Abstract—Recently, Jadbabaie et al. presented a social learning model, where agents update beliefs by combining Bayesian posterior beliefs based on personal observations and weighted averages of the beliefs of neighbors. For a network with fixed topology, they provided sufficient conditions for all the agents in the network to learn the true state almost surely. In this paper, we extend the model to networks with time-varying topologies. Under certain assumptions on weights and connectivity, we prove that agents eventually have correct forecasts for upcoming signals, and all the beliefs of agents reach a consensus. In addition, if there is no state that is observationally equivalent to the true state from the point of view of all agents, we show that the consensus belief of agents eventually reflects the true state.

I. INTRODUCTION

It is well known that beliefs affect decisions. For example, our beliefs or opinions shape what kind of clothes we choose to buy. Similarly, when we vote, our ideological beliefs will determine which candidate we decide to support. Because of the importance of beliefs, the study of how beliefs are formed has gradually become a hot topic in past decades, and created a new research area—social learning theory.

Social learning models can be classified as Bayesian and non-Bayesian. Bayesian models are those in which fully rational individuals use Bayes’ rule to form the best mathematical estimate of the relevant unknowns given their priors and understanding of the world [1]. Generally, Bayesian learning must consider the strategy of others in order to make optimal inference and decision. Most of these models can be formulated as a dynamic game with incomplete information, the analysis of which is quite challenging even for simple social network situations [2]–[7]. By contrast, individuals in non-Bayesian models are boundedly rational and form beliefs in naive ways, such as imitation and replication [8]–[12]. Most non-Bayesian models focus on investigating whether beliefs of the whole group can reach a consensus, i.e., agreement among agents with different initial beliefs. One problem facing non-Bayesian models, however, is that the consensus belief of the whole group (if it exists) is just a mixture of initial beliefs, which generally may not reveal the truth.

Most recently, Jadbabaie et al. [13] developed a non-Bayesian learning model, in which individuals use a simple updating rule which linearly combines their Bayesian posterior beliefs based on private signals and weighted averages of the beliefs of their neighbors. In this model, Bayesian deduction occurs in a personal situation, which is totally different from traditional Bayesian models, where other people’s actions and strategies must be considered. The major advantage of the model is that all the agents are boundedly rational such that the belief updating rule is quite simple, as that in most non-Bayesian learning models. Meanwhile, all agents can learn the truth, as they do in most Bayesian learning models, where much complicated deduction is needed to achieve this goal.

In [13], Jadbabaie et al. investigated the model in fixed and strongly connected networks. The underlying networks of interactions in the real world, however, are more likely to be time-varying. For example, neighboring relationships between individuals may be established or removed over time. Sometimes, even if the communications are maintained, the levels of trust are changed as individuals receive a growing amount of information. Motivated by this consideration, we extend Jadbabaie’s model to networks with time-varying topologies in this paper. With assumptions on the structure of networks, including joint connectivity and double stochasticity of networks, and positive weight threshold, we show that all agents eventually have correct forecasts for upcoming signals, and all the beliefs of agents reach a consensus. Furthermore, without a state that is observationally equivalent to the true state from the point of view of all agents, the consensus belief eventually reflects the true state.

The paper is organized as follows. In Section II, the social learning model in networks with time-varying topologies is described. In Section III, assumptions and main results are presented, and all proofs can be found in Section IV. Section V includes simulations to further verify our results. Finally, section VI contains concluding remarks.

II. MODEL DESCRIPTION

A. Network Structure

Consider a social network with time-varying topologies as a set of directed graphs: \( \{G_t = (V, E_t), t \geq 0\} \), where \( V = \{1, 2, \ldots, n\} \) is the node set and \( E_t \subset V \times V \) is the edge set at time \( t \). Each node in \( V \) represents an agent, and an edge connecting \( i \) to \( j \), denoted by the order pair \((i, j) \in E_t\), captures the fact that information flows from agent \( i \) to agent \( j \), and agent \( i \) is called a neighbor of agent \( j \). The set of neighbors of agent \( j \) at time \( t \) is denoted by \( N_j(t) = \{i \in V : (i, j) \in E_t\} \).
B. Possible States and Beliefs of Individuals

Let $\theta$ denotes a state of the world, and all the possible states compose a finite state set $\Theta = \{\theta_1, \theta_2, \cdots, \theta_m\}$, in which the true state is denoted by $\theta^*$. The belief of agent $i$ on state $\theta$ at time $t$ is denoted by $\mu_{i,t}(\theta)$, which is the probability that she believes state $\theta$ is true. Note that $\{\mu_{i,t}(\theta), \theta \in \Theta\}$ is a probability distribution over the state set $\Theta$.

C. Signal Structure

Conditional on the true state, at each time period $t > 0$, signal vector $s_t = (s_{t1}, \cdots, s_{tn}) \in S$ is generated according to the likelihood function $\ell(s_t|\theta^*)$, where signal $s_{it}$ is observed by agent $i$ at period $t$ and $S$ is the signal space. All of agent $i$'s observations compose her private signal set $s_i = \{s_{it}, t > 0\}$. For each observed signal $s_{it}$, agent $i$ has personal signal structure for each $\theta$, which is denoted by $\ell_i(s_{it}|\theta)$, representing the probability of $s_{it}$ arising if the true state is $\theta$ in agent $i$'s viewpoint. Furthermore, we assume that the signal structure of the true state $\theta^*$, i.e., $\ell_i(s_{it}|\theta^*)$, is the $i$-th marginal of $\ell(s_{it}|\theta^*)$, which means agents have exact knowledge of what will happen if the true state is $\theta^*$. Just as in the real world, observations do not necessarily contain valuable information that facilitates agents to infer the true state. In [13], observationally equivalent is used to denote the identification problem. The elements of the set $\tilde{\Theta}_i = \{\theta \in \Theta : \ell_i(s|\theta) = \ell_i(s|\theta^*)$ for all $s \in s_i\}$ are observationally equivalent to the true state $\theta^*$ from the point of view of agent $i$.

D. Belief Updating Rule

The belief updating rule can be described as, for all $\theta \in \Theta$

$$
\mu_{i,t+1}(\theta) = a_{ii}(t)\mu_{i,t}(\theta) - \frac{\ell_i(s_{i,t+1}|\theta)}{m_{i,t}(s_{i,t+1})} + \sum_{j \in N_i(t)} a_{ij}(t)\mu_{j,t}(\theta) \\
(1)
$$

where $m_{i,t}(s_{i,t+1}) = \sum_{\theta \in \Theta} \ell_i(s_{i,t+1}|\theta)\mu_{i,t}(\theta)$, which is the probability measure induced by agent $i$'s beliefs over the realization of her private signal $s_{i,t+1}$ in the next step, also known as agent $i$'s one-step-ahead forecast. Similarly, we define the k-step-ahead forecast of agent $i$ at time $t$ as

$$
m_{i,t}(s_{i,t+1}, \cdots, s_{i,t+k}) = \sum_{\theta \in \Theta} \left( \prod_{r=1}^{k} \ell_i(s_{i,t+r}|\theta) \right) \mu_{i,t}(\theta)
$$

which will be used in later analysis.

Belief updating rule (1) can be interpreted as that the individual updates her belief as a convex combination of the Bayesian posterior belief conditioned on her private signal and beliefs of her neighbors.

To facilitate the analysis, we rewrite (1) in matrix form as

$$
\mu_{i+1}(t) = A(t)\mu_{i}(t) + \text{diag}\left(\frac{\ell_i(s_{i,t+1}|\theta)}{m_{i,t}(s_{i,t+1})} - 1\right)\mu_{i}(t)
$$

where $\mu_{i}(\theta) = [\mu_{i,t}(\theta), \cdots, \mu_{n,t}(\theta)]^T$; $A(t)$ is called interaction matrix which has its entry on ith row and jth column as $a_{ij}(t)$; diag of a vector is a diagonal matrix with the vector on its diagonal.

E. Definitions of Social Learning

“Social learning” may have different meanings in different contexts. Here we adopt the definitions in [13], where two sort of learning have been defined.

Definition 1 [13]: The k-step-ahead forecasts of agent $i$ are eventually correct on a path $\{s_{i,t}\}_{t=1}^\infty$ if, along that path,

$$
m_{i,t}(s_{i,t+1}, \cdots, s_{i,t+k}) \rightarrow \prod_{r=1}^{k} \ell_i(s_{i,t+r}|\theta^*) \quad \text{as} \quad t \rightarrow \infty.
$$

Moreover, if agent $i$’s k-step-ahead forecasts are eventually correct for any natural number $k$ on some path, we say her beliefs weakly merge to the truth.

Definition 2 [13]: Agent $i$ asymptotically learns the underlying true state $\theta^*$ on a path $\{s_{i,t}\}_{t=1}^\infty$ if, along that path, $\mu_{i,t}(\theta^*) \rightarrow 1$ as $t \rightarrow \infty$.

Strictly speaking, the two notions of “social learning” are distinct and might not occur simultaneously. However, a general phenomenon is that learning the true state will suffice for weak merging to the truth, which is also the case here. More detailed discussion and examples about these two sorts of learning can be found in [14].

F. Learning in networks with fixed topologies

For a network with fixed topology, i.e., $G_t = G$ and $A(t) = A$, the main result in [13] can be summarized as follows:

Theorem 1 [13]: Suppose that

(a) The network is strongly connected.
(b) For all $i$ and $j$, $a_{ii} > 0$, and $a_{ij} > 0$ if agent $i$ has access to the belief held by agent $j$; otherwise, $a_{ij} = 0$.
(c) The interaction matrix $A$ is row stochastic, i.e., $\sum_{j=1}^n a_{ij} = 1$ for all $i$.

If there exists at least one agent with positive initial belief on the true state $\theta^*$, then,

1) the beliefs of all agents weakly merge to the truth almost surely;
2) all agents learn the true state almost surely.

III. MAIN RESULT

A. Assumptions

Now we consider the social learning model (1) with time-varying topologies. In this case, the network does not need to be strongly connected at every time slot, but some kind of connectivity is still required. We make the following joint connectivity assumption:

Assumption 1 (Joint Connectivity): The directed graph $(V,E_{\infty})$ is strongly connected, where $E_{\infty} = \{(i,j) \mid \text{There exists an integer } B \geq 1 \text{ such that edge } (i,j) \text{ appears at least once every } B \text{ consecutive time slots}\}.$
As a natural generalization of condition (b) in Theorem 1, we assume that there exists a positive weight threshold, i.e.,

**Assumption 2 (Weight Threshold):** There exists a scalar $0 < \eta < 1$ such that for all $i$ and $j$,

1) $a_{ij}(t) \geq \eta$ for all $t \geq 0$.
2) if agent $j$ is a neighbor of agent $i$ at time $t$, then $a_{ij}(t) \geq \eta$.

To guarantee the existence of a common left eigenvector of the interaction matrices corresponding to the unit eigenvalue, we extend the row stochasticity condition (c) in Theorem 1 to the following double stochasticity assumption:

**Assumption 3 (Double Stochasticity):** The interaction matrix is doubly stochastic, i.e., for any $i$ and $j$,

$$\sum_{i=1}^{n} a_{ij}(t) = \sum_{j=1}^{n} a_{ij}(t) = 1.$$

### B. Main Result

With these assumptions in hand, we now present the main result of our work on social learning in networks with time-varying topologies, and will cover the proof of it in the next section.

**Theorem 2:** Consider the social learning model (1). Suppose that Assumptions 1-3 hold, and there exists at least one agent with positive initial belief on the true state $\theta^*$. Then,

1) the beliefs of all agents weakly merge to the truth almost surely;
2) the beliefs of all agents converge to consensus almost surely.

In addition, suppose that there is no state $\theta \neq \theta^*$ that is observationally equivalent to $\theta^*$ from the point of view of all agents in the network. Then,

3) all agents learn the true state almost surely.

Note that learning the true state implies that all agents assign an asymptotic belief of one to the true state, and zero to other states, which can be viewed as consensus in a special situation, where only the true state $\theta^*$ in the intersection $\Theta_1 \cap \cdots \cap \Theta_n$. We also discuss a more general situation where more than one states are in the intersection. We point out in the second statement that beliefs of all agents on any state reach a consensus, no matter whether the state is in the intersection or not.

### IV. THEORETICAL ANALYSIS

#### A. Proof of Weak Merging to the Truth

Actually, we can trivially extend the proof of weak merging to the truth in [13] to networks with time-varying topologies as long as certain conditions hold. Therefore, we omit details here and just provide simple analysis of the proof.

In [13], the proof of weak merging to the truth depends on three conditions: (1) the interaction matrix $A$ has a fixed left eigenvector with all positive entries corresponding to the unit eigenvalue; (2) all agents have strictly positive self-reliances, i.e., $a_{ii} > 0$ for all $i$; (3) there exists an agent $i$ such that $\mu_{i,0}(\theta^*) > 0$ and, with the connectivity of the network, after long enough time, all the agents assign a strictly positive probability to the true state. That is to say, as long as these three conditions are satisfied, the method of proof in [13] can be applied to any situation. Now let us check what conditions are satisfied in our model. Firstly, under the double stochasticity assumption, i.e., $\sum_{i=1}^{n} a_{ij}(t) = \sum_{j=1}^{n} a_{ij}(t) = 1$, the interaction matrix $A(t)$ always has a fixed left eigenvector of all ones corresponding to the unit eigenvalue, even if $A(t)$ is time-varying. Secondly, Assumption 2 guarantees that all agents have strictly positive self-reliances, and the weights have a lower bound to guarantee the effectiveness of communication. At last, we also assume there exists an agent with positive initial belief assigned to the true state, and, combined with the joint connectivity in Assumption 1, all the agents have positive beliefs on the true state after at most $(n-1)B$ time slots. Summarily, all the conditions needed to perform the analysis in [13] to prove the weak merging to the truth are satisfied. Although some parts of the proof need to be modified slightly to fit our setting, one can easily repeat the proof in [13] with the remarks we have given.

#### B. Proof of Reaching a Consensus on Beliefs

We prove it in two parts. First, we focus on the set of states that are observationally equivalent to the true state from the point of view of all agents in the society. In the second part, we focus on the complement set.

**Part 1:** (For $\theta \in \Theta_1 \cap \cdots \cap \Theta_n$) We have already pointed out that the beliefs of all agents weakly merge to the truth almost surely, which implies that one-step-forecast is correct for any agent $i$, i.e.,

$$m_{i,t}(s_{t+1}^i) \rightarrow \ell_i(s_{t+1}^i|\theta^*) \text{ as } t \rightarrow \infty.$$ 

For any state $\theta \in \Theta_1 \cap \cdots \cap \Theta_n$, the above statement is also correct since $\ell_i(\theta) = \ell_i(\theta^*)$ for any observation. Then, according to the belief updating rule (2), we have

$$\mu_{i,t+1}(\theta) = A(t)\mu_i(\theta) + e_i(\theta) \tag{3}$$

where $e_i(\theta) = [e_{i,1}(\theta), \ldots, e_{n,1}(\theta)]^T$ and $e_{i,t}(\theta) = a_{ii}(t)\mu_{i,t}(\theta)\frac{\ell_i(s_{t+1}^i|\theta)}{m_{i,t}(s_{t+1}^i)} - 1$, which converges to zero almost surely as time goes on.

To describe the evolution of the beliefs, we introduce transition matrices

$$\Phi(t, s) = A(t)A(t-1)\cdots A(s)$$

for all $t$ and $s$ with $t \geq s$ where $\Phi(t, t) = A(t)$ for all $t$. Recent works have established explicit convergence rate results for these transition matrices [15]–[17].

Let Assumptions 1-3 hold. Then, according to [15], for all $i$, $j$ and all $t$, $s$ with $t \geq s$, we have

$$|\Phi(t, s)|_{ij} - \frac{1}{n} \leq \alpha \beta^{t-s},$$

where $|\Phi(t, s)|_{ij}$ is the entry of $\Phi(t, s)$ on $i$th row and $j$th column, $\alpha = 2\frac{1-\eta^{(n-1)B}}{1-\eta}$ and $\beta = [1-\eta^{(n-1)B}]^{1/(n-1)B} < 1$. 

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Based on (3), the relation between $\mu_{i,t+1}(\theta)$ and $\mu_{1,0}(\theta), \ldots, \mu_{n,0}(\theta)$ is given by

$$\mu_{i,t+1}(\theta) = \sum_{j=1}^{n} [\Phi(t, 0)]_{ij} \mu_{j,0}(\theta) + \sum_{r=1}^{t} \left( \sum_{j=1}^{n} [\Phi(t, r)]_{ij} e_{j,r-1}(\theta) \right) + e_{i,t}(\theta)$$

(4)

Then, we define an auxiliary sequence $\{z_t(\theta), t \geq 0\}$, where $z_t(\theta)$ is given by

$$z_t(\theta) = \frac{1}{n} \sum_{i=1}^{n} \mu_{i,t}(\theta)$$

for all $t$. Combining the two equation above implies

$$z_t(\theta) = \frac{1}{n} \sum_{j=1}^{n} \mu_{j,0}(\theta) + \frac{1}{n} \sum_{r=1}^{t-1} \left( \sum_{j=1}^{n} e_{j,r-1}(\theta) \right)$$

+ $\frac{1}{n} \sum_{j=1}^{n} e_{j,t-1}(\theta)$.

(5)

Using the relations in (4) and (5), we obtain for all $i$,

$$|\mu_{i,t}(\theta) - z_t(\theta)| = \left| \sum_{j=1}^{n} ([\Phi(t - 1, 0)]_{ij} - \frac{1}{n}) \mu_{j,0}(\theta) \right|$$

+ $\sum_{r=1}^{t-1} \sum_{j=1}^{n} ([\Phi(t - 1, r)]_{ij} - \frac{1}{n}) e_{j,r-1}(\theta)$

+ $|e_{i,t-1}(\theta)| - \frac{1}{n} \sum_{j=1}^{n} |e_{j,t-1}(\theta)|$

+ $\sum_{j=1}^{n} [\Phi(t - 1, 0)]_{ij} - \frac{1}{n} \sum_{j=1}^{n} e_{j,r-1}(\theta)$

+ $\sum_{j=1}^{n} \sum_{r=1}^{t-1} [\Phi(t - 1, r)]_{ij} - \frac{1}{n} \sum_{j=1}^{n} e_{j,r-1}(\theta)$

+ $|e_{i,t-1}(\theta)| - \frac{1}{n} \sum_{j=1}^{n} |e_{j,t-1}(\theta)|$

$$\leq \alpha \beta - 1 \sum_{j=1}^{n} |\mu_{j,0}(\theta)|$$

+ $\sum_{r=1}^{t-1} \alpha \beta^{r-1} \sum_{j=1}^{n} |e_{j,r-1}(\theta)|$

+ $|e_{i,t-1}(\theta)| - \frac{1}{n} \sum_{j=1}^{n} |e_{j,t-1}(\theta)|$

Since $\beta$ less than one and $e_{i,t}(\theta)$ converges to zero almost surely for all $i$ as time goes on, we have

$$\lim_{t \to \infty} \alpha \beta^{t-1} \sum_{j=1}^{n} |\mu_{j,0}(\theta)| = 0$$

$$\lim_{t \to \infty} \left( |e_{i,t-1}(\theta)| + \frac{1}{n} \sum_{j=1}^{n} |e_{j,t-1}(\theta)| \right) = 0$$

Furthermore, $\lim_{t \to \infty} \alpha \beta^{t-1} = 0$ for any $r$. Since $\sum_{r=1}^{t-1} \alpha \beta^{t-r-1} < \frac{\alpha}{1-\beta}$ and $\sum_{j=1}^{n} |e_{j,r-1}(\theta)| \to 0$ almost surely as $r \to \infty$, by Toeplitz lemma, we have

$$\lim_{t \to \infty} \sum_{r=1}^{t-1} \alpha \beta^{t-r-1} \sum_{j=1}^{n} |e_{j,r-1}(\theta)| = 0.$$

Thus,

$$\lim_{t \to \infty} \left| \mu_{i,t}(\theta) - z_t(\theta) \right| = 0$$

which implies $\mu_{i,t}(\theta) - \mu_{j,t}(\theta) \to 0$ almost surely for all $i, j$, and all $\theta \in \Theta_1 \cap \cdots \cap \Theta_n$.

In order to complete the proof of Part 1, all we need to show is the existence of $\lim_{t \to \infty} \mu_{i,t}(\theta)$. It is shown in [13] that $\sum_{i=1}^{n} v_i \mu_{i,t}(\theta)$ converges almost surely, where $[v_1, \ldots, v_n]$ is the left eigenvector corresponding to the unit eigenvalue of interaction matrix $A$. Given the argument in the analysis of the proof of weak merging to the truth, the claim is also true in our time-varying scenario, where $[v_1, \ldots, v_n] = [1, 1, \ldots, 1]$ is an eigenvector of $A(t)$ because of the double stochasticity. Therefore, $\sum_{i=1}^{n} \mu_{i,t}(\theta)$ converges almost surely. Since that $\mu_{i,t}(\theta) - \mu_{j,t}(\theta) \to 0$, $\lim_{t \to \infty} \mu_{i,t}(\theta)$ exists for all $i$ and all $\theta \in \Theta_1 \cap \cdots \cap \Theta_n$, and the value does not depend on $i$.

Part 2: (For $\theta \notin \Theta_1 \cap \cdots \cap \Theta_n$) We now consider the case that $\theta \notin \Theta_1$ for some agent $i$. Before processing next steps of the proof, we state a lemma from [13], which shows that there should exist a long enough signal sequence that is more probable under $\theta^*$ than any other state that is not observationally equivalent to $\theta^*$.

Lemma 4 in [13]: For any agent $i$, and any time $t$, there exists a positive integer $k_i$, a sequence of signals $(s_{t+1}, \ldots, s_{t+k_i})$, and constant $\delta_i \in (0, 1)$ such that

$$\prod_{t=1}^{k_i} \frac{\ell_i(s_{t+1}^i|\theta^*)}{\ell_i(s_{t+1}^i|\theta^*)} \leq \delta_i, \quad \forall \theta \notin \Theta_i$$

(6)

Recall that $m_{i,t}(s_{t+1}^i, \ldots, s_{t+k_i}^i) \to \prod_{r=1}^{k_i} \ell_i(s_{t+1}^i|\theta^*)$ almost surely for any natural number $k$. In particular, the claim is true for the sequence of signals $(s_{t+1}^i, \ldots, s_{t+k_i}^i)$ satisfying (6):

$$1 - \frac{m_{i,t}(s_{t+1}^i, \ldots, s_{t+k_i}^i)}{\prod_{r=1}^{k_i} \ell_i(s_{t+1}^i|\theta^*)} = 1 - \sum_{\theta \notin \Theta_i} \mu_{i,t}(\theta) \prod_{r=1}^{k_i} \frac{\ell_i(s_{t+1}^i|\theta^*)}{\ell_i(s_{t+1}^i|\theta^*)}$$

$$= 1 - \sum_{\theta \notin \Theta_i} \mu_{i,t}(\theta) \prod_{r=1}^{k_i} \frac{\ell_i(s_{t+1}^i|\theta^*)}{\ell_i(s_{t+1}^i|\theta^*)} - \sum_{\theta \notin \Theta_i} \mu_{i,t}(\theta)$$

$$= \sum_{\theta \notin \Theta_i} \mu_{i,t}(\theta) \left( 1 - \prod_{r=1}^{k_i} \frac{\ell_i(s_{t+1}^i|\theta^*)}{\ell_i(s_{t+1}^i|\theta^*)} \right) \to 0 \quad \text{a.s.}$$

1We would like to thank a referee for bringing a technical mistake in an earlier draft of the paper to our attention.
Since
\[ 1 - \prod_{r=1}^{k_i} \frac{\ell_i(s_{i,r+1}^{t+r} | \theta)}{\ell_i(s_{i,r}^{t+r} | \theta)} \geq 1 - \delta_i > 0 \quad \forall \theta \notin \bar{\Theta}_i, \]
and as a consequence, it must be the case that \( \mu_{i,t}(\theta) \rightarrow 0 \) as \( t \to 0 \) for any \( \theta \notin \bar{\Theta}_i \).

Now consider the belief updating rule of agent \( i \) given by (1), evaluated at some state \( \theta \notin \bar{\Theta}_i \):
\[ \mu_{i,t+1}(\theta) = a_{i,t}(\mu_{i,t}(\theta) \frac{\ell_i(s_{i,t+1}^{t+r} | \theta)}{m_i,t(s_{i,t+1}^{t+r})} + \sum_{j \in N_i(t)} a_{ij}(t) \mu_{j,t}(\theta) \]
We have already shown that \( \mu_{i,t}(\theta) \rightarrow 0 \) almost surely. Now we prove that any other agent \( j \) in the network also has \( \mu_{j,t}(\theta) \rightarrow 0 \) almost surely for the same \( \theta \) by contradiction.

Suppose that there exists an agent \( j \) in the network satisfies following statement with a positive probability: there exists an real number \( \sigma > 0 \) satisfying that \( \mu_{j,T}(\theta) > \sigma \) for an arbitrary large time \( T \). Then based on (1), any agent that is influenced by agent \( j \), say \( k \), has belief \( \mu_{k,T+1}(\theta) > \eta_0 \sigma \) at time \( T + 1 \), because of the lower bound of positive weight in Assumption 2. Considering the joint connectivity property in Assumption 1, we know that the belief of agent \( j \) affects the belief of agent \( i \) in at most \( (n-1)B \) time slots. Without loss of generality, suppose agent \( i \) is influenced directly or indirectly by agent \( j \) at time \( T+s \), where \( 1 \leq s \leq (n-1)B \). Then agent \( i \)'s belief \( \mu_{i,T+s}(\theta) > \eta_0 \sigma \geq \eta_0^{(n-1)B} \sigma \), where \( \eta_0^{(n-1)B} \sigma \) is a positive constant. Note that \( T + s \) can be arbitrary large, which means \( \mu_{i,t}(\theta) \), with a positive probability, cannot converge to zero as time goes on. This is incompatible with the fact that \( \mu_{i,t}(\theta) \rightarrow 0 \) almost surely. Thus it proves the claim that any other agent \( j \) in the network must have \( \mu_{j,i}(\theta) \rightarrow 0 \) almost surely. Consequently, any state \( \theta \notin \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n \) is assigned an asymptotic belief of zero.

Combining the results in both Part 1 and Part 2, we have shown that the beliefs of all agents converge to consensus almost surely.

C. Proof of Learning the True State

Recall that all agents assign an asymptotic belief of zero to states that are not observationally equivalent to \( \theta^* \) from the point of view of all individuals. And here we assume that there is no state \( \theta \neq \theta^* \) that is observationally equivalent to \( \theta^* \) from the point of view of all agents in the network. This implies \( \mu_{i,t}(\theta) \rightarrow 0 \) for all \( \theta \neq \theta^* \), and therefore, \( \mu_{i,t}(\theta^*) \rightarrow 1 \), which means all agents learn the true state almost surely.

V. SIMULATIONS

We consider a simple network of three agents. The topologies of the network at different times are shown in Fig. 1. The correspondingly time-varying interaction matrices are
\[
A(2t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(2t+1) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}
\]
for all \( t \geq 0 \). From Fig. 1 we know that the underlying network in each time is not strongly connected, but the union of networks in every two consecutive time slots is strongly connected.

Suppose that the state set is \( \Theta = \{ \theta_1, \theta_2, \theta_3 \} \), in which the true state \( \theta^* = \theta_3 \). Initial beliefs of any agent \( i \), i.e., \( \{ \mu_{i,0}(\theta_1), \mu_{i,0}(\theta_2), \mu_{i,0}(\theta_3) \} \) are adopted randomly in interval \( [0,1] \), satisfying \( \sum_{k=1}^{3} \mu_{i,0}(\theta_k) = 1 \). Signal set is \( \{ H, T \} \), generated according to the likelihood functions \( \ell(H|\theta_3) = 0.2 \) and \( \ell(T|\theta_3) = 0.8 \). We perform two simulations representing two different situations classified by whether the true state is the only state in \( \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n \), or not.

1) With multiple states in \( \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n \). In the first simulation, we suppose that all three agents view states \( \theta_1 \) and \( \theta_3 \) as observationally equivalent states, meanwhile \( \ell(H|\theta_2) = 0.5 \) for \( i = 1, 2, 3 \). The corresponding simulation is shown in Fig. 2.

The figure is plotted on a semilog scale to illustrate both detailed dynamic aggregation in the incipient stage and convergence in long run. It is shown that beliefs of all agents on state \( \theta_2 \) converge to 0, and beliefs on the true state \( \theta_3 \) and its observationally equivalent state \( \theta_1 \) converge to nonzero constants, which means, with other states observationally equivalent to the true state, agreement among agents is obtained, but agents cannot recognize the true state, i.e., asymptotic learning cannot be achieved.

2) With only the true state in \( \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n \). In the second simulation, we set that agent 1 and agent 2 view states \( \theta_1 \) and \( \theta_3 \) as observationally equivalent states, meanwhile \( \ell(H|\theta_2) = 0.5 \) for \( i = 1, 2 \). From the point of view of agent 3, states \( \theta_2 \) and \( \theta_3 \) are observationally equivalent, meanwhile \( \ell(3|\theta_3) = 0.8 \). Therefore, the intersection of all agents’ observationally equivalent states is only the true state \( \theta_3 \). Fig. 3 shows that beliefs of state \( \theta_1 \) and \( \theta_2 \) converge to 0, and beliefs of the true state \( \theta_3 \) converge to 1, which means the asymptotic learning has been achieved.

Note that, in the simulations, the interaction matrices are not doubly stochastic, which implies that double stochasticity is only a technique assumption, not a necessary one.

VI. CONCLUSIONS

In this paper, we extend a non-Bayesian social learning model from networks with fixed topologies to networks with time-varying topologies. With assumptions on the structure of networks, such as existence of weight threshold and joint connectivity of networks in finite time intervals, we
have proven that agents eventually have correct forecasts for upcoming signals. As in most non-Bayesian models, consensus in beliefs of all agents can be reached in the model. Moreover, the consensus belief eventually reflects the true state if there is no state observationally equivalent to the true state.

Note that we have provided sufficient conditions for successful learning, and some of them can be further relaxed, such as the double stochasticity. The constraints on weights and connectivity are necessary, at least during the initial time period of belief updating, to effectively exchange information. As time goes on, these assumptions may be redundant after individuals have correct forecasts or learn the truth. Thus, we conjecture that weights decreasing at a certain speed may be a weaker condition for successful learning.

References