Stabilizing any SISO LTI Plant with an Arbitrarily Large Unknown Delay and Unknown $B$ Matrix

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Abstract—Handling delays and uncertain parameters in control systems is difficult and of long-standing interest. In this paper, we consider the problem of bounded-input bounded-output stabilizing any single-input single-output, linear time-invariant, controllable/observable plant with an arbitrarily large unknown time delay and a large amount of uncertainty in the $B$ matrix of the state-space model. Despite being nonlinear time-varying, the proposed controller is relatively simple, the non-linearity is mild and the time variation is periodic; furthermore, the controller also guarantees the exponential decay of any initial conditions.

I. INTRODUCTION

In control system design, plant uncertainty is a common problem, and problems dealing with plant uncertainties have been studied for a long time. In this paper, we consider two types of uncertainty, namely, an unknown but upper bounded time delay and uncertainty in the $B$ matrix of the state-space model. Uncertain delays have many causes, for example, communication over a network or via a satellite, computational delays, physical transport delays, etc., and have been studied as part of control systems for a long time [12], [8], [7], [9], [4], [1], [2], [10]. Uncertainty in the $B$ matrix is a generalization of uncertainty in the high frequency gain [6], and translates to uncertainty in the zero locations.

Consider the problem of bounded-input bounded-output (BIBO) stabilizing a single-input single-output (SISO) continuous-time linear time invariant (LTI) plant with an uncertain delay. It is proven in [7] that there exists a maximum unknown delay for which stability can be maintained for a plant with an open right half plane pole when using static state feedback. When using output feedback, it is proven in [8] that this restriction is increased to any plant with a non-zero unstable pole; furthermore, explicit bounds (occasionally tight) are easily computed for any LTI plant. Finally, it was proven in [9] that a linear periodic finite dimensional output feedback controller is proposed which provides an arbitrarily large delay margin.

In this work, we not only have an unknown time delay, but also uncertainty in the $B$ matrix of the state-space model. Much work has been carried out on problems involving unknown delays and additional plant uncertainty, though little in the way of fundamental performance limitations (like [8] for time delays, [6] for gain margin, etc.). A common approach is to use linear matrix inequalities (LMIs) which can synthesize controllers and provide numerical calculations of loose upper bounds, [11], [4]; however, these results are difficult to generalize and provide little intuition. As far as we are aware, there are only two controllers which can tolerate an unknown time delay with an arbitrarily large upper bound as well as additional plant uncertainty, namely [5] and [2], and only the former provides BIBO stability. In [2], an infinite dimensional non-linear time varying adaptive controller is proposed and shown to achieve asymptotic tracking for any SISO LTI plant with an unknown arbitrarily long delay and with unknown $A$ and $B$ state-space matrices lying in a known, linearly parameterized observable/controllable set; however, the result was proven using state feedback and like many adaptive controllers, it does not provide BIBO stability. In our early work [5], we proposed a finite dimensional, non-linear, time varying controller which BIBO stabilizes any SISO LTI plant with an arbitrarily large unknown time delay and gain.

In this paper we focus on the open problem of BIBO stabilizing any controllable/observable LTI SISO continuous time plant with an unknown but upper bounded time delay and uncertainty in the $B$ matrix of the state-space model; while it is desirable to allow uncertainty in the $A$ matrix of the state-space model, at this time we cannot handle this form of uncertainty. Here, $B$ must lie in a compact set for which controllability of the corresponding state-space model is maintained. To solve this problem, we propose a finite dimensional non-linear periodic controller, which not only provides BIBO stability but also guarantees the exponential decay of the plant initial conditions even in the presence of noise. The general approach is based on that of our earlier work [5], where we considered only an unknown gain and delay, but with several new twists.

The paper is organized as follows. In Section II, a few mathematical preliminaries are introduced. In Section III, we introduce and set up the problem, as well as define our notion of stability. In Section IV we state our proposed controller, and in Section V we prove that our proposed controller works. In Section VI, we provide a simulation, and finally, in Section VII we provide some conclusions and discuss some future work. Due to space constraints, all proofs are omitted.

II. MATHEMATICAL PRELIMINARIES

We let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ denote the set of non-negative real numbers, $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}^+$ denote the set of non-negative integers and $\mathbb{C}$ denote the set of complex numbers. We define the infinity norm of a vector $x \in \mathbb{R}^n$ as $\|x\| = \max_{i=1,2,\ldots,n} |x|$; the corresponding induced norm of a matrix $A \in \mathbb{R}^{n \times m}$ is given by $\|A\| = \max_{x \in \mathbb{R}^m, \|x\|=1} \|Ax\|$. We let $\ell(\mathbb{R}^n)$ denote the set of $\mathbb{R}^n$ valued sequences on $\mathbb{Z}^+$, and we will write a sequence in $\ell(\mathbb{R}^n)$ using a Greek letter and $[\cdot]$ for the argument (i.e., $\psi[k]$). We define the infinity norm of a
sequence $\psi \in \ell(R^n)$ as $\|\psi\|_{\infty} = \sup_{k \geq 0} \|\psi[k]\|$, and we say that a sequence $\psi \in \ell_{\infty}(R^n)$ if $\|\psi\|_{\infty} < \infty$. We let $PC(R^n)$ denote the set of piecewise continuous functions from $R$ to $R^n$, and we write a function in $PC(R^n)$ using a Roman letter and (·) for the argument (i.e., $y(t)$). We define the infinity norm of a function $f \in PC(R^n)$ as $\|f\|_{\infty} = \sup_{t \geq 0} |f(t)|$, and we say that a function $f \in PC_{\infty}$ if $\|f\|_{\infty} < \infty$. To simplify notation, we will often drop the dimensions from our spaces. In order to discuss stability of a NLTV system, we will require a notion of the gain of such a system starting with zero initial conditions at time zero. To this end, the gain of $G : PC_{\infty} \rightarrow PC$ is defined by

$$\|G\| := \sup_{u \in PC_{\infty}, \|u\|_{\infty} \neq 0} \frac{\|Gu\|_{\infty}}{\|u\|_{\infty}}.$$

We use a special version\(^1\) of the “sign” function which maps $R$ to $\{-1, 1\}$:

$$\text{sgn}(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
-1 & \text{if } x < 0.
\end{cases}$$

### III. Problem Formulation

Let $A \in R^{n \times n}$ be an arbitrary matrix and then let $B$ be a compact set of vectors in $R^{n \times 1}$ such that $(A, B)$ is controllable for every $B \in B$. Our goal is to BIBO stabilize the following set of admissible plant models:

$$G := \left\{ \begin{array}{l}
\dot{x}(t) = Ax(t) + Bu_d(t - \tau) \\
y(t) = Cx(t)
\end{array} \right\}_{\tau \in [0, \tau], B \in B}.$$

Here the plant initial condition is $x(0) = x_0$ and $u(\theta) = u_0(\theta)$ for $\theta \in [-\tau, 0)$. We assume that $(C, A)$ is observable; since $B$ is compact, we observe that there exists a constant $\overline{\tau} > 0$ such that

$$\max_{B \in B} \|B\| = \overline{\tau}.$$

We consider the standard feedback structure: the controllers are input-output of the form

$$u = Ky.$$

To define stability, we introduce noise at the two plant/controller interfaces as shown in Figure 1. Due to the input delay, the input noise $d$ has an initial condition $d(\theta) = d_0(\theta)$ for $\theta \in [-\tau, 0)$, and we define the stacked noise vector $\overline{w}(t) := \begin{bmatrix} w(t) \\
d(t - \tau)
\end{bmatrix}$.\(^2\) With zero initial conditions on the plant, i.e., $x_0 = 0$, $u_0(\theta) = d_0(\theta) = 0$ for $\theta \in [-\tau, 0)$, we let $\Phi(\tau, B)$ be the closed loop map from $\begin{bmatrix} d \\
w
\end{bmatrix}$ to $\begin{bmatrix} y \\
u
\end{bmatrix}$.

**Definition 1:** We say that $K$ stabilizes $G$ if $\Phi(\tau, B)$ is uniformly bounded, i.e.,

$$\sup_{\tau \in [0, \tau], B \in B} \|\Phi(\tau, B)\| < \infty.$$

The goal of this paper is to design a controller $K$ which stabilizes $G$.

\(^1\)Normally $\text{sgn}(0)$ is defined to be zero.\(^2\)We delay $d$ by $\tau$ seconds so that $d$ from time $-\tau$ to $\infty$ is included in the norm of $\overline{w}$.

### IV. The Controller

For any plant $G \in \mathcal{G}$, we incorporate the input and output noise into the plant model, yielding

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t - \tau) + d(t - \tau) \\
y_w(t) &= Cx(t) + w(t),
\end{align*}$$

with initial conditions of $x(0) = x_0$, $u(\theta) = u_0(\theta)$ and $d(\theta) = d_0(\theta)$ for $\theta \in [-\tau, 0)$. Our proposed control scheme is given in Figure 2; it is periodic with period $T$. We will describe each block of the controller in detail, but for now, we present the basic idea. We start with a generalized hold $H$ and a higher frequency sampler $S$ as described in sub-section IV-A. In sub-section IV-B, we apply the sampler and hold to the plant to yield a discretized model, and using the sampler, we design a state estimator in sub-section IV-C. Using this discretized plant model, in sub-section IV-D we design an estimator for the discretized $B$ matrix, and in sub-section IV-E we propose our control law. Finally, we summarize the control scheme in sub-section IV-F.

#### A. The Sampler and Hold

We are interested in applying a generalized hold and an almost standard sampler to a plant in $\mathcal{G}$; a relevant question is “Since $(A, B)$ is controllable and $(C, A)$ is observable, is the same true for the discretized system?” If we are using a zero-order hold and a normal sampler, then from [3] the answer is yes so long as the sampling period is ‘non-pathological’; it will turn out that we have a very similar requirement, so it will be extremely useful to recall the definition of a pathological sampling period:

**Definition 2:** A sampling period $h_s$ is non-pathological (with respect to $A$) if, whenever $\mu \in C$ is an eigenvalue of $A$, none of the elements of $\left\{ \mu + \frac{2 \pi k}{h_s} : k \in N, k \neq 0 \right\}$ is an eigenvalue of $A$. Otherwise, we say that the sampling period $h_s$ is pathological.
As in [5], we are using a modified zero-order-hold, \( H \), which is parameterized by three positive quantities of time, \( T_1, T_2, \) and \( T_3 \), which satisfy the following:

i) \( T_3 > \tau \), and,

ii) \( T = T_1 + T_2 + T_3 \) is non-pathological (with respect to \( A \)).

We define the hold \( H : \ell(R) \rightarrow PC(R) \) for \( k \geq 0 \) by

\[
(Hu)(t) := \begin{cases} 0 & t \in [kT, kT + T_1) \\ \nu[k] & t \in [kT + T_1, kT + T_1 + T_2) \\ 0 & t \in [kT + T_1 + T_2, kT + T) \\ \end{cases}
\]

The purpose of the sampler \( S \) is to allow us to estimate the plant state at time \( kT \); to accomplish this goal, as in [5] we run the sampler at a period \( h \), and we impose the following three constraints on \( h \):

i) \( \frac{T}{h} \) must be an integer so that \( h \) is synchronous with \( T \).

ii) \( H \) must be non-pathological with respect to \( A \) so that \((C, e^{Ah})\) is guaranteed to be observable.

iii) \( h < \frac{T}{\delta} \) so that we obtain at least \( n \) samples over the interval \([kT, kT + T_1)\).\(^3\)

With these restrictions in place, we define the sampler \( S : PC(R) \rightarrow \ell(R) \) by

\[
(Sy)_w[jh] := y_w(jh), \quad j \in Z^+.
\]

B. Discretizing the Plant

We would now like to discretize the plant (1) using our sampler and hold:\(^4\)

\[
x(kT + T) = \chi[k + 1] = \begin{cases} x_0 & \text{if } k = 0 \\ \chi[k] & \text{if } k \geq 1 \\ \end{cases}
\]

\[
= : A_d = \chi[k]
\]

\[
e^{A(T-v)}Bu \left[ \int_{\tau}^{0} e^{-A(T-v)}Bdv + \int_{\tau}^{0} e^{A(T-v)}Bu \right] dv +
\]

\[
\int_{\tau}^{0} e^{A(T-v)}Bu\left[ \int_{\tau}^{0} e^{-A(T-v)}Bu \right] dv +
\]

\[
y_w(kT) = Cx(kT) + w(kT).
\]

Our discretized plant has the initial condition \( \chi_0 = x_0 \); it is also useful to note that

\[
\|\phi\| \leq \sup_{\theta \in [-\tau,0]} \|u_0(\theta)\|,
\]

and recalling that \( \|w(t)\| := \left[ \frac{w(t)}{d(t - \tau)} \right] \), it is easy to see that

\[
\|\omega\| = \|\chi[1]\| 
\]

\[
\|\zeta\| = \|\chi[1]\|.
\]

Since \((C, A)\) is observable by assumption, and \( T \) is chosen to be non-pathological with respect to \( A \), it follows from [3] that \((C, A_d)\) is observable. Verifying that \((A_d, B_d(\tau, B))\) is controllable requires some work. To this end, define the set of all possible \( B_d(\tau, B) \):

\[
B_d = \left\{ e^{A(T-T_1)} \left[ \int_{0}^{T_2} e^{-A\tau}Bdv; \tau \in [0, \tau], B \in B \right] \right\}.
\]

Proposition 1: \((A_d, B_d)\) is controllable for every \( B_d \in B_d \).

C. Estimating the State \( \chi \)

To estimate \( \chi[k] = x(kT) \) we will follow the approach outlined in [9]. During the initial part of the period, i.e., for \( t \in [kT, kT + T_1) \), the sampler \( S \) samples the output \( y_w(t) \) every \( h \) seconds, generating a total of \( n \) samples. Furthermore, we are guaranteed that the controlled input to the plant (even with the unknown delay) is zero over this time period.\(^5\)

Since we chose \( h \) to be non-pathological with respect to \( A \), it follows that \((C, e^{Ah})\) is observable. Assuming no noise, we have

\[
\begin{bmatrix}
y_w(kT) \\
y_w(kT + h) \\
\vdots \\
y_w(kT + (n-1)h)
\end{bmatrix} = \begin{bmatrix} C \\
C e^{Ah} \\
\vdots \\
C e^{Ah \cdot (n-1)}
\end{bmatrix} x(kT);
\]

our observability assumption ensures that \( O_h \) is invertible, so we can solve for \( x(kT) \), yielding \( x(kT) = \chi[k] \). Of course, we have noise in our actual system, so we set

\[
\hat{\chi}[k] = O_h^{-1}Y(kT), \quad k \geq 0,
\]

and then define the state estimation error \( \hat{\chi}[k] := \hat{\chi}[k] - \chi[k] \).

The following is easily proven:

Lemma 1: There exists a constant \( c_2 > 0 \) such that

\[
\|\hat{\chi}[k]\| \leq c_2 \|w\| + \sup_{\theta \in [-\tau,0]} \|u_0(\theta)\|, k \geq 0.
\]

D. Estimating \( B_d(\tau, B) \)

At this point, we make a major departure from our earlier paper [5] where we handled an uncertain delay and gain. In order to estimate \( B_d(\tau, B) \), we use the state equation (4) shifted backwards by \( h \) and solve for \( B_d(\tau, B) \); assuming that \( \nu[k - 1] \neq 0 \) and \( k \geq 2 \), we obtain

\[
\chi[k] = A_d\chi[k-1] + B_d(\tau, B)\nu[k-1] + \zeta[k-1]
\]

\[
\Leftrightarrow B_d(\tau, B) = \frac{\chi[k] - A_d\chi[k-1] - \zeta[k-1]}{\nu[k-1]}.
\]

\(^3\)Note that if there is no noise and zero initial conditions, then for every \( \tau \in [0, \tau] \), we have that \( u(t - \tau) = 0 \) for \( t \in [kT, kT + T_1) \); this property is critical for the design of our state estimator.

\(^4\)Due to the initial condition on the plant input, we have an extra term in the state equation, given by \( \delta[k]\int_{\tau}^{0} e^{-Av}Bu_0(v)dv \) (here \( \delta[k] \) is the standard discrete-time impulse sequence).

\(^5\)Due to the initial condition on the input \( u \), this is only true for \( k \geq 1 \), and of course, the plant will still be affected by noise over this time frame.
Since we do not know $\chi[k], \chi[k-1]$ nor $\zeta[k-1]$, in order to estimate $B_d(\tau, B)$ we use the state estimate $\hat{\chi}$ in place of the actual state and assume that the noise to zero. With arbitrary initial conditions of $\chi[-1] \in \mathbb{R}^n, \nu[-1] \in \mathbb{R}$, and $\beta_d[-1] \in \beta_d$, for $k \geq 0$ we define a crude estimate of $B_d(\tau, B)$ by:

$$\hat{B_d}[k] := \begin{cases} 
\frac{\hat{\chi}[k] - A_d \hat{\chi}[k-1]}{\nu[k-1]} & \text{if } \nu[k-1] \neq 0 \\
\hat{B_d}[k-1] & \text{if } \nu[k-1] = 0.
\end{cases}$$

(9)

Due to the noise, it is clear that $\hat{B_d}[k]$ may not lie in $B_d$, and, in fact, there is no guarantee that $(A_d, \hat{B_d}[k])$ is even controllable, so we will process $\hat{B_d}[k]$ to yield our final estimate $\hat{B_d}[k]$. In our earlier work [5], we projected the estimate of the delay $\tau$ onto the convex set $[0, \tau]$ and the estimate of the unknown gain $g$ onto the convex set $[1, \bar{g}]$; since each set is closed and convex, projecting guarantees that the estimation error decreases. In this work, we do not have this luxury since $B_d$ is not convex in general. However, we will be able to prove that $\hat{B_d}[k]$ approaches $B_d(\tau, B)$ as $\|\chi[k-1]\|_\infty \to \infty$, so we restrict our estimate $\hat{B_d}[k]$ to a larger set $\hat{B_d}$ which contains $B_d$ inside its interior (the largest open set inside $B_d$). We want to define $\hat{B_d}$ so that its elements are uniformly bounded and so that for every $\hat{B_d} \in \hat{B_d}$, we have that $(A_d, \hat{B_d})$ is controllable; we would also like to define $\hat{B_d}$ so that it is easy to check if an estimate $\hat{B_d}[k]$ lies inside $\hat{B_d}$. To this end, we define the controllability matrix for a pair $(A, B)$ as

$$C(A, B) := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix},$$

and then define

$$b_d := \min_{B_d(\tau, B) \in B_d} |\det C(A_d, B_d(\tau, B))|$$

and

$$\overline{b_d} := \max_{B_d(\tau, B) \in B_d} \|B_d(\tau, B)\|,$$

which we then use to define the following set:

$$\hat{B_d} = \left\{ B_d \in \mathbb{R}^{n \times 1} : \begin{bmatrix} \det C(A_d, B_d) \end{bmatrix} \geq \frac{b_d}{\overline{b_d}} \right\}.$$  

Clearly, $\hat{B_d} \subset \bar{B_d}$ and for every $B_d \in \hat{B_d}$ we have that $(A_d, B_d)$ is controllable, so now we need to show that $\hat{B_d}$ is compact and that the interior of $\hat{B_d}$ contains $B_d$.

**Lemma 2:** $\hat{B_d}$ is compact and there exists a constant $\varepsilon > 0$ such that

$$\{B_d + v : B_d \in \hat{B_d}, v \in \mathbb{R}^n, \|v\| < \varepsilon \} \subset \hat{B_d}.$$

With $\hat{B_d}$ defined as above, we ensure that our actual estimate lies in $\hat{B_d}$ as follows:

$$\hat{B_d}[k] = \begin{cases} 
\hat{B_d}[k] & \hat{B_d}[k] \in \hat{B_d} \\
\hat{B_d}[k-1] & \hat{B_d}[k] \notin \hat{B_d}.
\end{cases}$$

(10)

and then define the estimation error as $\hat{B_d}[k] := \hat{B_d}[k] - B_d(\tau, B)$. In the next section, we will prove that $\hat{B_d}[k] \to 0$ as $\|\chi[k-1]\|_\infty \to 0$, but before we do so, we first need to define the control law.

**E. The Control Signal**

Our control law is based on state feedback using pole placement. To this end, fix $\beta[2]$ to be an $n^{th}$ order monic Schur polynomial (with real coefficients), and for each $B_d \in \hat{B_d}$, define $F(B_d)$ to be the unique element of $\mathbb{R}^{1 \times n}$ which satisfies $\det [zI - A_d B_d F(B_d)] = \beta[z]^6$ and then define the set of all possible $F(B_d)$:

$$F := \left\{ F \in \mathbb{R}^{1 \times n} : A_d + B_d F \text{ is } \beta[z] \text{ for some } B_d \in \hat{B_d} \right\}.$$

For our control scheme, at each time step we will use our estimate $\hat{B_d}[k] \in \hat{B_d}$ to find an estimate of the desired feedback gain, so we define two special feedback gains, the first of which is the estimate at each time step:

$$\hat{F}[k] := F(\hat{B_d}[k]) \in \mathcal{F},$$

while the second is the feedback gain we would design if our estimate was perfect, i.e.,

$$F(\tau, B) := F(B_d(\tau, B)) \in \mathcal{F}.$$

We define the feedback gain error as

$$\hat{F}[k] := \hat{F}[k] - F(\tau, B),$$

and the following Lemma bounds this error.

**Lemma 3:** $\mathcal{F}$ is compact and there exists a constant $c_1 > 0$ so that for every $B_1, B_2 \in \hat{B_d}$ we have that

$$\|F(B_1) - F(B_2)\| \leq c_1 \|B_1 - B_2\|.$$

**Corollary 1:** There exists a constant $c_2 > 0$ so that

$$\|\hat{F}[k]\| \leq c_1 \|\hat{B_d}[k]\|.$$  

Since $\mathcal{F}$ and $\hat{B_d}$ are compact, it will be useful to upper bound the norms of $F \in \mathcal{F}$ and $B_d \in \hat{B_d}$:

$$\bar{F} := \sup_{F \in \mathcal{F}} \|F\|, \quad \bar{b_d} := \sup_{B_d \in \hat{B_d}} \|B_d\|.$$

Clearly, given any $B_d \in \hat{B_d}$, our ideal control for the system

$$\chi[k+1] = A_d \chi[k] + B_d \nu[k]$$

(12)

is given by $\nu[k] = F(B_d) \chi[k]$. However, in order to prove that $\|\hat{B_d}[k]\| \to 0$ as $\|\chi[k-1]\|_\infty \to 0$, we require an additional probing signal, yielding the following control law:

$$\nu[k] = F(B_d) \chi[k] + \rho \text{sgn}(F(B_d)\chi[k]) \|\chi[k]\|.$$  

(13)

We will now show that this stabilizes (12) if $\rho > 0$ is small enough. 7 Applying (13) to (12) results in the following closed loop system:

$$\chi[k+1] = (A_d + B_d F(B_d)) \chi[k] + \rho B_d \text{sgn}(F(B_d)\chi[k]) \|\chi[k]\|.$$  

(14)

6By the definition of $\hat{B_d}$, we have that $(A_d, \hat{B_d})$ is controllable, which implies that $F(\hat{B_d})$ is unique.

7We choose $\rho > 0$ so that both terms on the RHS of (13) have the same sign; this means, in particular, that $\nu[k] = 0$ if $\chi[k] = 0$. 8
We want to find the values of \( \rho > 0 \) such that the origin of (14) is a globally exponentially stable equilibrium point for every \( B_d \in \mathcal{B}_d \). To do so, first let \( P(B_d) \) be the unique positive definite solution to the Lyapunov equation

\[
(A_d + B_d F(B_d))' P(B_d) (A_d + B_d F(B_d)) - P(B_d) = -I.
\]

Using the fact that (15) is a linear equation, it is easy to prove:

**Proposition 2:** There exists positive constants \( c_3 \) and \( c_4 \) such that for every \( B_d \in \mathcal{B}_d \) and \( \chi \in \mathbb{R}^n \) we have that

\[
\begin{align*}
(i) & \quad c_3 I \leq P(B_d) \leq c_4 I \\
(ii) & \quad c_3 \leq \| P(B_d) \| \leq c_4 \\
(iii) & \quad c_3 \| \chi \|^2 \leq \chi' P(B_d) \chi \leq c_4 \| \chi \|^2.
\end{align*}
\]

We would also like to bound \( \| A_d + B_d F(B_d) \| \); since \( B_d \in \mathcal{B}_d, F(B_d) \in \mathcal{F} \), and \( \mathcal{B}_d \) and \( \mathcal{F} \) are compact sets, it follows that

\[
\overline{\sigma} := \sup_{B_d \in \mathcal{B}_d} \| A_d + B_d F(B_d) \| < \infty.
\]

With these constants, we can define

\[
\bar{\rho} := \frac{-nc_4 \sigma + \sqrt{nc_4(n c_4^2 + 1)}}{nc_3 b}.
\]

**Lemma 4:** If \( \rho \in (0, \bar{\rho}) \), then for all \( B_d \in \mathcal{B}_d \), the origin is a globally exponentially stable equilibrium point of (14).

**Remark 1:** The condition on \( \rho \) is equivalent to requiring that \( \rho > 0 \) and satisfies

\[
nc_4 \rho^2 + 2nc_4 \rho \bar{\sigma} < 1.
\]

We can now give our actual control law; it has the same general form as (13) except we replace the state with its estimate and we replace \( F(B_d) \) with our estimate \( \hat{F}[k] \). So, with \( \rho \in (0, \bar{\rho}) \) the control law is given by

\[
\begin{align*}
\nu[k] &= \hat{F}[k] \hat{\chi}[k] + \rho \text{sgn}(\hat{F}[k] \hat{\chi}[k]) \| \hat{\chi}[k] \| \\
u(t) &= (H \nu)(t), \quad t \in [kT, (k + 1)T], \quad k \geq 0.
\end{align*}
\]

Using (11) to bound \( \| \hat{F}[k] \| \), it follows that

\[
\| \nu[k] \| \leq \rho \| \hat{\chi}[k] \| + c_2 \| \overline{\sigma} \| \| \hat{\chi}[k] \|, \quad k \geq 0;
\]

this means, in particular, that \( \nu[k] = 0 \) if and only if \( \hat{\chi}[k] = 0 \). Furthermore, using Lemma 1 to bound \( \| \hat{\chi}[k] \| \), we can further refine (20), at least for \( k \geq 1 \):

\[
\| \nu[k] \| \leq \rho \| \chi[k] \| - c_2 \| \overline{\sigma} \| \| \chi[k] \| + c_2 \| \overline{\sigma} \| \| \chi[k] \|.
\]

**F. Summary of Proposed Controller K**

A description of the controller \( K \) is given by the following algorithm, with each step corresponding to the same numbered block in Figure 2. Our controller runs for \( k \in \mathbb{Z}^+ \) and has arbitrary initial conditions \( \hat{\chi}[-1] \in \mathbb{R}^n, \nu[-1] \in \mathbb{R}, \) and \( \hat{B}_d[-1] \in \mathcal{B}_d, \)

The condition on \( \rho \) may not belong to \( \ell_\infty \); however this signal is intermediary in nature and is used in the description of \( K \) to enhance clarity.
with \( \tau \in [0, 1] \) and
\[
B = \left\{ B = \begin{bmatrix} z \\ 1 \end{bmatrix} : z \in [0.9, -0.9] \cup [-1.1, -10] \right\}.
\]
This uncertainty set can be represented as follows:
\[
G = \left\{ \frac{e^{-\tau s}(s + z)}{(s + 1)(s - 1)} : \tau \in [0, 1], \ z \in [0.9, -0.9] \cup [-1.1, -10] \right\}.
\]
Clearly, this is a difficult to control set of plants with the potential of having a non-minimum phase zero to the left or right of an unstable pole. To design our controller, we set \( \rho = 10^{-7} \), we place the eigenvalues at 0.05 \( \pm 0.05 \)j, and we set \( h = 0.2 \). \( T_1 = 0.2 \), \( T_2 = 0.195 \), \( T_3 = 1.005 \) for an overall period \( T = 1.4 \). Using these values, we compute \( b_d = \min_{B_d(\tau, B) \in B_d} |\det C(A_d, B_d(\tau, B))| = 0.0138 \) and \( b_d = \max_{B_d(\tau, B) \in B_d} \|B_d(\tau, B)\| = 10.7759 \). Finally, to perform the simulations, we have an initial condition \( x(0) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \), \( u(\theta) = 0 \) for \( \theta \in [-1, 0] \), and we inject random noise starting from time zero with a maximum magnitude of \( 10^{-3} \). For the first simulation, the actual plant is given by
\[
G_1(s) = e^{-0.5s}(s - 0.9) \begin{bmatrix} s + 1 \\ s - 1 \end{bmatrix},
\]
which from [8] has an LTI delay margin of at most \( \frac{2}{\pi} \) seconds, with the results shown in Figure 3. For the second simulation, our actual plant is given by
\[
G_2(s) = e^{-s}(s - 1.1) \begin{bmatrix} s + 1 \\ s - 1 \end{bmatrix},
\]
which from [8] has an LTI delay margin of at most \( \frac{2}{\pi} \) seconds, and with the results shown in Figure 4.

As can be seen in both cases, despite trying to stabilize a plant at 4.5 and 5.5 times the LTI delay margins, our controller easily maintains stability and tolerates the injected noise despite not knowing the location of the zero.

VII. CONCLUSIONS

In this paper, we propose a control scheme that can BIBO stabilize any SISO LTI continuous time plant with an arbitrarily large unknown delay and an uncertain \( B \) state space matrix. The proposed controller uses a modified zero-order-hold to produce a discrete time model with a known discretized \( A \) matrix, which allows for a simple update law and estimate of the unknown delay and the discretized \( B \) matrix. This controller, while mildly non-linear and periodic, is relatively simple, and in addition to providing BIBO stability, it also guarantees the exponential decay of the initial conditions.

For future work, we would like to modify the controller to make it robust to some uncertainty in the \( A \) matrix, as well as prove that it can maintain stability in the face of time-varying parameters.

REFERENCES