

REGULAR L -FUZZY TOPOLOGICAL SPACES AND THEIR TOPOLOGICAL MODIFICATIONS

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ABSTRACT. For L a continuous lattice with its Scott topology, the functor ι_L makes every regular L -topological space into a regular space and so does the functor ω_L the other way around. This has previously been known to hold in the restrictive class of the so-called weakly induced spaces. The concepts of H -Lindelöfness (à la Hutton compactness) is introduced and characterized in terms of certain filters. Regular H -Lindelöf spaces are shown to be normal.

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1. Introduction. The two functors that provide a working link between the category $\mathbf{TOP}(L)$ of L -(fuzzy)-topological spaces and \mathbf{TOP} are the Lowen functors ι_L and ω_L . For a wide class of lattices L 's, ι_L is a right adjoint and left inverse of ω_L . Therefore, it is of interest to know how various L -topological invariants behave with respect to these functors.

In this paper, we show that when L is a continuous lattice with its Scott topology then ι_L maps the category $\mathbf{Reg}(L)$ of L -regular spaces onto the category \mathbf{Reg} of regular spaces. This improves upon and extends a result of Liu and Luo [6] which showed (with different but equivalent terminology) that ι_L maps weakly induced L -regular spaces to regular spaces (with L a completely distributive lattice with its upper topology). As a consequence, we have that $\omega_L(\mathbf{Reg})$ consists precisely of L -regular spaces of $\omega_L(\mathbf{TOP})$. Some generalities about L -regular spaces are included and stated in a slightly more general situation, viz. for L -topologies that admit a certain type of approximating relation. This captures complete L -regularity and zero-dimensionality.

We also introduce the concept of H -Lindelöfness (compatible with compactness in the sense of Hutton [2]) and characterize it in terms of closed filters. Finally, we prove that H -Lindelöf and L -regular spaces are L -normal.

2. Notation and some terminology. All the fuzzy topological concepts that concern us are standard. We nevertheless recall some of them.

Let $L = (L, ')$ be a complete lattice (bottom denoted 0) endowed with an order-reversing involution $'$. Thus L satisfies the de Morgan laws. For X a set, L^X is the set of all maps from X to L (called L -sets). Then $(L^X, ')$ is a complete lattice under pointwisely defined ordering and the order-reversing involution. The de Morgan laws

are also inherited by L^X . An L -topology on X is a family of elements of L^X (called open L -sets) such that any supremum and any finite infimum of open L -sets are open. The L -topology of an L -topological space (L -ts) X is denoted $o(X)$. Members of $\kappa(X) = \{k \in L^X : k' \in o(X)\}$ are called closed. For each $a \in L^X$, we let $\text{Int } a = \bigvee \{u \in o(X) : u \leq a\}$ and $\bar{a} = (\text{Int}(a'))'$. If X and Y are two L -ts's, then $f : X \rightarrow Y$ is continuous if uf (the composition of f and u) is in $o(X)$ whenever $u \in o(Y)$. The weakest L -topology on X making f continuous is denoted by $f^-(o(Y))$. We say that $S \subset L^X$ generates $o(X)$ if $o(X) = \bigcap \{T : S \subset T, \text{ an } L\text{-topology on } X\}$. If \mathcal{T} is a family of L -topologies on X , then the supremum L -topology $\bigvee \mathcal{T}$ is generated by $\bigcup \mathcal{T}$. In particular, $\bigvee_{j \in J} \pi_j^-(o(X_j))$ is the product L -topology on $\prod_{j \in J} X_j$ (π_j being the j th projection). The set of all restrictions $\{u \upharpoonright A : u \in o(X)\}$ is the subspace L -topology on $A \subset X$.

Given $\alpha, \beta \in L$ we let $\alpha \ll \beta$ whenever for any $B \subset L$ with $\beta \leq \bigvee B$ there is a finite $B_0 \subset B$ such that $\alpha \leq \bigvee B_0$. Then L is called continuous if $\alpha = \bigvee \{\beta \in L : \beta \ll \alpha\}$ for every $\alpha \in L$. We write $\downarrow \alpha = \{\beta \in L : \beta \ll \alpha\}$ and dually for $\uparrow \alpha$. Each continuous L has the interpolation property: $\alpha \ll \beta$ implies $\alpha \ll \gamma \ll \beta$ for some $\gamma \in L$. The Scott topology $\sigma(L)$ on a continuous L is one which has $\{\uparrow \alpha : \alpha \in L\}$ as a base. We write ΣL for $(L, \sigma(L))$ (see [1] for details).

We also recall that L is a frame provided $\alpha \wedge \bigvee B = \bigvee \{\alpha \wedge \beta : \beta \in B\}$ for every $\alpha \in L$ and $B \subset L$.

Given $a \in L^X$ and $\alpha \in L$, we let $[a \gg \alpha] = \{x \in X : a(x) \gg \alpha\}$, $[a \not\leq \alpha] = \{x \in X : a(x) \not\leq \alpha\}$, etc. The constant member of L^X with value α is denoted α as well, and $\alpha 1_A = \alpha \wedge 1_A$, where 1_A is the characteristic function of $A \subset X$. If $\mathcal{A} \subset L^X$, we let $\mathcal{A}' = \{a' : a \in \mathcal{A}\}$, $\bar{\mathcal{A}} = \{\bar{a} : a \in \mathcal{A}\}$, and similarly for $\text{Int } \mathcal{A}$. We include for record.

REMARK 2.1. Let L be a complete lattice and X a nonempty set. The following statements are equivalent:

- (1) L is continuous;
- (2) $a = \bigvee_{\alpha \in L} \alpha 1_{[a \gg \alpha]}$ for every $a \in L^X$;
- (3) $[a \not\leq \alpha] = \bigcup_{\beta \not\leq \alpha} [a \gg \beta]$ for every $a \in L^X$ and $\alpha \in L$.

3. L -topologies with approximating relation. Let $L = (L, ')$ be a complete lattice. An L -ts X is called L -regular [3] if for every $u \in o(X)$ there exists $\mathcal{V} \subset o(X)$ such that $u = \bigvee \mathcal{V}$ and $\bar{v} \leq u$ for all $v \in \mathcal{V}$. This is the case if and only if $u = \bigvee \mathcal{V} = \bigvee \bar{\mathcal{V}}$.

It is clear that X is L -regular if and only if for every basic open u one has $u = \bigvee \{v \in o(X) : \bar{v} \leq u\}$.

To avoid repetitions of some argument used in [5], we introduced an auxiliary relation $<$ on the L -topology $o(X)$ of an L -ts X .

DEFINITION 3.1. Let $<$ be a binary relation on $o(X)$ satisfying the following conditions for all $u, v, w_1, w_2 \in o(X)$:

- (1) $0 < u$;
- (2) $v < u$ implies $v \leq u$;
- (3) $w_1 \leq v < u \leq w_2$ implies $w_1 < w_2$;
- (4) $w_1 < u$ and $w_2 < u$ imply $w_1 \vee w_2 < u$;
- (5) $u < w_1$ and $u < w_2$ imply $u < w_1 \wedge w_2$.

We say X is \prec -regular if for each open u there exists $\mathcal{V} \subset o(X)$ such that $u = \bigvee \mathcal{V}$ and $v \prec u$ for all $v \in \mathcal{V}$.

EXAMPLES. (1) X is L -regular if and only if it is \prec -regular with $v \prec u$ defined by $\bar{v} \leq u$.

(2) X is completely L -regular [3] if and only if it is \prec -regular, where $v \prec u$ if and only if $v \leq L'_1 f \leq R_0 f \leq u$ for some $f \in C(X, I(L))$; see [5] for details and notice that (4) and (5) of Definition 3.1 require L to be meet-continuous (cf. Section 5).

(3) X is zero-dimensional if and only if it is \prec -regular and $v \prec u$, whenever $v \leq w \leq u$ for some closed and open w (cf. [9]).

PROPOSITION 3.2. *Let L be a complete lattice and let X be any of \prec -regular spaces of Example 3. The following hold*

- (1) *If $f : Y \rightarrow X$ is continuous, then Y is \prec -regular with respect to $f^-(o(X))$.*
 - (2) *Every subspace of X is \prec -regular.*
- If L is a frame, then*
- (3) *$u = \bigvee \{v : v \prec u\}$ for every subbasic open $u \in L^X$.*
 - (4) *If \mathcal{T} is a family of \prec -regular L -topologies on X , then $\bigvee \mathcal{T}$ is \prec -regular.*
 - (5) *\prec -regularity is preserved by arbitrary products.*

PROOF. The argument given in [5, Remark 2.5 and Lemma 2.3] for the case (2) of Example 3 goes unchanged in the remaining cases. □

PROPOSITION 3.3. *Let L be a continuous lattice. For X an L -topological space, the following are equivalent:*

- (1) *X is \prec -regular.*
- (2) *$u = \bigvee \{v : v \prec u\}$ for every (basic) open u .*
- (3) *$[u \gg \alpha] = \bigcup_{v \prec u} [v \gg \alpha]$ for every (basic) open u and $\alpha \in L$.*
- (4) *$[u \not\leq \alpha] = \bigcup_{v \prec u} [v \not\leq \alpha]$ for every (basic) open u and $\alpha \in L$.*

PROOF. (1) \implies (2). Obvious.

(2) \implies (3). Let $\alpha \ll u(x) = \bigvee \{v(x) : v \prec u\}$. Select $\beta \in L$ such that $\alpha \ll \beta \ll u(x)$. There is a finite family $\mathcal{V} \subset o(X)$ such that $\beta \leq (\bigvee \mathcal{V})(x)$ and $w \prec u$ for every $w \in \mathcal{V}$. Put $v = \bigvee \mathcal{V}$. Then $v \prec u$ and $\alpha \ll \beta \leq v(x)$. Thus $\alpha \ll v(x)$ with $v \prec u$. This proves the nontrivial inclusion of (3).

(3) \implies (4). If $u(x) \not\leq \alpha$, there is a β such that $\beta \ll u(x)$ and $\beta \not\leq \alpha$. By (3), $\beta \ll v(x)$ for some $v \prec u$. Then $v(x) \not\leq \alpha$, i.e., $[u \not\leq \alpha] \subset \bigcup_{v \prec u} [v \not\leq \alpha]$. The reverse inclusion is obvious.

(4) \implies (1). Let $u \neq 0$. Then $\Delta = \{(x, \beta) \in X \times L : u(x) \not\leq \beta\} \neq \emptyset$. For every pair $(x, \beta) \in \Delta$ select $v_{x\beta} \prec u$ such that $v_{x\beta}(x) \not\leq \beta$. Clearly, $\bigvee \{v_{x\beta} : (x, \beta) \in \Delta\} \leq u$. To show the converse, assume there exists $y \in X$ such that

$$y = \bigvee \{v_{x\beta}(y) : (x, \beta) \in \Delta\} \not\leq u(y). \tag{3.1}$$

Then $(y, y) \in \Delta$, hence $v_{yy}(y) \not\leq y$. But from (3.1) we have $v_{x\beta}(y) \leq y$ for all $(x, \beta) \in \Delta$, in particular $v_{yy}(y) \leq y$, a contradiction. □

REMARK 3.4. (1) The proof of (4) \implies (1) is a complete lattice proof. Since there is a direct and obvious complete lattice argument for (2) \implies (4), therefore (1) \iff (2) \iff (4) hold true for any complete lattice L .

(2) With L a complete chain without elements isolated from below (e.g., with $L = [0, 1]$), conditions (3) and (4) coincide. When expressed in terms of fuzzy points (these are L -sets of the form $\alpha 1_{\{x\}}$) and with $v < u$ if and only if $\bar{v} \leq u$, these conditions become the definitions of fuzzy regularity given by numerous authors, e.g., [10], thereby showing that all those definitions are equivalent to the one of Hutton-Reilly [3].

(3) For L a frame, the open L -set u in conditions (3) and (4) of Proposition 3.3 can be assumed to be in any family that generates the L -topology (on account of Proposition 3.2(3)); cf. [8, Lemma 3(iii)].

Now we show that the regularity axiom of Liu and Luo [6] is equivalent to the L -regularity for any complete L in which primes are order generating. We recall that $p \in L$ is called prime whenever $\alpha \wedge \beta \leq p$ implies $\alpha \leq p$ or $\beta \leq p$. The set of all primes is order generating if $\alpha = \bigwedge \{p \geq \alpha : p \text{ is prime}\}$ for every $\alpha \in L$. The dual concept is that of a coprime element. In our case, i.e., in $(L, ')$, an element $q \in L$ is coprime if and only if q' is prime. We have the following.

REMARK 3.5. Let L be a complete lattice in which primes are order generating. For X an L -ts, the following are equivalent:

- (1) X is L -regular.
- (2) (Liu and Luo [6]) for every $x \in X$, coprime q , and $k \in \kappa(X)$, whenever $k(x) \not\leq q$, there exists $h \in \kappa(X)$ such that $h(x) \not\leq q$ and $k \leq \text{Int } h$.

PROOF OF REMARK 3.5(2). Observe that condition (4) of Proposition 3.3 (cf. also Remark 3.4(1)) can be written as follows (with $v < u$ if and only if $\bar{v} \leq u$): $[u \not\leq p] = \bigcup_{\bar{v} \leq u} [v \not\leq p]$ for every open u and each prime p . And this is just the dual form of (2). □

4. The topological modifications of L -regular spaces. The main topic of this paper requires the lattice L to carry a topology such that $C(Y, L)$ is an L -topology for every topological space Y . Among examples of such lattices are the continuous lattices with their Scott topologies.

If L is a continuous lattice, then ΣL is a topological lattice (see [1, Chapter II, Corollary 4.16, Proposition 4.17]). The family $[Y, \Sigma L]$ of all continuous functions from a topological space Y to ΣL is, therefore, closed under finite suprema and finite infima (both formed in L^Y). However, by using the interpolation property of the relation \ll , for every $\alpha \in L$ and $\mathcal{U} \subset [Y, \Sigma L]$ one has $[\bigvee \mathcal{U} \gg \alpha] = \bigcup \{[\bigvee \mathcal{V} \gg \alpha] : \mathcal{V} \subset \mathcal{U} \text{ is finite}\}$, an open subset of Y . Thus $[Y, \Sigma L]$ is an L -topology on the set Y . For every topological space Y , $\omega_{\Sigma L} Y$ denotes the set Y provided with the L -topology $[Y, \Sigma L]$. One then says that $\omega_{\Sigma L} Y$ is *topologically generated* from Y .

Now, for X an L -topological space, let $t_{\Sigma L} X$ be the topological space with X as the underlying set and with the weak topology generated by $o(X)$ and ΣL , i.e., $t_{\Sigma L} X$ has $\bigvee \{u^-(\sigma(L)) : u \in o(X)\}$ as a topology. It is called the *topological modification* of X .

Then $\omega_{\Sigma L} : \mathbf{TOP} \rightarrow \mathbf{TOP}(L)$ and $t_{\Sigma L} : \mathbf{TOP}(L) \rightarrow \mathbf{TOP}$ (with preservation of mappings) are the Lowen functors (cf. [4, 5]).

We have $o(X) \subset o(\omega_{\Sigma L} \iota_{\Sigma L} X)$ and $\iota_{\Sigma L} \omega_{\Sigma L} = \text{id}_{\text{TOP}}$. Hence $\omega_{\Sigma L}$ is an injection. We also recall that if Y is a topological space, then χY denotes the set Y endowed with the L -topology $\{1_U : U \text{ open in } X\}$. Clearly, $\iota_{\Sigma L} \chi Y = Y$.

Sometimes it may be more convenient to write $(X, \omega_{\Sigma L}(T))$ for the space topologically generated from (X, T) , and similarly for $\iota_{\Sigma L}$.

LEMMA 4.1. *Let L be a continuous lattice. For every L -regular space X , $\iota_{\Sigma L} X$ is a regular topological space.*

PROOF. It suffices to show that every point of an arbitrary subbasic open set of $\iota_{\Sigma L} Y$ has an open neighborhood whose closure is in the set (this is Proposition 3.2(3) with $L = \{0, 1\}$). So, let u be open in X , $\alpha \in L$, and let $x \in [u \gg \alpha]$. By Proposition 3.3(3) there is an open v in X such that $\bar{v} \leq u$ and $x \in [v \gg \alpha]$. Select $\gamma \in L$ such that $\alpha \ll \gamma \ll v(x)$. Then

$$x \in [v \gg \gamma] \subset [\bar{v} \geq \gamma] \subset [u \gg \alpha]. \tag{4.1}$$

Now it suffices to note that, by Remark 2.1,

$$[\bar{v} \geq \gamma] = X \setminus [\text{Int}(v') \not\leq \gamma'] = X \setminus \bigcup_{\beta \not\leq \gamma'} [\text{Int}(v') \gg \beta]. \tag{4.2}$$

Thus $[\bar{v} \geq \gamma]$ is closed, hence $\iota_{\Sigma L} X$ is regular.

Now it is more convenient to write (X, T) for an L -ts X with the L -topology T . In [6], (X, T) is said to be *weakly induced* if $1_{[u \not\leq \alpha]} \in T$ for every $u \in T$ and $\alpha \in L$. Let $[T] = \{U \subset X : 1_U \in T\}$. In what follows, we write “ L -regular” on account of Remark 3.5. □

COROLLARY 4.2 [6]. *Let L be completely distributive. If (X, T) is a weakly induced L -regular space, then $(X, [T])$ is regular.*

PROOF. First, recall that a completely distributive L is continuous and the sets $\{\beta \in L : \beta \not\leq \alpha\}$ ($\alpha \in L$) form a subbase for its Scott topology (see [1, e.g., Chapter IV, Exercise 2.31 and Chapter III, Exercise 3.23]). Thus (X, T) is weakly induced if and only if $\iota_{\Sigma L}(T) \subset [T]$. Finally, notice that $[T] \subset \iota_{\Sigma L}(T)$ always since $[1_U \gg \alpha] \in \{\emptyset, U, X\}$ for every $\alpha \in L$. □

THEOREM 4.3. *Let L be a continuous lattice. Then the following hold:*

$$\iota_{\Sigma L}(\mathbf{Reg}(L)) = \mathbf{Reg}. \tag{4.3}$$

$$\omega_{\Sigma L}(\mathbf{Reg}) = \mathbf{Reg}(L) \cap \omega_{\Sigma L}(\mathbf{TOP}). \tag{4.4}$$

PROOF. (1) That $\iota_{\Sigma L}$ maps $\mathbf{Reg}(L)$ into \mathbf{Reg} is stated in Lemma 4.1. The mapping is onto since for any topological regular $X, \chi X$ is L -regular and $\iota_{\Sigma L} \chi X = X$.

(2) If X is a regular topological space and u is open in $\omega_{\Sigma L} X$, then for every $\alpha \in L$ there is a family \mathcal{W}_α of open subsets of X such that

$$[u \gg \alpha] = \bigcup \mathcal{W}_\alpha = \bigcup \overline{\mathcal{W}_\alpha}. \tag{4.5}$$

By Remark 2.1 and the first equality of (4.5), we obtain

$$\begin{aligned} u &= \bigvee_{\alpha \in L} \alpha 1_{[u \gg \alpha]} = \bigvee_{\alpha \in L} \left(\alpha \wedge \bigvee_{W \in \mathcal{W}_\alpha} 1_W \right) \\ &= \bigvee_{\alpha \in L} \bigvee_{W \in \mathcal{W}_\alpha} \alpha 1_W \leq \bigvee_{\alpha \in L} \bigvee_{W \in \mathcal{W}_\alpha} \overline{\alpha 1_W}. \end{aligned} \tag{4.6}$$

(Note that there is no distributivity used in arriving at the third equality: always $\alpha \wedge \bigvee B = \bigvee \{\alpha \wedge \beta : \beta \in B\}$ provided $B \subset \{0, 1\}$ as is the case above).

Since $\overline{\alpha 1_W} \leq \alpha 1_{\overline{W}}$, the same argument shows, by using the second equality of (4.5), that we actually have

$$u = \bigvee_{\alpha \in L} \bigvee_{W \in \mathcal{W}_\alpha} \alpha 1_W = \bigvee_{\alpha \in L} \bigvee_{W \in \mathcal{W}_\alpha} \overline{\alpha 1_W}. \tag{4.7}$$

This shows that $\omega_{\Sigma L} X$ is L -regular.

Conversely, if $\omega_{\Sigma L} X$ is L -regular, then $X = \iota_{\Sigma L} \chi X$ is regular by Lemma 4.1. □

REMARK 4.4. (1) Let L be a continuous frame (then it becomes completely distributive on account of the order reversing involution; cf. [1, Chapter I, Theorem 3.15]). Then the inclusion $\omega_{\Sigma L}(\mathbf{Reg}) \subset \mathbf{Reg}(L)$ obviously follows from Proposition 3.2(4). Indeed, for X a regular space, the L -topology of $\omega_{\Sigma L} X$ is the supremum of two L -regular L -topologies: the one of χX and the one consisting of all constant L -sets (cf. [5, Proposition 1.5.1(7)]).

(2) The equality (4.4) of Theorem 4.3 is available in [12] with $L = [0, 1]$ and in [6] with L completely distributive. Theorem 4.3 is also a supplement to the discussion about regularity in fuzzy topology given in [7].

(3) We recall that an L -ts X is an L - T_3 space if and only if it is L -regular and points of X can be separated by open L -sets. By [5, Remark 8.4], we obtain: $\iota_{\Sigma L}(L\text{-}T_3) = T_3$ and $\omega_{\Sigma L}(T_3) = L\text{-}T_3 \cap \omega_{\Sigma L}(\mathbf{TOP})$.

We close this section with some remarks about maximal L -regular spaces. Following [11], we say that X is *maximal* L -regular if the only L -regular L -topology on the set X which is stronger than the original one is L^X (the discrete L -topology).

PROPOSITION 4.5. *Let L be a continuous lattice. Every maximal L -regular space with a nondiscrete topological modification is topologically generated (from a maximal regular space).*

PROOF. Let (X, T) be maximal L -regular and let $\iota_{\Sigma L}(T)$ be nondiscrete. We have $T \subset \omega_{\Sigma L}(\iota_{\Sigma L}(T))$ and the latter L -topology is L -regular by Theorem 4.3. Assume $\omega_{\Sigma L}(\iota_{\Sigma L}(T)) = L^X$. Then, by acting with $\iota_{\Sigma L}$, we have $\iota_{\Sigma L}(T) = \iota_{\Sigma L}(L^X)$, a discrete topology. This contradiction shows that $T = \omega_{\Sigma L}(\iota_{\Sigma L}(T))$. Thus (X, T) is topologically generated from $(X, \iota_{\Sigma L}(T))$. The latter space is maximal regular. For, if $\iota_{\Sigma L}(T) \subsetneq S \subseteq \mathcal{P}(X)$ with S regular, then $T = \omega_{\Sigma L}(\iota_{\Sigma L}(T)) \subsetneq \omega_{\Sigma L}(S) \subsetneq \omega_{\Sigma L}(\mathcal{P}(X)) = L^X$. Since $\omega_{\Sigma L}(S)$ is L -regular, this contradicts the maximality of T (recall that $\omega_{\Sigma L}$ is injective). □

REMARK 4.6. From the above proof it is clear that Proposition 4.5 can be stated for any topological property \mathbf{P} and any L -topological property $L\text{-}\mathbf{P}$ for which there holds a counterpart of Theorem 4.3. This is, for instance, the case of complete L -regularity by [5, Theorem 8.5]. See also Remark 4.4(3).

5. H -Lindelöfness. An L -ts X is called H -Lindelöf if for every $k \in \kappa(X)$, whenever $k \leq \bigvee \mathcal{U}$ with $\mathcal{U} \subset \mathcal{o}(X)$, there exists a countable subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $k \leq \bigvee \mathcal{U}_0$. If \mathcal{U}_0 is finite, then X is called H -compact [2]. It is clear that H -Lindelöfness is preserved under continuous surjections. Also, the characterizations of H -compactness in terms of certain filters have their counterparts for H -Lindelöf spaces.

DEFINITION 5.1 (cf. [2]). Let $\mathcal{F} \subset L^X$ be nonempty and let $a \in L^X$. We say that:

- (1) \mathcal{F} has the countable intersection property relative to a if $\bigwedge \mathcal{F}_0 \not\leq a$ for every countable $\mathcal{F}_0 \subset \mathcal{F}$,
- (2) \mathcal{F} is a filter if it is closed under finite infima and such that if $f \in \mathcal{F}$ and $f \leq a$, then $a \in \mathcal{F}$. (A filter \mathcal{F} is called closed if $\mathcal{F} \subset \kappa(X)$.)

THEOREM 5.2. Let L be a complete lattice and let X be an L -ts. The following are equivalent:

- (1) X is H -Lindelöf.
- (2) Every family $\mathcal{K} \subset \kappa(X)$ with the countable intersection property relative to an open u satisfies $\bigwedge \mathcal{K} \not\leq u$.
- (3) Every closed filter \mathcal{K} with the countable intersection property relative to an open u satisfies $\bigwedge \mathcal{K} \not\leq u$.

PROOF. (1) \implies (2). Assume $\bigwedge \mathcal{K} \leq u$. Then $u' \leq \bigvee \mathcal{K}'$ and there is a countable $\mathcal{C} \subset \mathcal{K}'$ such that $u' \leq \bigvee \mathcal{C}$, a contradiction with the countable intersection property of \mathcal{K} .

(2) \implies (3). Obvious.

(3) \implies (1). Let $k \leq \bigvee \mathcal{U}$. Assume that \mathcal{U} does not have a countable subfamily which covers k . Let $\langle \mathcal{U}' \rangle$ be the closed filter generated by \mathcal{U}' , i.e., let

$$\langle \mathcal{U}' \rangle = \{f \in \kappa(X) : \exists \text{ finite } \mathcal{C}_f \subset \mathcal{U}' \text{ s.t. } \bigwedge \mathcal{C}_f \leq f\}. \tag{5.1}$$

We claim that $\langle \mathcal{U}' \rangle$ has the countable intersection property relative to k' . Suppose that this is not the case. Then for some countable $\mathcal{F} \subset \langle \mathcal{U}' \rangle$ one has $\bigwedge \mathcal{F} \leq k'$. Thus

$$k \leq \bigvee \mathcal{F}' \leq \bigvee_{f \in \mathcal{F}} \left(\bigwedge \mathcal{C}_f \right)' = \bigvee_{f \in \mathcal{F}} \left(\bigcup_{f \in \mathcal{F}} \mathcal{C}'_f \right) \tag{5.2}$$

and $\bigcup_{f \in \mathcal{F}} \mathcal{C}'_f$ is a countable subfamily of \mathcal{U} , a contradiction with our assumption about \mathcal{U} . Therefore $\langle \mathcal{U}' \rangle$ has the countable intersection property relative to k' , i.e., $\bigwedge \langle \mathcal{U}' \rangle \not\leq k'$. Hence $k \not\leq \bigvee \langle \mathcal{U}' \rangle'$ and since $\bigvee \mathcal{U} \leq \bigvee \langle \mathcal{U}' \rangle'$, we conclude that $k \not\leq \bigvee \mathcal{U}$. This contradiction completes the proof. \square

REMARK 5.3. There is no counterpart of Theorem 4.3 for H -Lindelöfness and Lindelöfness:

(1) The set $X = L = [0, 1]$ (with $\alpha' = 1 - \alpha$) equipped with the L -topology $[0, 1/4]^X \cup \{1_X\}$ is H -Lindelöf (as each open cover of a nonzero closed L -set must contain 1_X), while $\iota_{\Sigma L} X$ is an uncountable discrete space.

(2) An L -ts topologically generated from a Lindelöf space need not be H -Lindelöf. Indeed, let X be an uncountable Lindelöf topological space. Put $L = \mathcal{P}(X)$ with usual complement as its order-reversing involution (note that $\mathcal{P}(X)$ is a continuous lattice). Then the cover of 1_X consisting of all constant L -sets having values $\{x\}$ with $x \in X$

(these are all open in $\omega_{\Sigma L}X$) does not have a countable subcover. Therefore $\omega_{\Sigma L}X$ fails to be H -Lindelöf.

(3) However, if $\omega_{\Sigma L}X$ is H -Lindelöf, then X is Lindelöf. Indeed, χX carries a weaker L -topology than $\omega_{\Sigma L}X$, so that χX is H -Lindelöf, and the latter is equivalent to the statement that X is a Lindelöf space.

(4) All the above discussion applies unchanged to the case of H -compactness and compactness.

It is clear that for any complete L , every H -compact and L -regular space X is L -normal, i.e., whenever $k \leq u$ (k is closed and u is open), there exists an open v with $k \leq v \leq \bar{v} \leq u$ [3]. In what follows we show that H -compactness can be replaced by H -Lindelöfness provided L is meet-continuous, i.e., for every $\alpha \in L$ and every directed subset $\mathcal{D} \subset L$ there holds: $\alpha \wedge \bigvee \mathcal{D} = \bigvee \{\alpha \wedge \delta : \delta \in \mathcal{D}\}$. We recall that every continuous L is meet-continuous [1]. Also, on account of the order-reversing involution, the dual law is valid too.

THEOREM 5.4. *Let L be a meet-continuous lattice. Then every L -regular and H -Lindelöf space is L -normal.*

PROOF. Let k be closed, u be open, and $k \leq u$ in an L -regular H -Lindelöf space X . By L -regularity there exist $\mathcal{U} \subset \mathcal{O}(X)$ and $\mathcal{K} \subset \mathcal{K}(X)$ such that $u = \bigvee \mathcal{U} = \bigvee \bar{\mathcal{U}}$ and $k = \bigwedge \mathcal{K} = \bigwedge \text{Int} \mathcal{K}$ (the latter on account of the de Morgan laws). By H -Lindelöfness, there exist two countable subfamilies $\mathcal{U}_0 \subset \mathcal{U}$ and $\mathcal{K}_0 \subset \mathcal{K}$ such that $k \leq \bigvee \mathcal{U}_0$ and (again by the de Morgan laws) $\bigwedge \mathcal{K}_0 \leq u$. Thus

$$k \leq \bigvee \mathcal{U}_0 \leq \bigvee \bar{\mathcal{U}}_0 \quad \text{and} \quad k \leq \bigwedge \text{Int} \mathcal{K}_0 \leq \bigwedge \mathcal{K}_0 \leq u. \quad (5.3)$$

The rest of the proof is exactly the same as that of [5, Theorem 9.11] which shows that second countability plus L -regularity implies L -normality. Note that the proof in [5] uses a result holding for L a meet-continuous lattice. \square

REMARK 5.5. By [5, Lemma 3.7], every second countable L -ts is H -Lindelöf for any complete L . Therefore Theorem 5.4 extends [5, Theorem 9.11].

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