback invariance (14) indicates that applying a fixed feedback controller after identification but before feedback control cannot change the complexity of the uncertainty set.

4) Examples of identification problems with the feedback invariant property include those from a fixed-length observation in identification $\eta$-width problems [23]. For those problems with the feedback invariant property, (15) claims that applying a fixed feedback before identification will not change the complexity of uncertainty sets.

Control complexities will be computed for some stabilization problems in the following sections.

IV. CONTROL COMPLEXITY: STABILIZATION OF PLANTS WITH UNCERTAIN GAINS

In robust stabilization problems, we may simply postulate the performance index as

$$f(G, F) = \begin{cases} 0, & \text{if } F \in \Pi(G) \\ \infty, & \text{if } F \notin \Pi(G). \end{cases}$$

Let

$$\Omega = \mathcal{G}(U) = \{kG_0; k \in U = [a, b], 0 < a < b < \infty\}$$

with $G_0$ a finite-dimensional nominal plant in $\mathfrak{B}_c$. The following results are valid for both discrete-time systems ($\mathfrak{B} = H^\infty(D)$) and continuous-time systems ($\mathfrak{B} = H^\infty(H)$). In this special case of interval uncertainty, balls of variable radii are reduced to intervals of variable lengths. For stabilization problems, the coverings $\{[\alpha_j; \beta_j], j = 1, \ldots, n\}$ of $U = [a, b]$ induce a covering of $\Omega$ in the form of $\mathcal{M} = \{M_j, j = 1, \ldots, n\}$ with

$$M_j = \{cG_0; c \in [\alpha_j, \beta_j], 0 < \alpha_j < \beta_j < \infty\}.$$ 

Khargonekar and Tannenbaum [7] obtained a necessary and sufficient condition for $M_j$ to be robustly stabilizable by a fixed controller in $\mathfrak{B}_c$, namely

$$\frac{\beta_j}{\alpha_j} \leq \gamma$$

where $\gamma > 1$ is the optimal gain margin which depends only on the nominal plant $G_0$. $\gamma$ and the corresponding optimal or suboptimal controllers and can be readily obtained by means of Nevanlinna–Pick interpolation [7].

If $\gamma = \infty$, then $\Omega$ can be robustly stabilized by a fixed controller and $n_C(U, \rho) = 1$ for any $\rho < \infty$. Assume, therefore, $\gamma < \infty$.

Suppose an identification scheme is applied to identify the plant in (16). If the identification can only locate the gain $k$ to a precision of $\delta$, i.e., $k \in (\hat{k} - \delta, \hat{k} + \delta)$, then the posterior gain uncertain is

$$U_p(\delta) = \{U(\hat{k}); U(\hat{k}) = (\hat{k} - \delta, \hat{k} + \delta), \hat{k} \in [a, b]\}.$$ 

It is noted that

$$U_p(\delta_1) \preceq U_p(\delta_2), \quad \text{whenever } \delta_1 \preceq \delta_2. \quad (18)$$ 

Namely, better identification will produce smaller posterior uncertainty sets in the sense of refiners.

**Theorem 1:**

1) If $\delta < \frac{a(\gamma - 1)}{(\gamma + 1)}$, then

$$N_C(U_p, \rho) = \left[ \log \frac{b + \delta}{a - \delta} \right] \log \gamma.$$ 

2) If $\delta \geq \frac{a(\gamma - 1)}{(\gamma + 1)}$, then

$$N_C(U_p, \rho) = \infty.$$ 

*Proof:* In the Appendix.

It is noted from (19) that

$$\lim_{\delta \to \infty} N_C(U_p, \rho) = \left[ \frac{\log b}{\log \gamma} \right] := n_C(U, \rho).$$

In other words, the complexity approaches the limit when the identification error approaches zero.

This example demonstrates clearly that the Kolmogorov entropy is not appropriate for characterizing control complexity. Indeed, for the compact interval $[a, b]$ its Kolmogorov $\varepsilon$-complexity $n^K(\varepsilon)$ or $\varepsilon$-entropy $e^K = \log n^K$ is finite and $n^K = \lfloor (b - a)/\varepsilon \rfloor$. In contrast

$$n_C(U, \rho) = \left[ \frac{\log b}{\log \gamma} \right] \to \infty, \quad \text{as } a \to 0.$$ 

Hence, the norm compactness and control compactness are fundamentally different. This example reflects one of the main motivations for introducing a new complexity measure for control systems.

V. CONTROL COMPLEXITY: STABILIZATION OF PLANTS WITH ADDITIVE UNSTRUCTURED UNCERTAINTIES

Concentrate now on the case of unstructured additive uncertainty $^5$

$$\Omega = \mathcal{G}(U) = \{G_0 + W \Delta, \Delta \in U \subseteq H^\infty(H)\} \quad (20)$$ 

$^5$It is easy to extend the discussions here to the cases $\mathfrak{B} = H^\infty(D), \mathfrak{P}, L^1$, etc.
where $G_0 \in B_{c}$ is a nominal plant which has a coprime factorization $G_0 = N_0D_0^{-1}$ with $N_0$, $D_0 \in \mathcal{H}\infty(H)$ and for some $X_0, Y_0 \in \mathcal{H}\infty(H)$

$$N_0X_0 + D_0Y_0 = 1.$$ 

$W \in \mathcal{H}\infty(H)$ is a fixed weighting function, and the uncertainty set $U$ will be specified later on. A covering of $U$ by open balls $V = \{V_j = Ball(W, r_j), j = 1, \ldots, n\}$ induces a covering of $\Omega$ by $M = \{M_j = G_0 + \lambda V_j, j = 1, \ldots, n\}$.

**Lemma 1 [6]:** $M_j = G_0 + \lambda V_j$ is robustly stabilizable if and only if

$$r_j \leq \mu_0^{-1},$$

where

$$\mu_0 = \inf_{Q \in \mathcal{H}\infty(H)} |W D_0 \{X_0 + D_0(Q))|\infty.$$

It is interesting to observe that the condition (21) is independent of the uncertainty center $\Delta_j \in \mathcal{H}\infty(H)$. We are now ready to establish a relationship between the control complexity and the Kolmogorov entropy. Let $\epsilon^K(U, \epsilon)$ be the Kolmogorov $\epsilon$-entropy of $U$, i.e., the minimum cardinality of coverings of $U$ by open balls of uniform radius $\epsilon$ [21], [8]. Formally

$$n^K(U, \epsilon) = \inf \left\{ n : \cup \subseteq \sum_{i=1}^{n} Ball(\Delta_i, \epsilon) \right\},$$

for some $\Delta_1, \ldots, \Delta_n \in \mathcal{H}\infty$.

$$e^K(U, \epsilon) = \log n^K(U, \epsilon).$$

Suppose that an identification scheme is applied to the plant in (20). If the scheme can only identify the plant to a precision of $\delta$ in the norm, then posterior information on $U$ is given by

$$U_p = \{Ball(0, \delta); \delta \in U\}.$$ 

We would like to establish an estimate of $N_G(U_p, \rho)$.

**Theorem 2:** Suppose $\delta > 0$ and $\mu_0$ is defined as in Lemma 1. For any $0 \leq \delta < \infty$

1) if $\delta < \mu_0^{-1}$, then

$$e^K(U, \mu_0^{-1}) \leq \log N_G(U_p, \rho) \leq e^K(U, \mu_0^{-1} - \delta).$$

2) If $\delta \geq \mu_0^{-1}$, then

$$N_G(U_p, \rho) = \infty.$$ 

**Proof:** In Appendix.

It is known that for norm compact $U$, $e^K(U, \mu_0^{-1} - \delta) \rightarrow e^K(U, \mu_0^{-1})$ as $\delta \rightarrow 0$. In this sense, Theorem 2 gives an asymptotically accurate estimate of $N_G(U_p, \rho)$ when $\delta$ is small.

$\text{VI. Illustrative Examples}$

In this section, we will illustrate the concepts and results of the previous sections by several numerical examples. In these examples, the resulting control tables will have the minimal sizes (when balls are used to model the uncertainty sets).

**Example 5:** Let $\Omega = \{kG_0; k \in [0.1, 100]\}$ with

$$G_0 = \frac{1}{2} \frac{s - z_0}{s - p_0},$$

where $z_0 = 2, p_0 = 0.5$. By [7], $c_{\text{max}} = (\frac{(z_0 - p_0)(z_0 + p_0)}{((1 + c_{\text{max}})(1 - c_{\text{max}}))} = 16$ is the optimal achievable gain margin. Since $b/\alpha = 1000 > 16$, we cannot robustly stabilize the uncertainty set by a single LTI controller.

Suppose that $k$ can be accurately measured, then the control complexity of $\Omega$ is

$$n = \left[ \frac{\log b}{\log \gamma} \right] = \left[ \frac{\log 1000}{\log 16} \right] = \left[ \frac{9.966}{4} \right] = 3.$$ 

Hence, we will need at least three controllers to cover the uncertainty range $[0.1, 100]$. Controllers that achieve optimal gain margins can be shown to take the form $F = c((s+\beta)/(s+\beta+1))$, $0 < \beta < 2$. When $\beta$ is very close to two, a gain margin close to $\gamma = 16$ is achieved. We will select the following three controllers:

$$F_1 = \frac{3.268s + 1.7}{1.7s + 1}, \quad F_2 = \frac{0.309s + 1.8}{1.8s + 1}, \quad F_3 = \frac{0.03125s + 1.6}{1.6s + 1}$$

with respective robustness ranges $U_1 = [0.000, 1.040], U_2 = [0.9, 11.686], U_3 = [10, 102]$. Collectively, $U \subseteq U_1 \cup U_2 \cup U_3$.

For a demonstration of deriving a potential gain-scheduling scheme from the table, we now allow $k$ to be time varying, albeit slowly. When $k(t)$ is in the robustness range of the controller, the closed-loop system is exponentially stable with decaying rates in $e^{-\gamma t}$ as $\sigma_1 \geq 0.125$, $\sigma_2 \geq 0.0572$, $\sigma_3 \geq 0.1818$. This, together with the overlapping of the robustness ranges, implies that these controllers can be used in gain scheduling.
control when $k$ varies sufficiently slowly. The gain-scheduling or switching logic is shown in Fig. 3.

Fig. 4 demonstrates simulation results in which plant and controllers are implemented as follows. The plant is

$$y(t) = -k(t) \left( \frac{s-2}{2s-1} u \right) + d(t)$$

where $((s-2)/(2s-1))u$ denotes the application of an operator to the signal $u$, whereas the left multiplication by $k(t)$ is the usual multiplication in the time-domain. $d(t)$ is a random noise uniformly distributed in $[-10, 10]$. The controllers are defined as follows: for $i = 1, 2, 3$

$$\dot{x} = -\frac{1}{\beta_i}x + \left( \frac{1}{\beta_i} - \frac{1}{\beta_i} \right)y$$

$$u = -x - \frac{c}{\beta_i} y.$$

The plant initial value $y(0) = 150$, and the time-varying gain

$$k(t) = 50.05 + 49.95 \cos \omega t.$$ 

For $\omega = 2$ and sampling interval $T = 0.01$ second, the trajectories of control signals, plant output, and controller switching are shown.

It should be emphasized that if $k$ varies faster, it may become necessary to increase the number of controllers, indicating that uncertain time variation will affect complexity of an uncertainty set. Beyond a certain rate of variation, the number of controllers is no longer relevant in analyzing the behavior of the controlled system. In this case, one is forced to deal with the analysis of a switched system. See, e.g., [9] for many challenging issues in switching control problems.

**Example 6:** For a plant with additive uncertainty

$$\Omega = \frac{s-1}{s-2} + \frac{s+1}{s+2} U.$$
we can compute that \( \mu_0 = 3 \), and
\[
X_0 = \frac{2\mu_0}{\mu_0 - z_0} = 4 \quad Y_0 = 1 - X_0 = -3 \\
Q = \frac{s + 2}{4(s + 1)} = -\frac{s + 2}{s + 1}.
\]
For an uncertainty set Ball(\( \Delta_j \), 1/\( \mu_0 \)), the controller
\[
F = \frac{X_0 + D_0 Q}{Y_0 - W \Delta_j X_0 - (N_0 + W \Delta_j D_0) Q}
\]
\[
= \frac{4 + s - 2}{s + 1} \left( -\frac{s + 2}{s + 1} \right)
\]
\[
= -3 - \frac{s + 1}{s + 2} + \frac{s - 1}{s + 2} + \frac{s + 1}{s + 2} + \frac{s - 2}{s + 2} \Delta_j
\]
\[
= -2(s + 2) - 3(s + 1) \Delta_j
\]
can robustly stabilize it. In particular, for Ball(0, 1/3)
\[
F = \frac{3(s + 2)}{-2(s + 2)} = -1.5.
\]
Now, suppose that \( U \) contains both parametric and unstructured uncertainties
\[
U = \{ a + Ball(0, 0.1) : a \in [0, 0.7] \}
\]
where Ball(0, 0.1) is the ball of center 0 and radius 0.1 in \( H^\infty \). Suppose that \( a \) is identified with error \( \delta = 0.05 \). Then, the output of identification will be the uncertainty set \( \{ \hat{a} + \delta + \Delta : |\delta| \leq 0.05, ||\Delta||_\infty \leq 0.1 \} \). Hence
\[
\bar{U}_p = \{ \hat{a} + \delta + \Delta : |\delta| \leq 0.05, ||\Delta||_\infty \leq 0.1 \} : \hat{a} \in [0.05, 0.65] \}
\]
Since \( \mu_0^{-1} = 1/3 \), we will try to cover \( \bar{U}_p \) by balls of radius 1/3 = 0.333. Since the diameter of \( \bar{U} = 0.9 > 2/\mu_0^{-1} = 0.667 \), we cannot cover \( \bar{U}_p \) by a single ball of radius 0.333. Choose \( \Delta_1 = 0.23 \), \( \Delta_2 = 0.54 \). We will show that
\[
\bar{U}_p \subseteq \{ Ball(0.23, 0.333), Ball(0.54, 0.333) \}.
\]
(23)
Indeed, if \( U \in \bar{U}_p \), then \( \bar{U}_p \) = \( \{ \hat{a} + \delta + \phi : |\delta| \leq 0.05, ||\delta||_\infty \leq 0.1 \} \), with 0.05 \( \leq \hat{a} \leq 0.65 \). If 0.05 \( \leq \hat{a} \leq 0.41 \), then
\[
||\hat{a} + \delta + \phi - \Delta_1||_\infty \leq ||\hat{a} - \Delta_1|| + ||\delta|| + ||\phi||_\infty
\]
\[
\leq 0.18 + 0.05 + 0.1 = 0.33.
\]
Hence, \( \bar{U}_p \subseteq Ball(0.23, 0.333) \). If 0.36 \( \leq \hat{a} \leq 0.65 \), then
\[
||\hat{a} + \delta + \phi - \Delta_2||_\infty \leq ||\hat{a} - \Delta_2|| + ||\delta|| + ||\phi||_\infty
\]
\[
\leq 0.18 + 0.05 + 0.1 = 0.33.
\]
Hence, \( \bar{U}_p \subseteq Ball(0.54, 0.333) \). Since \( U \) is arbitrary in \( \bar{U}_p \), (23) is valid.

Correspondingly, we can construct the two robust controllers as
\[
F_1 = \frac{3(s + 2)^2}{-2(s + 2)^2 - 3(s + 1) \cdot 0.23} = -\frac{3(s + 2)^2}{2s^2 + 8s + 6.09}
\]
\[
F_2 = \frac{3(s + 2)^2}{-2(s + 2)^2 - 3(s + 1) \cdot 0.54} = -\frac{3(s + 2)^2}{2s^2 + 9.64s + 9.64}
\]
Arguments similar to those in Example 5 can be made in developing a gain-scheduled scheme by switching between \( F_1 \) and \( F_2 \) when \( a \) and \( \phi \) vary with time sufficiently slowly. When \( a \) is the only time-varying component, a switching logic is given in Fig. 5.

Fig. 6 demonstrates simulation results that show the trajectories of control signals, plant output, and controller switching. [\( \square \)]

VII. RELATIONS BETWEEN IDENTIFICATION AND CONTROL COMPLEXITIES

The relationship between fidelity of information acquisition (identification) and complexity of information processing (control) established in the previous sections can be employed to establish a complexity relationship between identification and control.

The control complexity expresses the complexity of uncertainty sets as a function of identification errors. The identification errors are, in turn, functions of identification complexity. As a result, metric complexities of identification and control are related. The better the identification is, the less complex the control can be, or equivalently, the more complex control is, the less complex the identification needs to be. Let us illustrate this idea with the following examples.

A. Effects of Identification Errors on Control
Consider an uncertain discrete-time system with prior information
\[
\Omega = \{ G(k) = kG_0 : k \in U = [a, b], 0 < a < b < \infty \}.
\]
Suppose the control complexity \( N_G(U_p, \rho) \) is constrained by \( n_0 \)
\[
N_G(U_p, \rho) \leq n_0
\]
due to, say, communication channel capacity, computer time or space limitations. We would like to know how accurate identification must be.

Suppose \( \gamma \) is computed as in Theorem 1. By Theorem 1, the identification error must satisfy
\[
\frac{(b + \delta)(\gamma - 1) - 2\delta \gamma}{(a - \delta)(\gamma - 1) - 2\delta} \leq \gamma^{n_0}
\]
i.e.,
\[
\delta \leq \frac{(\alpha \gamma^{n_0} - b)(\gamma - 1)}{(\gamma^{n_0} - 1)(\gamma + 1)} := \beta_0
\]
(24)
where \( \alpha \gamma^{n_0} - b \geq 0 \) by Theorem 1.
Suppose certain identification experiments are applied to the system with additive noise at the times $0, 1, \ldots, T-1$ of observation

$$y(l) = k(G_0x(l)) + v(l), \quad l = 0, 1, \ldots, T-1$$

where the probing input $x$ is bounded by $\|x\|_\infty \leq M$, the noise is bounded by $\|v\|_\infty \leq \varepsilon$, and $g$ is the impulse response of the nominal discrete-time plant.

Define $P(T) = \sum_{l=0}^{T-1} |g(l)|$ which is monotone increasing and unbounded since $G_0$ is unstable. We claim that the inequality (24) can be satisfied if and only if

$$\frac{\varepsilon}{P(T)M} \leq \beta_0. \quad (25)$$

The inequality (25) elaborates a relationship among identification observation length $T$, signal-to-noise ratio (SNR) $M/\varepsilon$, robust stabilizability of the system $\gamma$, and complexity of control $n_0$.

To show (25), observe that $T$ consecutive input–output measurements $\{x(i), y(i) : i = 0, 1, \ldots, T-1\}$ are related by

$$y(i) = k \sum_{\tau=0}^{i} g(i-\tau)x(\tau) + v(i), \quad i = 0, 1, \ldots, T-1.$$ 

This implies that

$$k = \frac{\sum_{\tau=0}^{i} g(i-\tau)x(\tau) + v(i)}{\sum_{\tau=0}^{i} g(i-\tau)x(\tau)}.$$ 

The identification error at each time instant is

$$\delta(i) = \frac{v(i)}{\sum_{\tau=0}^{i} g(i-\tau)x(\tau)}.$$ 

The worst-case error occurs when all measurements give the same center, and in this case the optimal worst-case identification error can be obtained

$$\delta = \inf_{\|h\|_\infty \leq M} \inf_{0 \leq l \leq T-1} \sup_{\|v\|_\infty \leq \varepsilon} \frac{|\delta(i)|}{P(T)M}.$$ 

Therefore, the inequality (24) is satisfied if and only if

$$\frac{\varepsilon}{P(T)M} \leq \beta_0.$$ 

B. Identification $\eta$-Width and Control

Suppose a priori information of an uncertain discrete-time plant is given by a general $\Omega = \mathcal{G}(\mathcal{U})$, $\mathcal{U} \subseteq \mathcal{L}$, and $\delta(\eta)$ is the identification $\eta$-width of $\mathcal{U}$ introduced in [23], i.e., the best achievable worst-case identification error based on $\eta$ consecutive observations. For a given level $\rho$ of performance specification $f$ and identification error $\delta$, $N_G(\mathcal{U}(\rho, \delta))$ is the corresponding control complexity. Note that $\delta(\eta)$ is a monotone decreasing function of $\eta$ (the length of observation), and $N_G(\mathcal{U}(\rho, \delta), \rho)$ a monotone decreasing function of $\delta$. Now, the function

$$N(\eta) = N_G(\mathcal{U}(\rho, \delta(\eta)), \rho)$$

is a monotone decreasing function of $\eta$. For a given identification time complexity $\eta$, $N(\eta)$ gives the minimum complexity of control required to achieve the specified performance level $\rho$. Then, the inverse function of $N(\eta)$

$$n = \phi(N_0) = \inf\{l \in \mathbb{N}_+: N(l) \leq N_0\}$$

gives the minimal time complexity of identification which is needed to achieve the performance specifications when the complexity of control is limited by $N_0$.

One possible application of such tradeoff between complexities of identification and control might be allocation of resources to identification and control. Suppose $c_1(n)$ and $c_2(N)$ are cost functions of implementing identification experiments and control mechanism, respectively. Then, optimization of the combined cost function

$$c(n) = c_1(n) + c_2(N(n))$$

will provide a candidate for optimal allocation of resources to achieve a given level of performance specifications.

VIII. CONCLUDING REMARKS

Despite tremendous progress in control design methodologies, complexity issues in feedback control remain largely unexplored. While intuitive perceptions of complex plants and controllers have been employed in practical systems, rigorous formulation and investigation of complexity problems in an information-based framework turn out to be an extremely challenging task. One reason for this difficulty is that it requires a clarifying and rigorous characterization of the ability of feedback mechanism in achieving robust performance. Recent advance in metric characterization of feedback robustness provides first time a solid foundation
on which complexity issues may be analyzed rigorously in an information framework. This paper is a preliminary effort in this pursuit.

The main approach employed in this paper is an extension of the classical mathematical notion of Kolmogorov entropy, by accommodating the inherent feedback robustness in the problem formulation. The key ideas employed in this paper follow the fundamental philosophy of George Zames in developing a comprehensive information-based theory of feedback, identification, and adaptation.

There remain many open issues. For instance, it is intuitively apparent that for time-varying plants, their variation rates have significant bearing on identification and control complexities. At this moment, it is not clear how such intuition can be clarified into a rigorous argument. Finally, the information-based framework discussed in this paper is naturally related to communication problems. Control-communication interaction in this framework seems to be an extremely interesting area for further investigation.

APPENDIX

Proof of Theorem 1

1) We will prove (19) by showing the optimality of the covering

\[ V = \{ V_i = [\alpha_i, \beta_i), i = 1, \ldots, n \} \]

for \( \mathcal{U}_p \), where \( \alpha_i = a - \delta, \beta_i = \alpha_{i-1} + \gamma, \) and \( \alpha_i = \beta_i - 2\delta, \beta_i = \alpha_i + \gamma, i = 2, \ldots, n, \) and

\[ n = \inf \left\{ j \in \mathbb{Z}_+ : \beta_j \geq b + \delta \right\} \]

\[ = \left\lfloor \log \left( \frac{(b + \delta)(\gamma - 1) - 2\delta \gamma}{(a - \delta)(\gamma - 1) - 2\delta} \right) \log \gamma \right\rfloor . \]

Since \( \beta_i / \alpha_i = \gamma, V_i, i = 1, 2, \ldots, n \) are robustly stabilizable. Moreover, for any \( \mathcal{U}(\hat{k}) = (\hat{k} - \delta, \hat{k} + \delta), \) select \( i_0 = \max \{ i \in \mathbb{Z}_+ : \alpha_i \leq \hat{k} - \delta \}. \)
Then, we have \( \mathcal{U}(k) \subseteq [\alpha_0, \beta_0] \). Therefore, \( \mathcal{U}_p \subseteq \mathcal{V} \). It follows that
\[
N_G(\mathcal{U}_p, \rho) \leq n = \left\lceil \frac{\log \left( \frac{(b + \delta)(\gamma - 1) - 2\delta}{(a - \delta)(\gamma - 1) - 2\delta} \right)}{\log \gamma} \right\rceil.
\] (26)

The optimality of \( \mathcal{V} \) is shown by contradiction. Suppose \( \mathcal{V}^0 = \{V^i_1 = [a_0, b_0^i], i = 1, 2, \ldots, n_0 \} \) is another \( \rho \)-covering of \( \mathcal{U}_p \) with \( n_0 < n \). Assume, without loss of generality, that \( a_0^1 \leq \cdots \leq a_0^n \).

Since \( \mathcal{V}^0 \) is a \( \rho \)-covering of \( \mathcal{U}_p \), \( \mathcal{V}(a) \subseteq \mathcal{V}^0 \) and \( \mathcal{V}(a) \subseteq a - \delta \).

Also, \( \beta_0^1 \leq \cdots \leq \beta_0^l \leq a - \delta \). Consequently, \( \mathcal{U}(\hat{k}) \not\subseteq \mathcal{V}^0 \) for all \( \hat{k} > (a - \delta) \gamma - \delta \), which implies
\[
\left\{ \mathcal{V}(i) : a_i = 1, \ldots, n_0 \right\} \not\subseteq \mathcal{V}^0.
\]

Now, induction leads to
\[
\left\{ \mathcal{V}(i) : a_i = 1, \ldots, n_0 \right\} \not\subseteq \mathcal{V}^0;
\]

Since \( \beta_0^i < b + \delta \), i.e.,
\[
(a - \delta) \gamma^{n-1} - 2\delta \gamma^{n-2} - \frac{1}{\gamma - 1} - \delta < b
\]
we have
\[
\left\{ \mathcal{V}(i) : a_i = 1, \ldots, n_0 \right\} \not\subseteq \mathcal{V}^0.
\]

This, together with (18) and the monotonicity of \( N_G(\mathcal{U}_p, \rho) \), implies
\[
N_G(\mathcal{U}_p, \rho) = \infty
\]
for \( \delta \geq (a(\gamma - 1)/(\gamma + 1)) \).

\( \Box \)

Proof of Theorem 2
\begin{enumerate}
\item Suppose \( \mathcal{K} = \{K_i = Ball_0(K_i, \mu_0^{-1} - \delta); K_i \in \mathcal{U}_p \} \) is an optimal \( \mu_0^{-1} \)-covering of \( \mathcal{U} \) in the Kolmogorov sense, i.e.,
\[
k = 2^{K(\mathcal{U}_p, \mu_0^{-1} - \delta)}
\]
then for any \( K \in \mathcal{U} \), there exists \( Ball_0(K_i, \mu_0^{-1} - \delta) \in \mathcal{K} \) such that
\[
K \in Ball_0(K_i, \mu_0^{-1} - \delta)
\]
which implies that
\[
Ball_0(K, \delta) \subseteq Ball_0(K_i, \mu_0^{-1} - \delta).
\] (27)
\end{enumerate}

Now, define \( \mathcal{V} = \{V_i = Ball_0(K_i, \mu_0^{-1}) : i = 1, \ldots, k \} \). By (27)
\[
\mathcal{U}_p \subseteq \mathcal{V}
\]
which implies
\[
\log N_G(\mathcal{U}_p, \rho) \leq cK(\mathcal{U}, \mu_0^{-1} - \delta).
\]

To prove the lower bound, we observe that by Lemma 1, for \( 0 \leq \rho < \infty \), \( \mathcal{V} = \{V_j = Ball_0(\Delta_j, \gamma_j), j = 1, \ldots, n \} \) is a \( \rho \)-covering of \( \mathcal{V} \) if and only if \( \gamma_j \leq \mu_0^{-1} - \delta \). \( \mathcal{V} \) covers \( \mathcal{U} \). Or equivalently, \( \mathcal{V} \) is a \( \mu_0^{-1} \)-covering of \( \mathcal{U} \) in the sense of Kolmogorov. Therefore, we conclude, after taking minimum of cardinalities of all such coverings,
\[
\log N_G(\mathcal{U}_p, \rho) \geq cK(\mathcal{U}, \mu_0^{-1} - \delta).
\]

2) If \( \delta \geq \mu_0^{-1} \), then by Lemma 1, \( \mathcal{G}(Ball_0(K, \delta)) \in \mathcal{U}_p \) is not robustly stabilizable. Therefore, \( N_G(\mathcal{U}_p, \rho) = \infty \).

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