On the Lyapunov-based second-order SMC design for some classes of distributed parameter systems

YURY ORLOV
CICESE Research Center, Ensenada, Mexico

ALESSANDRO PISANO*, STEFANO SCODINA AND ELIO USAI
Department of Electrical and Electronic Engineering, University of Cagliari, Cagliari, Italy

*Corresponding author: pisano@diee.unica.it

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This paper addresses the Lyapunov-based design of second-order sliding mode controllers in the domain of distributed parameter systems (DPSs). To the best of our knowledge, the recent authors’ publications (Orlov et al., 2010, Continuous state-feedback tracking of an uncertain heat diffusion process. Syst. Control Lett., 59, 754–759; Orlov et al., 2011, Exponential stabilization of the uncertain wave equation via distributed dynamic input extension. IEEE Trans. Autom. Control, 56, 212–217; Pisano et al., 2011, Tracking control of the uncertain heat and wave equation via power-fractional and sliding-mode techniques. SIAM J. Control Optim., 49, 363–382) represent the seminal applications of second-order sliding mode control techniques to DPSs. A Lyapunov-based framework of analysis was found to be appropriate in the above publications. While reviewing the main existing results in this new field of investigation, the paper provides the novelty as well and gives several hints and perspectives for the generalization, listing some open problems.

Keywords: distributed parameter systems; heat equation; wave equation; second-order sliding mode control; Lyapunov analysis.

1. Introduction

Sliding mode control has long been recognized as a powerful control method to counteract non-vanishing external disturbances and unmodelled dynamics when controlling dynamical systems of finite and infinite dimension (see Utkin, 1992).

Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented (see Levaggi, 2002; Orlov, 2000, 2009; Orlov & Utkin, 1987; Orlov et al., 2004) and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart. Other robust control paradigms have been fruitfully applied in the infinite-dimensional setting such as adaptive and model-reference control (see Krstic & Smyshlyaev, 2008b; Demetriou et al., 2009), geometric and Lyapunov-based design (see Christofides, 2001) and $H_\infty$ and LMI-based design (see Fridman & Orlov, 2009). It should be noted that the latter paradigms are capable of ‘attenuating’ vanishing disturbances only, whereas the former discontinuous control is additionally capable of ‘rejecting’ persistent disturbances with an a priori known bound on their $L_2$-norm.

In the present paper, we consider generalized uncertain forms of two popular parabolic and hyperbolic infinite-dimensional dynamics, the heat and wave equations, under the effect of an external smooth disturbance. In some recent authors’ publications (see Orlov et al., 2010, 2011; Pisano et al., 2011), two finite-dimensional robust control algorithms, namely, the ‘super-twisting’ and ‘twisting’
second-order sliding mode controllers (2-SMCs) (see Fridman & Levant, 1996; Levant, 1993, for details on these controllers), have been generalized to the infinite-dimensional setting and applied for controlling heat and wave processes, respectively. The mentioned 2-SMCs are of special interest because in the finite-dimensional setting they significantly improve the performance of sliding mode control systems, in terms of accuracy and chattering avoidance, as compared to the standard ‘first-order’ sliding mode control techniques (see Bartolini et al., 2002).

In this paper, we enlarge the class of controlled dynamics as compared to existing publications (cf. Orlov et al., 2010, 2011; Pisano et al., 2011) by considering generalized forms of the heat and wave equations. More precisely, we consider the presence of some additional terms in the plant equation (dispersion and damping terms) and, furthermore, we let all the system parameters (diffusivity and dispersion coefficients, for the heat equation, and the wave velocity, the damping coefficient and the dispersion coefficient, for the wave equation) to be spatially varying and uncertain. We additionally put the constraint that the distributed control input must be a continuous (although possibly non-smooth) function of the space and time variables.

The rest of the paper is structured as follows. Some notations are introduced in the remainder of Section 1. Section 2 presents the problem formulation for the tracking control of the generalized heat equation and describes the associated solution, based on a proper combination of distributed versions of proportional-integral (PI) and super-twisting control. Section 3 presents the problem formulation for the tracking control of the generalized wave equation and describes the associated solution, based on a proper combination of distributed versions of proportional-derivative (PD) and twisting control with dynamic input extension. Section 4 illustrates some relevant numerical simulation results. Finally, Section 5 gives some concluding remarks and draws possible directions of improvement of the proposed results.

Notation. In general, the notation used throughout is fairly standard (see Curtain & Zwart, 1995, for details). $L^2(a, b)$, with $a \leq b$, stands for the Hilbert space of square integrable functions $z(\zeta), \zeta \in [a, b]$, equipped with the $L^2$-norm

$$||z(\cdot)||_2 = \sqrt{\int_a^b z^2(\zeta) d\zeta}. \quad (1.1)$$

$W^{l,2}(a, b)$ denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta), \zeta \in [a, b]$, with square integrable derivatives $z^{(l)}(\zeta)$ up to the order $l \geq 1$. A non-standard notation stands for

$$\Omega^{4,2}(0, 1) = \{\omega(\xi) \in W^{4,2}(0, 1); \sqrt{|\omega(\xi)|} \text{sign}(\omega(\xi)) \in W^{2,2}(0, 1)\}. \quad (1.2)$$

2. Supertwisting synthesis of reaction–diffusion processes

Consider the space- and time-varying scalar field $Q(\xi, t)$ evolving in a Hilbert space $L_2(0, 1)$, where $\xi \in [0, 1]$ is the monodimensional (1D) space variable and $t \geq 0$ is the time. Let it be governed by the following perturbed reaction–diffusion equation with spatially varying parameters

$$Q_t(\xi, t) = [\theta_1(\xi) Q_\xi(\xi, t) ]_\xi + \theta_2(\xi) Q(\xi, t) + u(\xi, t) + \psi(\xi), \quad (2.1)$$

where $\theta_1(\cdot) \in C^1(0, 1)$ is a positive-definite spatially varying parameter called ‘thermal conductivity’ (or, more generally, ‘diffusivity’), $\theta_2(\cdot) \in C^1(0, 1)$ is another spatially varying parameter called ‘dispersion’ (or ‘reaction constant’), $u(\xi, t)$ is the modifiable source term (the distributed control input) and
\( \psi(\xi) \) represents an ‘uncertain’ time-independent spatially distributed disturbance source term, which is assumed to be of class \( W^{2,2}(0, 1) \), i.e.

\[ \psi(\xi) \in W^{2,2}(0, 1). \quad (2.2) \]

The time-independent spatially varying diffusivity and dispersion coefficients \( \theta_1(\xi) \) and \( \theta_2(\xi) \) are supposed to be uncertain, too. We consider non-homogeneous mixed boundary conditions (BCs)

\[ Q(0, t) - a_0 \frac{\partial Q}{\partial \xi}(0, t) = Q_0(t) \in W^{1,2}(0, \infty), \quad (2.3) \]
\[ Q(1, t) + a_1 \frac{\partial Q}{\partial \xi}(1, t) = Q_1(t) \in W^{1,2}(0, \infty), \quad (2.4) \]

with some positive uncertain constants \( a_0, a_1 \). The initial conditions (ICs)

\[ Q(\xi, 0) = \omega_0(\xi) \in \Omega^{4,2}(0, 1) \quad (2.5) \]

are assumed to be of class (1.2) and to meet the same BCs.

Since non-homogeneous BCs are in force, a solution of the above boundary-value problem is defined in the mild sense (see Curtain & Zwart, 1995) as that of the corresponding integral equation, written in terms of the strongly continuous semigroup, generated by the infinitesimal plant operator.

The control task is to make the scalar field \( Q(\xi, t) \) to asymptotically decay, in the \( L_2 \)-norm, to an \textit{a priori} given reference \( \xi(\xi) \in \Omega^{4,2}(0, 1) \), satisfying the BCs (2.3).

### 2.1 Robust control of the reaction–diffusion process

Consider the deviation variable

\[ x(\xi, t) = Q(\xi, t) - \xi(\xi), \quad (2.6) \]

whose \( L_2 \)-norm will be driven to zero by the designed feedback control. The dynamics of the error variable (2.6) is easily derived as

\[ x_t(\xi, t) = [\theta_1(\xi)x(\xi, t)]_\xi + \theta_2(\xi)x(\xi, t) + u(\xi, t) + \eta(\xi), \quad (2.7) \]

with the ‘augmented’ disturbance

\[ \eta(\xi) = [\theta_1(\xi)Q(\xi, t)]_\xi + \theta_2(\xi)\xi(\xi) + \psi(\xi) \quad (2.8) \]

and the next ICs and homogeneous mixed BCs

\[ x(\xi, 0) = \omega_0(\xi) - \xi(\xi, 0) \in \Omega^{4,2}(0, 1), \quad (2.9) \]
\[ x(0, t) - a_0x(0, t) = x(1, t) + a_1x(1, t) = 0. \quad (2.10) \]

Assume what follows.

**Assumption 2.1** There exist possibly uncertain constants \( \Theta_{1m}, \Theta_{1M} \) such that

\[ 0 < \Theta_{1m} \leq \theta_1(\xi) \leq \Theta_{1M}, \quad \text{for all} \ \xi \in [0, 1], \quad (2.11) \]
and there is an \textit{a priori} known constant $\Theta_{2M}$ such that

$$|\theta_2(\xi)| \leq \Theta_{2M}, \quad \text{for all } \xi \in [0, 1].$$  \hspace{1cm} (2.12)

It should be noted that the assumptions on the ICs and BCs, made above, allow us to deal with strong, sufficiently smooth solutions of the uncertain error dynamics (2.7–2.10) in the open loop when no control input is applied.

In order to stabilize the error dynamics, it is proposed a dynamical distributed controller defined as follows:

$$u(\xi, t) = -\lambda_1 \sqrt{|x(\xi, t)|} \text{sign}(x(\xi, t)) - \lambda_2 x(\xi, t) + v(\xi, t),$$

$$v_t(\xi, t) = -W_1 \frac{x(\xi, t)}{\|x(\xi, t)\|_2} - W_2 x(\xi, t), \quad v(\xi, 0) = 0,$$  \hspace{1cm} (2.13, 2.14)

which can be seen as a distributed version of the finite-dimensional super-twisting 2-SMC (see Fridman & Levant, 1996; Levant, 1993) complemented by the two additional proportional and integral linear terms with gains $\lambda_2$ and $W_2$, respectively. For ease of reference, the combined distributed super-twisting/PI controller (2.13–2.14) will be abbreviated as DSTPI.

The non-smooth nature of the DSTPI controller (2.13–2.14), that undergoes discontinuities on the manifold $x = 0$ due to the discontinuous term $\frac{x(\xi, t)}{\|x(\xi, t)\|_2}$, requires appropriate analysis about the meaning of the corresponding solutions for the resulting discontinuous feedback system. The precise meaning of the solutions of (2.7), (2.9) and (2.10) with the piecewise continuously differentiable control input (2.13–2.14) can be defined in a generalized sense (see Orlov, 2009) as a limiting result obtained through a certain regularization procedure, similar to that proposed for finite-dimensional systems (see Filippov, 1988; Utkin, 1992). According to this procedure, the strong solutions of the boundary-value problem are only considered whenever they are beyond the discontinuity manifold $x = 0$, whereas in a vicinity of this manifold the original system is replaced by a related system, which takes into account all possible imperfections (e.g. delay, hysteresis, saturation, etc.) in the new input function $u^\delta(\xi, \xi, t)$, for which there exists a strong solution $x^\delta(\xi, t)$ of the corresponding boundary-value problem with the smoothed input $u^\delta(x, \xi, t)$. In particular, a relevant approximation occurs when the discontinuous term $U(x) = \frac{x(\xi, t)}{\|x(\xi, t)\|_2}$ is substituted by the smooth approximation $U^\delta(x) = \frac{x(\xi, t)}{\|x(\xi, t)\|_2 + \delta}$, where $\delta$ is a positive coefficient.

A generalized solution of the system in question is then obtained through the limiting procedure by diminishing $\delta$ to zero, thereby making the characteristics of the new system approach those of the original one. As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a ‘sliding mode’.

\textbf{Remark 2.1} The existence of generalized solutions, thus defined, has been established within the abstract framework of Hilbert space-valued dynamic systems (cf., e.g. Orlov, 2009, Theorem 2.4), whereas the uniqueness and well-posedness appear to follow from the fact that in the system in question, no sliding mode occurs but in the origin $x = 0$. While being well recognized for second-order sliding mode control algorithms if confined to the finite-dimensional setting, this fact, however, remains beyond the scope of the present investigation.
The performance of the closed-loop system is analysed in the next theorem.

**Theorem 2.1** Consider the perturbed diffusion/dispersion equations (2.1–2.3) satisfying Assumption 2.1. Then, the distributed control strategy (2.6), (2.13–2.14), with the parameters \( \lambda_1, \lambda_2, W_1 \) and \( W_2 \) selected according to

\[
\lambda_2 \geq \Theta_2 M, \quad W_1 > 0, \quad \lambda_1 > 0, \quad W_2 > 0,
\]

guarantees the global asymptotic stability in the \( L_2 \)-space of the solution \( x(\xi, t) \) of the closed-loop system (2.7–2.10).

**Proof of Theorem 2.1.**
Define the auxiliary variable

\[
\delta(\xi, t) = v(\xi, t) + \eta(\xi).
\]

System (2.7) with the control law (2.13–2.14) yields the following closed-loop dynamics in the new \( x - \delta \) coordinates

\[
x_t(\xi, t) = [\theta_1(\xi)x_\xi(\xi, t)]_\xi - \lambda_1 \sqrt{|x(\xi, t)|} \text{sign}(x(\xi, t)) - (\lambda_2 - \theta_2(\xi))x(\xi, t) + \delta(\xi, t),
\]

\[
\delta_t(\xi, t) = -W_1 \frac{x(\xi, t)}{\|x(\cdot, t)\|_2} - W_2 x(\xi, t).
\]

In order to simplify the notation, the dependence of the system coordinates from the space and time variables \((\xi, t)\) will be mostly omitted from this point on. Setting

\[
s = x_t = [\theta_1(\xi)x_\xi(\xi, t)]_\xi - \lambda_1 \sqrt{|x|} \text{sign}(x) - (\lambda_2 - \theta_2(\xi))x + \delta,
\]

consider the following Lyapunov functional

\[
V_1(t) = 2W_1 \|x\|_2 + W_2 \|x\|_2^2 + \frac{1}{2} \|\delta\|_2^2 + \frac{1}{2} \|s\|_2^2,
\]

inspired from the finite-dimensional treatment of Moreno & Osorio (2008). The time derivative of \( V_1(t) \) is given by

\[
\dot{V}_1(t) = \frac{2W_1}{\|x\|_2} \int_0^1 x_s d\xi + 2W_2 \int_0^1 x_s d\xi + \int_0^1 \delta_s d\xi + \int_0^1 s_s d\xi,
\]

where

\[
s_t = [\theta_1(\xi)x_\xi(\xi, t)]_\xi - \frac{1}{2} \lambda_1 \frac{s}{\sqrt{|x|}} - (\lambda_2 - \theta_2(\xi))s - W_1 \frac{x}{\|x\|_2} - W_2 x
\]

is formally obtained by differentiating (2.19) along the solutions of (2.17), (2.18).

**Remark 2.2** To verify that the differentiation of (2.19) is legible, it suffices to note that the non-linear term of the state variable, that appear in (2.17), is clearly differentiable beyond the parameterized region.
\[ D_{\xi_0, \xi_1} = \{ x \in L_2(0, 1): x(\xi_0, t) = 0 \forall \xi \in (\xi_0, \xi_1) \subset [0, 1], t \in (t_0, t_1) \subset [0, \infty) \}, \]

where the system, while being confined to \( \xi \in (\xi_0, \xi_1) \subset [0, 1] \), possesses a local equilibrium on the time interval \( t \in (t_0, t_1) \subset [0, \infty) \), whereas within \( D_{\xi_0, \xi_1} \), the non-differentiable term of the second term and possibly (iff \( \xi_0 = 0, \xi_1 = 1 \)) the fourth term in (2.19), become identically zero for all \( \xi \in (\xi_0, \xi_1), t \in (t_0, t_1) \). Thus, standard arguments, used in smooth partial differential equation (PDE) (similar to that of Curtain & Zwart, 1995, Theorem 3.1.3; Krasnoselskii et al., 1976, Theorem 23.2), apply here beyond the origin to make sure that under the regularity assumption \(^1\) (2.9), imposed on the ICs, (2.17) can be differentiated in time, thereby validating (2.22) everywhere except the origin where the well-posedness of (2.21–2.22) is trivially validated.

Substituting (2.18) and (2.22) into (2.21) and rearranging yield

\[
\dot{V}(t) = \frac{2W_1}{\|x\|_2} \int_0^1 xs \, d\xi + 2W_2 \int_0^1 x \, d\xi - \frac{W_1}{\|x\|_2} \int_0^1 \delta \, d\xi - W_2 \int_0^1 x \, d\xi + \int_0^1 s[\theta_1(\xi)x_\xi] \, d\xi
\]

\[
- \frac{1}{2} \lambda_1 \int_0^1 \frac{s^2 \, d\xi}{\sqrt{|x|}} - \left[ \lambda_2 - \theta_2(\xi) \right] \int_0^1 s \, d\xi.
\]

which can be manipulated as follows by virtue of Assumption 2.1:

\[
\dot{V}(t) \leq \frac{2W_1}{\|x\|_2} \int_0^1 x(\delta - s) \, d\xi - W_2 \int_0^1 x(\delta - s) \, d\xi + \int_0^1 s[\theta_1(\xi)x_\xi] \, d\xi
\]

\[
- \frac{1}{2} \lambda_1 \int_0^1 \frac{s^2 \, d\xi}{\sqrt{|x|}} - \left[ \lambda_2 - \theta_2M \right] \int_0^1 s \, d\xi.
\]

By (2.19), one has

\[
\delta - s = \lambda_1 \sqrt{|x|} \text{sign}(x) + [\lambda_2 - \theta_2(\xi)]x - [\theta_1(\xi)x_\xi],
\]

\[
\delta + s = 2s + \lambda_1 \sqrt{|x|} \text{sign}(x) + [\lambda_2 - \theta_2(\xi)]x - [\theta_1(\xi)x_\xi].
\]

Due to this, and considering once more Assumption 2.1, (2.24) can further be manipulated as

\[
\dot{V}(t) \leq - \frac{W_1 \lambda_1}{\|x\|_2} \int_0^1 x \sqrt{|x|} \text{sign}(x) \, d\xi - \frac{W_1 \lambda_2 - \theta_2M}{\|x\|_2} \int_0^1 x \, d\xi + \frac{W_1}{\|x\|_2} \int_0^1 [\theta_1(\xi)x_\xi] \, d\xi
\]

\[
- W_2 \lambda_1 \int_0^1 x \sqrt{|x|} \text{sign}(x) \, d\xi - W_2 \lambda_2 - \theta_2M \int_0^1 x \, d\xi + W_2 \int_0^1 [\theta_1(\xi)x_\xi] \, d\xi
\]

\[
+ \int_0^1 s[\theta_1(\xi)x_\xi] \, d\xi - \frac{1}{2} \lambda_1 \int_0^1 \frac{s^2 \, d\xi}{\sqrt{|x|}} - \left[ \lambda_2 - \theta_2M \right] \int_0^1 s \, d\xi.
\]

\(^1\)This assumption ensures that \( s(\xi, 0) \), specified according to (2.19), is of class \( W^{2,1}(0, 1) \), so that Remark 2.1 remains in force to argue the existence of generalized solutions \( s(\xi, t) \) of (2.22).
By taking into account the BCs (2.10) and their time derivatives, standard integration by parts yields

\[
\int_0^1 x[\theta_1(\xi)x_\xi]\, d\xi = -\int_0^1 \theta_1(\xi)x_\xi^2\, d\xi + \theta_1(1)x(1, \xi)x_\xi(1, t) - \theta_1(0)x(0, \xi)x_\xi(0, t) \\
\leq -\Theta_1m\|x_\xi\|^2 \|\xi\|^2 - \theta_1(1)\frac{x^2(1, t)}{a_1} - \theta_1(0)\frac{x^2(0, t)}{a_0},
\]

(2.28)

\[
\int_0^1 s[\theta_1(\xi)s_\xi]\, d\xi = -\int_0^1 \theta_1(\xi)s_\xi^2\, d\xi + \theta_1(1)s(1, \xi)s_\xi(1, t) - \theta_1(0)s(0, \xi)s_\xi(0, t) \\
\leq -\Theta_2m\|s_\xi\|^2 - \theta_1(1)\frac{s^2(1, t)}{a_1} - \theta_1(0)\frac{s^2(0, t)}{a_0}.
\]

(2.29)

Additional straightforward manipulations of (2.27) taking into account (2.28) and (2.29) yield

\[
\dot{V}_1(t) \leq -W_1[\lambda_2 - \Theta_{2M}]\|x\|_2 - W_2[\lambda_2 - \Theta_{2M}]\|x\|_2^2 - W_1\Theta_1m\|x_\xi\|^2 \|\xi\|^2 - W_1\Theta_1m\Theta_1m\|x_\xi\|^2 - W_2\Theta_1m\|x_\xi\|^2
\]

\[
- \frac{W_1}{\|x\|^2}\theta_1(0)\frac{x^2(0, t)}{a_0} - W_2\Theta_1m\|x_\xi\|^2 - W_2\theta_1(1)\frac{x^2(1, t)}{a_1} - W_2\theta_1(0)\frac{x^2(0, t)}{a_0}
\]

\[
- [\lambda_2 - \Theta_{2M}]\|s\|_2^2 - \Theta_1m|s_\xi|^2 - \theta_1(1)\frac{s^2(1, t)}{a_1} - \theta_1(0)\frac{s^2(0, t)}{a_0}
\]

\[
- W_2\lambda_1\int_0^1 |x|^{3/2}\, d\xi - \frac{1}{2}\lambda_1\int_0^1 \frac{s^2\, d\xi}{|x|} - \frac{W_1\lambda_1}{\|x\|_2}\int_0^1 \sqrt{|x|}\|x|\, d\xi. 
\]

(2.30)

It is worth noting that by virtue of the tuning inequality \(\lambda_2 > \Theta_{2M}\) in (2.15), all terms appearing in the right-hand side of (2.30) are negative definite. To complete the proof, it remains to demonstrate that

\[
\|x(\cdot, t)\|_2 \to 0 \quad \text{as} \quad t \to \infty.
\]

(2.31)

For this purpose, let us integrate the relation

\[
\dot{V}(t) \leq -W_1[\lambda_2 - \Theta_{2M}]\|x\|_2,
\]

(2.32)

straightforwardly resulting from the negative definiteness of all terms in the right-hand side of (2.30), to conclude that

\[
\int_0^\infty \|x(\cdot, t)\|_2\, dt < \infty.
\]

(2.33)

The inequality \(\dot{V}_1(t) \leq 0\) guarantees that \(V_1(t) \leq V_1(0)\) for any \(t \geq 0\). From this, and considering (2.20), one can conclude that the \(L_2\)-norm of \(s = x_t\) fulfills the estimation

\[
\|x_t\|_2^2 \leq 2V_1(0), \quad \forall \, t \geq 0.
\]

(2.34)
Thus, the integrand \( \phi(t) = \|x(\cdot, t)\|_2 \) of (2.33) possesses a uniformly bounded time derivative

\[
\dot{\phi}(t) = \int_0^1 x(\xi, t) \frac{dx}{\|x\|_2} \leq \|x\|_2 \leq \sqrt{2R}
\]

on the semi-infinite time interval \( t \in [0, \infty) \), where \( R \) is any positive constant such that \( R \geq V_1(0) \). Convergence (2.31) is then verified by applying the Barbalat lemma (see Khalil, 2002). Since the Lyapunov functional (2.20) is radially unbounded, the global asymptotic stability of the closed-loop system (2.7–2.10) is thus established in the \( L_2 \)-space. Theorem 2.1 is proved.

**Remark 2.3** If the spatially varying profiles \( \theta_1(\xi), \theta_2(\xi) \) of the system parameters are known, then a trivial modification of the suggested controller can be made in order to ensure the same convergence property (2.31) with a time-dependent reference \( Q_r(\xi, t) \in W^{2,2} \), too. The corresponding modified controller is

\[
u_t(\xi, t) = -W_1 \frac{x(\xi, t)}{\|x(\cdot, t)\|_2} - W_2 x(\xi, t), \quad v(\xi, 0) = 0,
\]

with the control parameters subject to the same tuning conditions (2.15) and the additional feedforward term

\[
u_{ff}(\xi, t) = Q^T(\xi, t) - [\theta_1(\xi)Q^T(\xi)]_\xi - \theta_2(\xi)Q^T(\xi).
\]

The proof can be easily developed by observing that the resulting external disturbance, affecting the corresponding error system, remains time independent so that the line of reasoning used in the proof of Theorem 2.1 is applicable here as well. The detailed proof is thus omitted for brevity.

### 3. Twisting synthesis of perturbed wave processes

We consider a class of uncertain infinite-dimensional systems whose \((y, y_t)\) solution is defined in the Hilbert space \( L_2(0, 1) \times L_2(0, 1) \) and is governed by the next hyperbolic PDE

\[
y_{tt}(\xi, t) = [v^2(\xi)y_{\xi}]_\xi + \theta_1(\xi)y(\xi, t) + \theta_2(\xi)y_t(\xi, t) + u(\xi, t) + \psi(\xi, t) \quad (3.1)
\]

with spatially varying parameters, where \( y \in L_2(0, 1) \) and \( y_t \in L_2(0, 1) \) are the state variables, \( \xi \in [0, 1] \) is the monodimensional (1D) spatial variable and \( t \geq 0 \) is the time. The spatially varying coefficient \( v^2(\cdot) \in C^1(0, 1) \) represents the squared value of the wave velocity, and \( \theta_1(\cdot) \in C^1(0, 1) \) and \( \theta_2(\cdot) \in C^1(0, 1) \) are referred to, respectively, as the dispersion and damping coefficients. \( u(\xi, t) \in L_2(0, 1) \) is the modifiable source term (the distributed control input), and \( \psi(\xi, t) \in L_2(0, 1) \) represents an uncertain spatially distributed disturbance source term, which, in contrast to the heat process, is admitted to be time varying. The spatially varying (but time-independent) parameters are supposed to be uncertain, too, and satisfying the next assumption.
ASSUMPTION 3.1 There exist a priori known constants $\gamma_m$, $\gamma_M$, $\Theta_1$ and $\Theta_2$ such that

$$0 < \gamma_m \leq v^2(\xi) \leq \gamma_M, \quad |\theta_1(\xi)| \leq \Theta_1, \quad |\theta_2(\xi)| \leq \Theta_2, \quad \forall \xi \in [0, 1].$$  \hfill (3.2)

We consider non-homogeneous mixed BCs

$$y(0, t) - \beta_0 y_\xi(0, t) = Y_0(t) \in W^{1,2}(0, \infty), \quad (3.3)$$

$$y(1, t) + \beta_1 y_\xi(1, t) = Y_1(t) \in W^{1,2}(0, \infty), \quad (3.4)$$

with some positive uncertain constants $\beta_0$, $\beta_1$ and the ICs

$$y(\xi, 0) = \varphi_0(\xi) \in W^{4,2}(0, 1), \quad y_\xi(\xi, 0) = \varphi_1(\xi) \in W^{4,2}(0, 1), \quad (3.5)$$

where $\varphi_0(\cdot)$, $\varphi_1(\cdot)$ are also assumed to meet the BCs imposed on the system. As in the diffusion equation case, non-homogeneous BCs are in general admitted, which is why a solution of the above boundary-value problem is defined in the mild sense (see Curtain & Zwart, 1995). The control task is to make the position $y(\xi, t)$ and the velocity $y_\xi(\xi, t)$ to exponentially track an a priori given reference signal $y^r(\xi, t)$ and, respectively, its velocity $y^r_\xi(\xi, t)$ in the $L_2$-space, regardless of whichever admissible disturbance $\psi(\xi, t)$ affects the system. It is assumed throughout that the reference signal $y^r(\xi, t)$ and its time derivatives are smooth enough in the sense that

$$y^r(\cdot, t) \in W^{4,2}(0, 1), \quad y^r_\xi(\cdot, t) \in W^{4,2}(0, 1), \quad y^r_{\xi\xi}(\cdot, t) \in L_2(0, 1), \quad \forall t \geq 0. \quad (3.6)$$

Apart from this, the reference signal is assumed to meet the actual BCs (3.3–3.4).

The ‘deviation variables’

$$\tilde{y}(\xi, t) = y(\xi, t) - y^r(\xi, t), \quad \tilde{y}_\xi(\xi, t) = y_\xi(\xi, t) - y^r_\xi(\xi, t) \quad (3.7)$$

are then to eventually be driven to zero in $L_2$-norm by the controller to be designed. By differentiating (3.7) and making appropriate substitutions and manipulations, one derives the next PDE governing the corresponding error dynamics

$$\tilde{y}_{\xi\xi}(\xi, t) = [v^2(\xi)\tilde{y}_\xi^2(\xi) + \theta_1(\xi)\tilde{y}(\xi, t) + \theta_2(\xi)\tilde{y}_\xi(\xi, t) + u(\xi, t) + \psi(\xi, t) - y^r_\xi(\xi, t) + [v^2(\xi)\tilde{y}^r_\xi(\xi) + \theta_1(\xi)y^r(\xi, t) + \theta_2(\xi)y^r_\xi(\xi, t) \quad (3.8)$$

with the ICs

$$\tilde{y}(\xi, 0) = \varphi_0(\xi) - \varphi^r_0(\xi) \in W^{4,2}(0, 1), \quad \tilde{y}_\xi(\xi, 0) = \varphi_1(\xi) - \varphi^r_1(\xi) \in W^{4,2}(0, 1), \quad (3.9)$$

and homogeneous BCs

$$\tilde{y}(0, t) - \beta_0 \tilde{y}_\xi(0, t) = \tilde{y}(1, t) + \beta_1 \tilde{y}_\xi(1, t) = 0. \quad (3.10)$$

The assumptions on the ICs and BCs, made above, allow us to deal with strong, sufficiently smooth solutions of the uncertain error dynamics (3.8–3.10) in the open loop when no control input is applied. Just in case, the open-loop system locally possesses a unique (twice differentiable in time) strong
solution \( \tilde{y}(\xi, t) \) which is defined in a standard manner (see Curtain & Zwart, 1995) as an absolutely continuous function, almost everywhere satisfying the corresponding PDE rather than its integral counterpart.

The class of reference signals and admissible disturbances is specified in the next assumptions.

**Assumption 3.2** There exist *a priori* known constants \( H_0, \ldots, H_4 \) such that the reference trajectory \( y^r(\xi, t) \) and its spatial and temporal derivatives meet the following inequalities for all \( t \geq 0 \):

\[
\|y^r(\cdot, t)\|_2 \leq H_0, \quad \|y^r_t(\cdot, t)\|_2 \leq H_1, \quad \|y^r_{tt}(\cdot, t)\|_2 \leq H_2, \quad (3.11)
\]
\[
\|v^2(\xi) y^r_{\xi}(\cdot, t)\|_2 \leq H_3, \quad \|v^2(\xi) y^r_{\xi t}(\cdot, t)\|_2 \leq H_4. \quad (3.12)
\]

**Assumption 3.3** There exist *a priori* known constants \( \Psi_0, \Psi_1 \) such that the disturbance \( \psi(\xi, t) \) and its temporal derivative meet the following inequalities for all \( t \geq 0 \):

\[
\|\psi(\cdot, t)\|_2 \leq \Psi_0, \quad \|\psi_t(\cdot, t)\|_2 \leq \Psi_1. \quad (3.13)
\]

### 3.1 Distributed sliding manifold design

Define the distributed sliding variable \( \sigma \in L_2(0, 1) \) as follows:

\[
\sigma(\xi, t) = \tilde{y}_t(\xi, t) + c \tilde{y}(\xi, t), \quad c > 0. \quad (3.14)
\]

The motion of the system constrained on the sliding manifold \( \sigma(\xi, t) = 0 \) is governed by the corresponding simple first-order ordinary differential equation \( \tilde{y}_t(\xi, t) + c \tilde{y}(\xi, t) = 0 \) with the spatial variable \( \xi \) to be viewed as a parameter, whose solution \( \tilde{y}(\xi, t) \), along with its time derivative, exponentially tends to zero in \( L_2(0, 1) \)-norm. Hence, the control task can be reduced to the simplified problem of steering to zero the \( L_2 \)-norm of the distributed sliding variable.

In order to simplify the notation, the dependence of the system signals from the space and time variables \( (\xi, t) \) will be mostly omitted from this point on. Consider the first- and second-order time derivatives of the above defined distributed sliding variable \( \sigma \):

\[
\sigma_t = \tilde{y}_{tt} + c \tilde{y}_t, \quad \sigma_{tt} = \tilde{y}_{ttt} + c \tilde{y}_{tt}. \quad (3.15)
\]

Differentiating the error dynamics (3.8), it derives

\[
\tilde{y}_{ttt} = [v^2(\xi) \tilde{y}_{\xi\xi}]_\xi + \theta_1(\xi) \tilde{y}_t + \theta_2(\xi) \tilde{y}_{tt} + u_t + \psi_t - y^r_{tt} + [v^2(\xi) y^r_{\xi\xi}]_\xi + \theta_1(\xi) y^r_t + \theta_2(\xi) y^r_{tt}. \quad (3.16)
\]

Then substituting (3.8) and (3.16) into the second term of (3.15) yields, after some manipulations,

\[
\sigma_{tt} = [v^2(\xi) (\tilde{y}_{\xi\xi} + c \tilde{y}_{\xi})]_\xi + \theta_1(\xi) [\tilde{y}_t + c \tilde{y}] + \theta_2(\xi) [\tilde{y}_{tt} + c \tilde{y}_t] + u_t + cu - y^r_{tt} - cy^r_{tt} + \bar{\psi}, \quad (3.17)
\]

where

\[
\bar{\psi} = \psi_t + c \psi + [v^2(\xi) (\tilde{y}^r_{\xi\xi} + c \tilde{y}^r_{\xi})]_\xi + \theta_1(\xi) (\tilde{y}^r_t + c \tilde{y}^r) + \theta_2(\xi) (\tilde{y}^r_{tt} + c \tilde{y}^r_{tt}) \quad (3.18)
\]
is an uncertain ‘augmented’ disturbance depending on both the disturbance \(\psi\) and the reference trajectory \(y'\), and their derivatives.

By exploiting Assumptions 3.1, 3.2 and 3.3, it follows that the next restriction on the \(L_2\)-norm of the augmented disturbance \(\psi\) holds for all \(t \geq 0\)

\[
\|\psi(\cdot, t)\|_2 \leq M \equiv \mathcal{P}_1 + c\mathcal{P}_0 + (H_4 + cH_3) + \Theta_1(H_1 + cH_0) + \Theta_2(H_2 + cH_1).
\] (3.19)

After simple additional manipulations, one obtains that the sliding variable \(\sigma\) is governed by a PDE which is formally equivalent to the original wave equation (3.1), with a new fictitious control variable \(\nu\) which dynamically depends on the original plant control input \(u\), according to

\[
\sigma_{tt} = \left[v^2(\zeta)\sigma_{\zeta}\right]_{\zeta} + \theta_1(\zeta)\sigma + \theta_2(\zeta)\sigma_t + \nu - y_{ttt}^r - cy_{tt}^r + \psi,
\] (3.20)

\[
\nu = u_t + cu,
\] (3.21)
equipped with the appropriate ICs and the next ‘homogeneous’ BCs

\[
\sigma(0, t) - \beta_0\sigma_\zeta(0, t) = \sigma(1, t) + \beta_1\sigma_\zeta(1, t) = 0.
\] (3.22)

3.2 Combined PD/sliding mode control of the wave process

In order to stabilize the uncertain dynamics (3.20), (3.22), the following distributed controller

\[
\nu = y_{ttt}^r + cy_{tt}^r - W_1\sigma - W_2\sigma_t - \lambda_1\frac{\sigma}{\|\sigma(\cdot, t)\|_2} - \lambda_2\frac{\sigma_t}{\|\sigma_t(\cdot, t)\|_2}
\] (3.23)
is proposed for generating the fictitious control \(\nu\). Controller (3.23) can be viewed as a mixed linear/sliding mode control algorithm, with a feedforward term, a linear PD-type feedback term, and with the discontinuous feedback term being a distributed version of the finite-dimensional ‘twisting’ controller, which belongs to the class of so-called 2-SMCs (see Levan, 1993).

It is worth to discuss how the actual control input \(u(\zeta, t)\) should be recovered from \(u(\zeta, t)\) once the latter has been computed according to (3.23). In relation (3.21), the spatial variable \(\zeta\) can be viewed as a fixed parameter. By virtue of this fact, (3.21) can be interpreted as a continuum of first-order ODEs whose parameterized solutions give rise to the actual control input to be applied to the wave equation. The transfer function block \(\frac{1}{s^2+c}\) (with \(s\) being the Laplace variable and \(c\) being the positive constant in (3.14)) can effectively represent the relation between signals \(u(\zeta, t)\) (considered as the block input) and \(u(\zeta, t)\) (considered as the block output). The plant control \(u(\zeta, t)\), obtained at the output of a dynamical filter driven by the discontinuous control \(v(\zeta, t)\), will therefore be a continuous signal with a discontinuous time derivative \(u_t(\zeta, t) = v(\zeta, t) - cu(\zeta, t)\), according to the associated block scheme in Fig. 1.

The solution concept of the wave process (3.20–3.22) subject to the control strategy (3.23), (3.7), (3.14) is defined in the same manner as that of the diffusion process dealt with in Section 2, i.e. by means of an appropriate limiting procedure. The exponential stability in the space \(L_2(0, 1)\) of the system in question is demonstrated in Theorem 3.1, given below.

**Theorem 3.1** Consider the generalized wave equation (3.1) along with the ICs and BCs (3.5) and (3.3), and whose parameters, reference trajectory and external disturbance satisfy Assumptions 3.1, 3.2 and 3.3. Consider the associated error variable (3.7) and the sliding variable (3.14). Then, the distributed control strategy (3.23) with the parameters \(W_1, W_2, \lambda_1\) and \(\lambda_2\) such that

\[
W_1 > \Theta_1, \quad W_2 > \Theta_2, \quad \lambda_2 > M, \quad \lambda_1 > \lambda_2 + M,
\] (3.24)
guarantees the exponential decay of the $L_2$-norms $\|\tilde{y}(\cdot, t)\|_2$ and $\|\tilde{y}_t(\cdot, t)\|_2$ of the solutions of (3.20–3.22).

**Proof of Theorem 3.1.**

Refer to the sliding variable dynamics (3.20) along with the BCs (3.22). The closed-loop sliding variable dynamics is easily obtained by substituting (3.23) into (3.20), which yields

$$
\sigma_{tt} = [v^2(\xi)\sigma_{\xi}]_\xi - (W_1 - \theta_1(\xi))\sigma - (W_2 - \theta_2(\xi))\sigma_t - \lambda_1 \frac{\sigma}{\|\sigma(\cdot, t)\|_2} - \lambda_2 \frac{\sigma_t}{\|\sigma_t(\cdot, t)\|_2} + \tilde{\psi}.
$$

(3.25)

Consider the following Lyapunov functional $\tilde{V}(t)$:

$$
\tilde{V}(t) = \frac{1}{2} \|\sqrt{W_1 - \theta_1(\xi)}\sigma\|_2^2 + \lambda_1 \|\sigma\|_2^2 + \frac{1}{2} \|\sigma_t\|_2^2 + \frac{1}{2} \|v(\xi)\sigma_{\xi}\|_2^2 + \frac{1}{2} \frac{v^2(0)}{\beta_0} \sigma_t(0, t)^2 + \frac{1}{2} \frac{v^2(1)}{\beta_1} \sigma(1, t)^2.
$$

(3.26)

Its positive definiteness can be concluded by considering the first and the second tuning inequality (3.24), along with Assumption 3.1, that imply the next conditions

$$
W_1 - \theta_1(\xi) > W_1 - \Theta_1 > 0, \quad W_2 - \theta_2(\xi) > W_2 - \Theta_2 > 0, \quad \forall \xi \in [0, 1].
$$

(3.27)

The time derivative of $\tilde{V}(t)$ is given by

$$
\dot{\tilde{V}}(t) = \int_0^1 (W_1 - \theta_1(\xi))\sigma\sigma_t \, d\xi + \frac{\lambda_1}{\|\sigma\|_2} \int_0^1 \sigma\sigma_t \, d\xi + \int_0^1 \sigma_t\sigma_{tt} \, d\xi + \int_0^1 v(\xi)\sigma_{\xi}\sigma_{\xi, t} \, d\xi
$$

$$
\quad + \int_0^1 \frac{v^2(0)}{\beta_0} \sigma(0, t)\sigma_t(0, t) + \frac{v^2(1)}{\beta_1} \sigma(1, t)\sigma_t(1, t).
$$

(3.28)

By evaluating (3.28) along the solutions of (3.25), it turns out after some simplifications that

$$
\dot{\tilde{V}}(t) = -\int_0^1 (W_2 - \theta_2(\xi))\sigma\sigma_t \, d\xi - \lambda_2 \|\sigma_t\|_2^2 + \int_0^1 \sigma_t\tilde{\psi} \, d\xi + \frac{v^2(0)}{\beta_0} \sigma(0, t)\sigma_t(0, t)
$$

$$
\quad + \frac{v^2(1)}{\beta_1} \sigma(1, t)\sigma_t(1, t) + \int_0^1 [v^2(\xi)\sigma_{\xi}]_\xi \sigma_t \, d\xi + \int_0^1 v^2(\xi)\sigma_{\xi}\sigma_{\xi, t} \, d\xi.
$$

(3.29)

The last term in the right-hand side of (3.29) can be integrated by parts, and taking into account the homogeneous BCs (3.22) yields

$$
\int_0^1 v^2(\xi)\sigma_{\xi}\sigma_{\xi, t} \, d\xi = v^2(1)\sigma_{\xi}(1, t)\sigma_t(1, t) - v^2(0)\sigma_{\xi}(0, t)\sigma_t(0, t) - \int_0^1 [v^2(\xi)\sigma_{\xi}]_\xi \sigma_t \, d\xi
$$

$$
\quad - \frac{v^2(1)}{\beta_1} \sigma(1, t)\sigma_t(1, t) - \frac{v^2(0)}{\beta_0} \sigma(0, t)\sigma_t(0, t) - \int_0^1 [v^2(\xi)\sigma_{\xi}]_\xi \sigma_t \, d\xi.
$$

(3.30)
which leads to the next simplified form of \( \dot{V}(t) \):

\[
\dot{V}(t) = -\int_0^1 (W_2 - \theta_2(\xi)) \sigma_t \, d\xi - \lambda_2 \|\sigma_t\|_2 + \int_0^1 \sigma_t \psi \, d\xi. \tag{3.31}
\]

By employing the Hölder integral inequality and Cauchy–Schwartz inequality (see Krstic & Smyshlyaev, 2008c), and taking into account (3.19), one derives that

\[
\left| \int_0^1 \sigma_t \psi \, d\xi \right| \leq \int_0^1 |\sigma_t \psi| \, d\xi \leq \|\sigma_t\|_2 \|\psi\|_2 \leq M \|\sigma_t\|_2. \tag{3.32}
\]

Then, by (3.31) and (3.32) it follows that

\[
\dot{V}(t) \leq -(W_2 - \Theta_2)\|\sigma_t\|_2^2 - (\lambda_2 - M) \|\sigma_t\|_2, \tag{3.33}
\]

which implies, considering (3.24), that the Lyapunov functional \( \tilde{V}(t) \) is a non-increasing function of time, i.e.

\[
\tilde{V}(t_2) \leq \tilde{V}(t_1), \quad \forall t_2 \geq t_1 \geq 0. \tag{3.34}
\]

Denote

\[
\mathcal{D}_R = \{ (\sigma, \sigma_t) \in L_2(0, 1) \times L_2(0, 1): \tilde{V}(\sigma, \sigma_t) \leq R \}. \tag{3.35}
\]

Clearly, by virtue of (3.34), taking any \( R \geq \tilde{V}(0) \), the resulting domain \( \mathcal{D}_R \) will be ‘invariant’ for the error system trajectories. Our subsequent analysis will take into account that the states \((\sigma, \sigma_t)\) belong to the domain \( \mathcal{D}_R \) starting from the initial time \( t = 0 \) on. Note that the knowledge of the constant \( R \) is not required.

We now demonstrate a simple lemma that will be used along the proof.

**Lemma 3.1** If the states \((\sigma, \sigma_t)\) belong to the domain \( \mathcal{D}_R \) (3.35), then the following estimates hold:

\[
\|\sigma_t\|_2^2 \leq \sqrt{2R} \|\sigma_t\|_2, \tag{3.36}
\]

\[
\int_0^1 \sigma_t \psi \, d\xi \geq -\frac{1}{2} \left[ \frac{R}{\lambda_1} \|\sigma\|_2^2 + \|\sigma_t\|_2^2 \right]. \tag{3.37}
\]

**Proof of Lemma 1.** Equation (3.36) comes from the following trivial chain of implications:

\[
\tilde{V}(t) \leq R \quad \Rightarrow \quad \frac{1}{2} \|\sigma_t\|_2^2 \leq R \quad \Rightarrow \quad \|\sigma_t\|_2 \leq \sqrt{2R} \quad \Rightarrow \quad \|\sigma_t\|_2^2 \leq \sqrt{2R} \|\sigma_t\|_2. \tag{3.38}
\]

A similar procedure results in

\[
\tilde{V}(t) \leq R \quad \Rightarrow \quad \lambda_1 \|\sigma\|_2 \leq R \quad \Rightarrow \quad \|\sigma\|_2 \leq \frac{R}{\lambda_1}. \tag{3.39}
\]
By applying the well-known inequality $ab \geq -\frac{1}{2}(a^2 + b^2)$, it follows that
\[
\int_0^1 \sigma \sigma_t \, d\xi \geq -\frac{1}{2}[\|\sigma\|_2^2 + \|\sigma_t\|_2^2] = -\frac{1}{2}[\|\sigma\|_2\|\sigma\|_2 + \|\sigma_t\|_2^2].
\] (3.40)

Being coupled together, relations (3.38–3.40) yield (3.37), which proves Lemma 1. □

Now consider the ‘augmented’ functional
\[
V_R(t) = \tilde{V}(t) + \kappa_R \int_0^t \sigma \sigma_t \, d\xi + \frac{1}{2} \kappa_R \|\sqrt{W_2 - \theta_2(\xi)}\sigma\|_2^2
\]
\[
= \frac{1}{2} \|\sqrt{W_1 - \theta_1(\xi)}\sigma\|_2^2 + \lambda_1 \|\sigma\|_2 + \frac{1}{2} \|\sigma_t\|_2^2 + \frac{1}{2} \|\nu(\xi)\sigma_\xi\|_2^2
\]
\[
+ \frac{1}{2} \frac{\nu^2(0)}{\beta_0} \sigma^2(0, t) + \frac{1}{2} \frac{\nu^2(1)}{\beta_1} \sigma^2(1, t) \kappa_R \int_0^t \sigma \sigma_t \, d\xi + \frac{1}{2} \kappa_R \|\sqrt{W_2 - \theta_2(\xi)}\sigma\|_2^2,
\] (3.41)

where $\kappa_R$ is a positive constant. In light of Lemma 1, function $V_R(t)$ can be estimated as
\[
V_R(t) \geq \frac{1}{2} (W_1 - \Theta_1 + \kappa_R(W_2 - \Theta_2)) \|\sigma\|_2^2 + \left(\lambda_1 - \frac{\kappa_R R}{2\lambda_1}\right) \|\sigma\|_2
\]
\[
+ \frac{1}{2} (1 - \kappa_R) \|\sigma_t\|_2^2 + \frac{1}{2} \|\nu(\xi)\sigma_\xi\|_2^2 + \frac{1}{2} \frac{\nu^2(0)}{\beta_0} \sigma^2(0, t) + \frac{1}{2} \frac{\nu^2(1)}{\beta_1} \sigma^2(1, t).
\] (3.42)

Since $W_1 - \Theta_1 > 0$ and $W_2 - \Theta_2 > 0$, as previously noted, then, provided that the positive coefficient $\kappa_R$ is selected sufficiently small according to
\[
0 < \kappa_R \leq \min \left\{ \frac{2\lambda_1^2}{R}, 1 \right\},
\] (3.43)

the augmented functional (3.41) proves to be positive definite within the invariant domain $D_R$, and it can be then used as a candidate Lyapunov functional to analyse the stability of the error dynamics. Let us compute the time derivative of $V_R(t)$ along the solutions of (3.25). Simple manipulations yield
\[
\dot{V}_R(t) = -\|\sqrt{W_2 - \theta_2(\xi)}\sigma_t\|_2^2 - \lambda_2 \|\sigma_t\|_2 + \int_0^1 \sigma_t \bar{\psi} \, d\xi
\]
\[
+ \kappa_R \|\sigma_t\|_2^2 + \kappa_R \int_0^1 [\nu^2(\xi)\sigma_\xi] \sigma \, d\xi - \kappa_R \|\sqrt{W_1 - \theta_1(\xi)}\sigma\|_2^2 - \kappa_R \lambda_1 \|\sigma\|_2
\]
\[
- \frac{\kappa_R \lambda_2}{\|\sigma_t\|_2} \int_0^1 \sigma \sigma_t \, d\xi + \kappa_R \int_0^1 \sigma \bar{\psi} \, d\xi.
\] (3.44)

Let us compute upperbounds to the sign-indefinite terms of (3.44). Equation (3.32) has previously been derived, which is rewritten in a similar form with the signal $\sigma$ replacing $\sigma_t$:
\[
\kappa_R \left| \int_0^1 \sigma \bar{\psi} \, d\xi \right| \leq \kappa_R \int_0^1 |\sigma \bar{\psi}| \, d\xi \leq \kappa_R M \|\sigma\|_2.
\] (3.45)
Apart from this, the next inequality can readily be derived by employing the Cauchy–Schwartz inequality (see Krstic & Smyshlyaev, 2008c):

\[
\frac{\kappa \lambda_2}{\|\sigma_1\|_2} \int_0^1 \sigma \sigma_t \, d\xi \leq \frac{\kappa \lambda_2}{\|\sigma_1\|_2} \int_0^1 |\sigma \sigma_t| \, d\xi \leq \frac{\kappa \lambda_2}{\|\sigma_1\|_2} \sqrt{\int_0^1 \sigma^2 \, d\xi} \sqrt{\int_0^1 \sigma_t^2 \, d\xi} = \kappa \lambda_2 \|\sigma\|_2. \quad (3.46)
\]

Then, by substituting (3.45) and (3.46) into (3.44), considering Assumption 3.1 and noticing that the equality

\[
\int_0^1 [v^2(\xi)\sigma_1^2] \, d\xi = -\frac{v^2(1)}{\beta_1} - \sigma^2(1, t) - \frac{v^2(0)}{\beta_0} - \sigma^2(0, t) - \|v^2(\xi)\sigma_1^2\|_2 \quad (3.47)
\]

holds due to the BCs (3.22), the following estimate can be written:

\[
\dot{V}_R(t) \leq -(W_2 - \Theta_2)\|\sigma_1\|_2^2 - \rho \|\sigma_t\|_2 - \kappa_R(\lambda_1 - \lambda_2 - M)\|\sigma\|_2
\]

\[
- \kappa_R(W_1 - \Theta_1)\|\sigma_t\|_2 - \kappa_R \gamma_m \|\sigma_1\|_2 \leq \frac{v^2(1)}{\beta_1} - \sigma^2(1, t) - \frac{v^2(0)}{\beta_0} - \sigma^2(0, t), \quad (3.48)
\]

\[
\rho = (\lambda_2 - M - \kappa_R \sqrt{2R}). \quad (3.49)
\]

Therefore, employing the parameter tuning conditions (3.24) and introducing one more restriction

\[
\kappa_R \leq \min \left\{ \frac{2\lambda_2^2}{R}, 1, \frac{\lambda_2 - M}{\sqrt{2R}} \right\} \quad (3.50)
\]

about the coefficient \(\kappa_R\) beyond (3.43), it readily follows that all terms appearing in the right-hand side of (3.48) are negative definite. It can then be concluded that

\[
\dot{V}_R(t) \leq -c_R(\|\sigma\|_2 + \|\sigma_1\|_2^2 + \|\sigma_t\|_2 + \|\sigma_1\|_2^2 + \|\sigma_1\|_2^2 + \sigma^2(1, t) + \sigma^2(0, t)), \quad (3.51)
\]

\[
c_R = \min \left\{ W_2 - \Theta_2, \rho, \kappa_R(W_1 - \Theta_1), \kappa_R(\lambda_1 - \lambda_2 - M), \kappa_R \gamma_m, \frac{\gamma_m}{\beta_1}, \frac{\gamma_m}{\beta_0} \right\}. \quad (3.52)
\]

Applying the Cauchy–Schwartz integral inequality and considering (3.39) yield

\[
\left| \int_0^1 \sigma \sigma_t \, d\xi \right| \leq \|\sigma \sigma_t\|_1 \leq \|\sigma\|_2 \|\sigma_t\|_2 \leq \frac{R}{\lambda_1} \|\sigma_t\|_2. \quad (3.53)
\]

Now substituting (3.53) into (3.41) and upper-estimating further the resulting right-hand side in light of Assumption 3.1 yield the next estimation

\[
\dot{V}_R(t) \leq \omega_R(\|\sigma_2\|_2 + \|\sigma_t\|_2 + \|\sigma_1\|_2 + \|\sigma_t\|_2 + \|\sigma_1\|_2^2 + \sigma^2(1, t) + \sigma^2(0, t)), \quad (3.54)
\]
\[ w_R = \max \left\{ \frac{1}{2} (W_1 - \Theta_1 + \kappa_R (W_2 - \Theta_2)), \lambda_1, \frac{1}{2}, \kappa_R \frac{R}{\lambda_1}, \frac{1}{2} \gamma_M, \frac{\gamma_M}{\beta_1}, \frac{\gamma_M}{\beta_0} \right\}. \] (3.55)

Hence, the following differential inequality holds:

\[ \dot{V}_R(t) \leq -c_1 V_R(t), \quad c_1 = \frac{c_R}{w_R} > 0, \] (3.56)

thereby ensuring the exponential decay of \( V_R(t) \) to zero. By (3.42) and (3.43), then it can be found \( p_R > 0 \) such that

\[ V_R(t) \leq p_R (\|\sigma\|_2^2 + \|\sigma_t\|_2^2 + \|\sigma_{\zeta}\|_2^2 + \sigma^2(1, t) + \sigma^2(0, t)). \] (3.57)

It means that \( \|\sigma\|_2, \|\sigma_t\|_2, \|\sigma_{\zeta}\|_2, \sigma^2(0, t) \) and \( \sigma^2(1, t) \) tend to zero exponentially as \( t \to \infty \). It remains to prove that the \( L_2 \)-norm of the tracking error \( \tilde{y}(\xi, t) \) and that of its derivative tend exponentially to zero. To this end, note that the inequality

\[ \|\sigma(\cdot, t)\|_2 \leq c_2 V_R(t), \quad c_2 = \frac{2 \lambda_1}{2 \lambda_1^2 - \kappa_R R}, \] (3.58)

is straightforwardly derived from (3.42), whereas, by (3.14), the spatiotemporal evolution of \( \tilde{y}(\xi, t) \) is governed by

\[ \tilde{y}_t(\xi, t) = -c \tilde{y}(\xi, t) + \sigma(\xi, t), \quad c > 0. \] (3.59)

In (3.59), the sliding variable \( \sigma(\xi, t) \) can be viewed as an external driving input, exponentially decaying in \( L_2 \)-norm according to (3.58). Computing the time derivative of the Lyapunov functional \( W(t) = \|\tilde{y}\|_2 \) along dynamics (3.59) yields

\[ \dot{W}(t) = -c W(t) + \frac{1}{W(t)} \int_0^1 \tilde{y}(\xi, t) \sigma(\xi, t) d\xi. \] (3.60)

Since

\[ \left| \int_0^1 \tilde{y} \sigma d\xi \right| \leq \|\tilde{y}\|_2 \|\sigma\|_2 \leq W(t) \|\sigma\|_2, \] (3.61)

by combining (3.58), (3.60) and (3.61) it follows that

\[ \dot{W}(t) \leq -c W(t) + \|\sigma\|_2 \leq -c W(t) + c_2 V_R(t). \] (3.62)

It is now clear that relations (3.56–3.59), and (3.62), coupled together, ensure the exponential decay of \( \|\tilde{y}(\cdot, t)\|_2 \) and \( \|\tilde{y}_t(\cdot, t)\|_2 \). Theorem 3.1 is thus proved. \( \square \)

Remark 3.1 The proposed dynamical controller (3.23) makes explicit use for feedback of the first- and second-order output derivatives \( y_t(\xi, t) \) and \( y_{tt}(\xi, t) \), which might be not available for direct measurement. The problem of estimating the output derivatives by means of output measurement of \( y(\xi, t) \) given in real time is of great interest in itself. Possible solutions could be devised by suitably generalizing to
the DPS setting some existing approaches to real-time differentiation such as the high-gain observers (see Dabroom & Khalil, 1999) or the sliding mode differentiators (see Levant, 2003) and combining them with the suggested controller. In turn, the use of the state observers from Orlov et al. (2004) could allow one to successfully address the practically relevant case of the output feedback design. The careful analysis of these aspects is a challenging task which is beyond the scope of the present paper and it is going to be addressed in the context of next research activities.

4. Numerical simulations

For solving the PDEs governing the closed-loop systems, standard finite-difference approximation method is used by discretizing the spatial solution domain $\zeta \in [0, 1]$ into a finite number of $N$ uniformly spaced solution nodes $\zeta_i = ih$, $h = 1/(N + 1)$, $i = 1, 2, \ldots, N$. The value $N = 100$ has been used in the present simulations. The resulting 100th-order discretized system is implemented in Matlab-Simulink and solved by fixed-step Euler integration method with constant step $T_s = 10^{-4}s$.

4.1 Reaction–diffusion equation

Consider the perturbed reaction–diffusion equation (2.1) with the spatially varying parameters given by

$$
\theta_1(\zeta) = 0.1 + 0.02 \sin(1.3\pi \zeta), \quad (4.1)
$$

$$
\theta_2(\zeta) = 1 + 0.1 \sin(3.5\pi \zeta), \quad (4.2)
$$
mixed-type BCs

$$
Q(0, t) - a_0 Q_\zeta(0, t) = Q(1, t) + a_1 Q_\zeta(1, t) = 20 - 5\pi \quad (4.3)
$$

and ICs

$$
Q(\zeta, 0) = 20 + 10 \sin(6\pi \zeta). \quad (4.4)
$$

We choose the next spatially varying reference profile

$$
Q_r(\zeta, t) = 20 + 5 \sin(\pi \zeta), \quad (4.5)
$$

which meets the actual BCs. A spatially varying disturbance term is considered in the form

$$
\psi(\zeta) = 5 \sin(2.5\pi \zeta). \quad (4.6)
$$

By (4.2), the bound $\Theta_{2M}$ in (2.12) can be readily overestimated by any $\Theta_{2M} > 1.1$. Then, the controller gains are set in accordance with (2.15) to the values

$$
W_1 = 20, \quad \lambda_1 = 20, \quad W_2 = 20, \quad \lambda_2 = 20. \quad (4.7)
$$

The left plot in Fig. 2 depicts the solution $Q(\zeta, t)$, which converges to the given reference profile as confirmed by the contractive evolution of the tracking error $L_2$-norm $\|x(\cdot, t)\|_2$ shown in Fig. 2 (right). Figure 3 depicts the control input $u(\zeta, t)$ which, as expected, appears to be a smooth function of both time and space. The attained results confirm the validity of the presented analysis.
4.2 Generalized wave equation

Consider the perturbed equation (3.1) with spatially varying parameters:

\[
\nu^2(\xi) = 0.1 + 0.02 \sin(2\pi \xi),
\]

\[
\theta_1(\xi) = -(1 + \sin(1.2\pi \xi)),
\]

\[
\theta_2(\xi) = -(5 + 3 \sin(3\pi \xi))
\]

and mixed BCs

\[
y(0, t) - y_\xi(0, t) = y(1, t) + y_\xi(1, t) = 0.
\]

The bounds \(\Theta_1 = 2, \Theta_2 = 8\) to the uncertain system parameters (see Assumption 3.1) are taken into account for the controller tuning. The ICs in (3.5) are set to \(\varphi_0(\xi) = 10 \sin(6\pi \xi), \varphi_1(\xi) = 0\). The reference profile is set to \(y' (\xi, t) = 2 \sin(\pi \xi) \sin(\pi t)\). The bounds \(H_0 = 2, H_1 = 6, H_2 = 20, H_3 = 3, H_4 = 96\) to the norms of its derivatives as in (3.11–3.12) are considered. The disturbance is set to \(\psi(\xi, t) = 10 \sin(5\pi \xi) \sin(2\pi t)\). The upperbounds \(\Psi_0 = 10, \Psi_1 = 63\) are considered in the restrictions (3.13).

The distributed sliding manifold \(\sigma(\xi, t)\) has been implemented with the parameter \(c = 2\). Parameter \(M\) in (3.19) is chosen as \(M = 400\). The controller parameters are set in accordance with
Figure 4. Different views of the solution $y(\xi, t)$ of the controlled generalized wave equation.

Figure 5. Generalized wave equation test. Left plot: distributed control $u(\xi, t)$. Right plot: tracking error $L_2$-norm $\|\tilde{y}(\cdot, t)\|_2$.

(3.24) as $W_1 = 2$, $W_2 = 10$, $\lambda_2 = 500$ and $\lambda_1 = 1000$. Figure 4 reports two different views of the solution $y(\xi, t)$. Figure 5 shows the corresponding plots of the distributed control $u(\xi, t)$ and of the tracking error $L_2$-norm $\|\tilde{y}(\cdot, t)\|_2$. Good performance of the proposed control algorithm is concluded from the graphics that confirm the theoretical properties of the proposed distributed controller. The continuity of the applied distributed control input particularly follows from the inspection of Fig. 5 (left).

5. Concluding remarks

The ‘super-twisting’ and ‘twisting’ 2-SMC algorithms have been used in conjunction with linear PI and PD controllers, respectively, in a distributed system and control set-up. The two resulting schemes have been applied to solve the tracking control problems for heat and wave processes subject to persistent disturbances of arbitrary shapes and with spatially varying uncertain plant parameters. By means of appropriate ad hoc Lyapunov functionals, the stability of the resulting error dynamics is
proven in the $L_2$-space. Along with this, the proposed infinite-dimensional treatments retain robustness features against non-vanishing matching disturbances similar to those possessed by their finite-dimensional counterparts. Finite-time convergence of the proposed algorithms, which would be the case if confined to a finite-dimensional treatment, cannot be proved using the proposed Lyapunov functionals, and it remains among other actual problems to be tackled in the future within the present framework.

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