What is an Ideal Logic for Reasoning with Inconsistency?

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Reasoning with Inconsistency

• Inconsistency handling is at the heart of reasoning under uncertainty.

• Many AI applications incorporate logics that tolerate inconsistent data in a non-trivial way.

  • Inconsistency measurements e.g., by Priest’s 3-valued logic LP [Ohller, 2004]
  • Database integration systems e.g., by Marcos-Carnielli’s LFI1 [de-Amo et at 2002]
  • Preference modeling e.g., by Belnap’s four-valued logic [Perny & Tsoukias 1998]
  • Many other areas ...

What should be the exact nature of a formal system for tolerating inconsistent information, and how to choose one for specific needs?
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   What is a Logic, Multi-Valued Semantics by Matrices

3. Basic Properties of Ideal Paraconsistent Logics
   Maximal Paraconsistency, Expressivity, Containment in Classical Logic

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**What is a Logic?**

A (Tarskian) *consequence relation* for a language $L$ is a binary relation $\vdash$ (between theories in $L$ and formulas in $L$), satisfying:

- **Reflexivity:** if $\psi \in \Gamma$ then $\Gamma \vdash \psi$
- **Monotonicity:** if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$
- **Transitivity:** if $\Gamma \vdash \psi$ and $\Gamma' \cup \{\psi\} \vdash \varphi$ then $\Gamma \cup \Gamma' \vdash \varphi$

A consequence relation $\vdash$ is called:

- **Structural:** if $\Gamma \vdash \psi$ implies $\theta(\Gamma) \vdash \theta(\psi)$ for every substitution $\theta$
- **Non-trivial:** if $\Gamma \not\vdash \psi$ for some nonempty $\Gamma$ and a formula $\psi$
- **Finitary:** if whenever $\Gamma \vdash \psi$, also $\Gamma' \vdash \psi$ for a *finite* $\Gamma' \subseteq \Gamma$

A *(propositional) logic* is a pair $(L, \vdash)$, where $L$ is a propositional language and $\vdash$ is a structural, non-trivial, and finitary consequence relation.
Multi-Valued Semantics

The most standard semantic way of defining logics is by matrices.

A *matrix* for a language $L$ is a triple $M = (V, D, O)$, where:

- $V$ is a nonempty set of *truth values*
- $D$ is a nonempty proper subset of $V$, called the *designated values*
- $O$ includes a function $\hat{\circlearrowleft} : V^n \rightarrow V$ for every $n$-ary connective $\hat{\circlearrowleft}$ of $L$.

Standard definitions for the induced semantic notions:

- $M$-*valuation* :
  $\nu : WFF(L) \rightarrow V$, where $\nu(\hat{\circlearrowleft}(\psi_1 \ldots \psi_n)) = \hat{\circlearrowleft}(\nu(\psi_1) \ldots \nu(\psi_n))$

- $M$-*model of* $\psi$ :
  $\text{mod}(\psi) = \{\nu \mid \nu(\psi) \in D\}$.

- $M$-*model of* $\Gamma$ :
  $\text{mod}(\Gamma) = \bigcap_{\psi \in \Gamma} \text{mod}(\psi)$.

The induced consequence relation:

$\Gamma \vdash_M \psi$ if every $M$-model of $\Gamma$ is an $M$-model of $\psi$ \hspace{1cm} [\text{mod}(\Gamma) \subseteq \text{mod}(\psi)]$
Multi-Valued Semantics (Cont’d.)

Examples

1. Propositional classical logic  \( \text{CL} = (\{t, f\}, \{t\}, \{
eg, \lor, \land\}) \).

2. Priest’s 3-valued logic  \( \text{LP} = (\{t, m, f\}, \{t, m\}, \{
eg, \lor, \land\}) \)

3. Kleene’s 3-valued logic  \( \text{KL} = (\{t, m, f\}, \{t\}, \{
eg, \lor, \land\}) \)

Theorem [Shoesmith & Smiley, 1971]: For every propositional language \( \text{L} \) and a finite matrix \( \text{M} \) for \( \text{L} \), \( \mathcal{S}_M = (\text{L}, \vdash_M) \) is a propositional logic.
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1. Paraconsistency

Should *any* proposition be inferred from a contradiction?
[Jaskowski, Nelson, Anderson, Belnap, da-Costa, and many other since early 1950’s]

To handle inconsistent information one needs *a logic that allows contradictory, yet non-trivial theories*. Logics of this sort are called *paraconsistent*.

*formally:*

A logic $\mathcal{L} = (L, \vdash)$, where $L$ is a language with a unary operator $\neg$ is *$\neg$-paraconsistent*, if $\{\psi, \neg\psi\} \not\vdash \varphi$ for some formulas $\psi$ and $\varphi$. 
2. **Maximality**

... but paraconsistency by itself is often not sufficient. A useful paraconsistent logic should also be *maximal* [da-Costa, 1974]

**Intuition:** By trying to further extend a paraconsistent logic (without changing the language), one loses the property of paraconsistency.

**formally:**

$L = (L, \vdash)$ is *maximally paraconsistent*, if it is paraconsistent, and every logic $L' = (L, \vdash')$ that *properly extends* $L$ (i.e., $\vdash \subseteq \vdash'$), is not paraconsistent.
3. Expressivity

a) The language of a paraconsistent logic should have a negation connective that is entitled of this name.

b) Maximally paraconsistent logic may not have a sufficient expressive power:

**Fact:** The three-valued logic induced from the matrix $SE = ( \{t, m, f\}, \{t, m\}, \{\neg\} )$ i.e., whose only connective is Sette’s negation ($\neg t = f$, $\neg m = t$, $\neg f = t$), is maximally paraconsistent. [Arieli, Avron, Zamansky, *Studia Logica* 2011]
3. Expressivity (Cont’d.)

a) The language of a paraconsistent logic should have a negation connective that is entitled of this name.

A bivalent $\neg$-interpretation for a language $L$ is a function $F$ that
a) associates a two-valued truth-table with each connective of $L$, and
b) $F(\neg)$ is the classical truth table for negation.

$M_F$ - the two-valued matrix for $L$ induced by $F$.  
$M_F = (\{t,f\}, \{t\}, \{F(\diamond) | \diamond \in L\})$

A logic $\mathcal{L} = (L, \vdash)$ is $\neg$-contained in classical logic, if $\Gamma \vdash \phi \Rightarrow \Gamma \vdash_{M_F} \phi$ for some bivalent $\neg$-interpretation $F$ for $L$.

**Proposition:** No two-valued paraconsistent matrix is $\neg$-contained in classical logic.
3. Expressivity (Cont’d.)

b) Sufficient expressive power.

A binary connective \( \supset \) is a *proper implication* for a logic \( \mathcal{L} = (L, \vdash) \) if the classical deduction theorem holds for \( \supset \) and \( \vdash \): \( \Gamma, \psi \vdash \phi \) iff \( \Gamma \vdash \psi \supset \phi \).

[reducing inferences to theoremhoods]

A logic \( \mathcal{L} = (L, \vdash) \) is called *normal*, if it \( \neg \)-contained in classical logic and has a proper implication.

[The adequacy of the expressive power of normal logics:]

**Proposition**: For any normal logic (that is \( F \)-contained in classical logic), \( F(\neg) \) and \( F(\supset) \) form a functionally complete set (i.e., any two-valued connective is definable in term of them).
4. Maximality Relative to Classical Logic

Intuition: A useful paraconsistent logic should retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories.

Formally:

A logic $\mathcal{L} = (\mathbb{L}, \vdash)$ is maximal relative to classical logic if there is some bivalent $\neg$-interpretation $F$, such that:

- $\mathcal{L}$ is $F$-contained in classical logic, and
- Any logic that is obtained by adding to $L$ a non-provable classical $F$-tautology $\psi$, includes all the classical $F$-tautologies.
Ideal Paraconsistent Logics

A propositional logic is called \textit{ideal} (for reasoning with inconsistency), if it is

\begin{itemize}
\item \textit{\neg} paraconsistent,
\item \textit{normal} (i.e., \textit{\neg}-contained in classical logic and has a proper implication),
\item \textit{maximal relative to classical} logic, and
\item maximally paraconsistent.
\end{itemize}

Two notions of maximality:
1. Absolute maximality w.r.t. paraconsistency
2. Maximal faithfulness to classical logic
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Are There Any Ideal Paraconsistent Logics?

Theorem: For every n>2 there is a family of n-valued ideal logics, each one of which is not equivalent to any k-valued logic with k<n.

Construction: for every n>3, let:

$L = \{\text{FALSE, } \neg, \diamond, \supset, \ldots\},$

$L = \{\text{FALSE, } \neg, \diamond, \supset, \ldots\},$

$M = (V, D, O) – \text{a matrix for } L, \text{ where:}$

$V = \{ t, f, m, u_1, \ldots, u_{n-3} \}, \quad D = \{ t, m \}$

$\neg t = f, \quad \neg f = t, \quad \neg x = x \text{ otherwise}$

$\diamond t = f, \quad \diamond f = t, \quad \diamond m = u_1, \quad \diamond u_1 = u_2, \quad \ldots, \quad \diamond u_{n-3} = m$

$a \supset b = t \text{ if } a \notin D, \quad a \supset b = b \text{ otherwise}$

$\text{any other connective } * \text{ is classically closed } \quad [ \forall i \ a_i \in \{t,f\} \Rightarrow * (a_1 \ldots a_m) \in \{t,f\} ]$

$\mathcal{L}_M = (L, \vdash_M)$ is an ideal n-valued paraconsistent logic that is not equivalent to any logic with less than n elements.
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Ideally Paraconsistent 3-Valued Logics

One of the oldest and best known approaches towards paraconsistency is based on three-valued logics (The simplest semantic framework for paraconsistent reasoning).

**Proposition:** Let $M$ be a three-valued matrix for a propositional language $L$ that is $\neg\neg$-contained in classical logic. Then $\mathcal{L}_M$ is paraconsistent iff $M$ is isomorphic to a matrix $(V,D,O)$ in which $V = \{t, T, f\}$, $D = \{t, T\}$, $\neg t = f$, $\neg f = t$ and $\neg T \in \{T, t\}$.

**Theorem:** Every three-valued paraconsistent logic is ideal iff it is normal.

[improvement of the result in IJCAI paper – maximal paraconsistency is guaranteed by normality]

**Thus:** In the three-value case the normality of a paraconsistent logic guarantees maximal paraconsistency and maximality relative to classical logic.
A three-valued propositional logic is ideal if it is

- \textit{\neg}-paraconsistent,
- \textit{normal} (i.e., \textit{\neg}-contained in classical logic and has a proper implication),
- \textit{maximal relative to classical} logic, and
- maximally paraconsistent.

Two notions of maximality:
1. Absolute maximality w.r.t. paraconsistency
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Inconsistency maintenance

Expressivity

Guaranteed!
1. **Sette’s logic P1**

P1 = (\{t,m,f\}, \{t,m\}, \{\neg, \lor, \land, \rightarrow\})

- The \{\neg, \lor\}-fragment, the \{\neg, \land\}-fragment, the \{\neg, \rightarrow\}-fragment, and all the other fragments containing Sette’s negation are also ideal \neg-paraconsistent logics.
Examples (Cont’d.)

2. The Logic PAC

PAC = (\{t,m,f\}, \{t,m\}, \{\neg, \lor, \land, \supset\})

3. The Logic J3

PAC + the propositional constant FALSE
Examples (Cont’d.)

4. Logics of Formal Inconsistency (LFIs)

[da-Costa, Carnielli-Marcos]

LFI = (\{t,m,f\}, \{t,m\}, \{\neg, \circ, \lor, \land, \rightarrow\})

\begin{array}{|c|c|c|c|}
\hline
\lor & t & f & m \\
\hline
 t & t & t & t|m \\
 f & t & f & t|m \\
 m & t|m & t|m & t|m \\
\hline
\end{array}

\begin{array}{|c|c|c|c|}
\hline
\land & t & f & m \\
\hline
 t & t & f & t|m \\
 f & f & f & f \\
 m & t|m & f & t|m \\
\hline
\end{array}

\begin{array}{|c|c|c|c|}
\hline
\rightarrow & t & f & m \\
\hline
 t & t & f & t|m \\
 f & t & t & t|m \\
 m & t|m & f & t|m \\
\hline
\end{array}

\begin{array}{|c|c|}
\hline
\neg & \circ \\
\hline
 t & f \\
 f & t \\
 m & t|m \\
\hline
\end{array}

• Interpretations: 2 for \neg, 2^5 for \lor, 2^3 for \land, and 2^4 for \rightarrow. Altogether 8,192 (8K) LFI’s.
• All of these LFIs are ideal paraconsistent logics.
...But

5. Priest’s 3-Valued Logic LP is not ideal

LP = ({t,m,f}, {t,m}, {¬, ∨, ∧,})

- LP does not have a (primitive or defined) proper implication, thus it is not normal.

- LP is maximally paraconsistent and maximal relative to classical logic, but it is not ideal.
Summary and Conclusion

• We define in precise terms the basic properties that an `ideal propositional paraconsistent logic’ is expected to have, and investigate the relations between them.

• Identification of known logics (in particular, three-valued ones) that meet these requirements.

• For any n>2 ideal n-valued paraconsistent logics do exist (by a constructive construction).

• A directive (rather than a definite) approach on how to choose useful paraconsistent logics; Incorporation of further considerations for choosing appropriate logics for specific needs.

• Future work:
  • Some open questions (e.g., does maximal paraconsistency always imply maximality relative to classical logic? [as in the three-valued case]),
  • Nonmonotonic reasoning.