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Sandor Type Fuzzy Inequality Based on the (s,m)-Convex Function in the Second Sense

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Abstract: Integral inequalities play critical roles in measure theory and probability theory. Given recent profound discoveries in the field of fuzzy set theory, fuzzy inequality has become a hot research topic in recent years. For classical Sandor type inequality, this paper intends to extend the Sugeno integral. Based on the (s,m)-convex function in the second sense, a new Sandor type inequality is proposed for the Sugeno integral. Examples are given to verify the conclusion of this paper.

Keywords: Sandor type inequality; (s,m)-convex function in the second sense; Sugeno integral

1. Introduction

Fuzzy sets are an important measurement tool for processing uncertain information. Fuzzy measurements and fuzzy integrals, which were originally introduced by Sugeno in 1974 [1], are important analytical methods of measuring uncertain information [2–4]. The Sugeno integral has been applied to many fields, such as management decision-making and control engineering [5–9]. Because of the special integral operator of the Sugeno integral, it is limited in many practical problems. To overcome this shortcoming, many scholars have replaced the Sugeno integral in large or small operations with new operators, and they have proposed various types of fuzzy integrals, such as the Shilkret integral [10], the bipolar level Choquet integral [11], the level-dependent Sugeno integral [12], etc. In recent years, some famous integral inequalities have been generalized to fuzzy integrals (cf. [13–16]). The study of inequalities for the Sugeno integral, which was initiated by Roman-Flores et al. [17], is the most popular.

The convexity method, i.e., establishing inequalities for convex functions, is one of the most powerful tools in establishing analytic inequalities. Particularly, there are many important applications in the study of higher transcendental functions [18]. Convex functions have become the theoretical basis and a powerful tool in game theory, mathematical programming theory, economics, etc. Some famous convex functions have been put forward, and in recent years they have been used to establish integral inequalities. For example, Gill et al. [19] extended Hadamard's Inequality on the basis of r-convex functions. Sarikaya and Kiris [20] established interesting and important inequalities based on Hermite-Hadamard type inequality for the s-convex in the second sense. Muddassar and Bhatti [21] established generalizations of Hadamard type inequalities through differentiability for the s-convex function. Micherda and Rajba [22] introduced a new class of (k,h)-convex functions defined on k-convex domains, and they proved new inequalities of Hermite-Hadamard and Fejér types for such mappings.

Sandor type inequality is an important integral inequality for convex functions. Most extended Sensor type inequalities are established on the basis of definite integrals, but the research based on fuzzy integral is still scarce [23–27]. Sándor [23] first introduced Sandor type inequality for definite integrals with respect to convex functions. Caballero and Sadarangani [25] extended Sandor type inequality for the Sugeno integral with convex functions. Based on other kinds of convex functions, for the Sugeno integral, other Sandor type inequalities have also been established. For example, Li et al. [24] derived Sandor type inequalities for the Sugeno integral with respect to general (α,m,r)-convex functions, Yang et al. [26] derived Sandor type inequality for the Sugeno integral with respect to (α,m) convex functions, and Lu et al. [27] obtained Sandor type inequality for the Sugeno integral with respect to r-convex functions. We find that various kind convex functions have different Sensor type inequalities. Studies of Sendor type inequalities can give good estimates of the Sugeno integral under various convex functions. (s,m)-Convex functions in the second sense are important convex functions, and an ordinary convex function is a special case [28]. Many authors have been interested in achieving inequality for this function under definite integrals (see [29–32]). In this paper, we will extend the Sandor type inequality for (s,m)-convex functions in the second sense for the Sugeno integral. Some examples are given to illustrate the results. The organization of this article is as follows: Section 2 will recall the definitions and properties of the Sugeno integral and of the convex function. Section 3 will establish some new Sandor type inequalities for (s,m)-convex functions in the second sense based on the Sugeno integral. Finally, conclusions are drawn in Section 4.

2. Preliminaries

In this section, some definitions and properties of the Sugeno integral and (s,m)-convex function in the second sense are presented. In this article, we always denote by R the set of real numbers. Let X be anon-empty set and let $R_+ = [0, \infty)$, Σ be a $\sigma-$ algebra of subsets of X.

Definition 1. *In reference* [33], *a mapping* $\mu : \Sigma \to R_+$ *is a non-negative set function, and it is called a non-additive measure if it satisfies the following properties:*

(1) $\mu(\Phi) = 0$.
(2) $A, B \in \Sigma$ *and* $A \subset B$ *imply* $\mu(A) \le \mu(B)$.
(3) *For any* $n \ge 1$, $A_n \in \Sigma$ *and* $A_1 \subset A_2 \subset \cdots$, *imply* $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.
(4) *For any* $n \ge 1$, $A_n \in \Sigma$, $A_1 \supset A_2 \supset \cdots$ *and* $\mu(A_1) < +\infty$, *imply* $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Remark 1. *The triple* (X, Σ, μ) *is called a fuzzy measure space. If* $f : X \to R_+$ *is a non-negative real-valued function, for any* $\alpha \ge 0$, *denote* $F_\alpha = \{x | x \in X, f(x) \ge \alpha\} = \{f \ge \alpha\}$, *then it is easily shown that if* $\alpha \le \beta$, *then* $F_\beta \subset F_\alpha$.

Definition 2. *In references* [1,33], *let* (X, Σ, μ) *be a fuzzy measure space, let* $f : X \to R_+$ *be a non-negative measurable function, and* $A \in \Sigma$; *thus, the Sugeno integral (or the fuzzy integral) of* f *on set* A *is defined as*

$$(S)\int_A f d\mu = \bigvee_{\alpha \ge 0} [\alpha \wedge \mu(A \cap F_\alpha)] \tag{1}$$

When $A = X$,

$$(S)\int_X f d\mu = \bigvee_{\alpha \ge 0} [\alpha \wedge \mu(F_\alpha)] \tag{2}$$

Here $\vee$ *and* $\wedge$ *denote the operations sup and inf on* $R_+ = [0, \infty)$, *respectively.*

Proposition 1. *In reference* [33], *let* (X, Σ, μ) *be a fuzzy measure space,* $A, B \in \Sigma$, *and let* f, g *be non-negative measurable functions defined on* X. *Thus,*

(1) $(S)\int_A f d\mu \le \mu(A)$;
(2) *For any constant* $k \in R_+$, *then* $(S)\int_A k d\mu \le k \wedge \mu(A)$;

(3) *If* $f \leq g$ *on the set* A, *then* $(S)\int_A f d\mu \leq (S)\int_A g d\mu$;

(4) *If* $A, B \in \Sigma$ *and* $A \subset B$, *then* $(S)\int_A f d\mu \leq (S)\int_B f d\mu$;

(5) *If* $\mu(A \cap \{f \geq \alpha\}) \geq \alpha$, *then* $(S)\int_A f d\mu \geq \alpha$;

(6) *If* $\mu(A \cap \{f \geq \alpha\}) \leq \alpha$, *then* $(S)\int_A f d\mu \leq \alpha$;

(7) $(S)\int_A f d\mu < \alpha \Leftrightarrow$ *There exists* $\gamma < \alpha, s.t. \mu(A \cap \{f \geq \gamma\}) < \alpha$;

(8) $(S)\int_A f d\mu > \alpha \Leftrightarrow$ *There exists* $\gamma > \alpha, s.t. \mu(A \cap \{f \geq \gamma\}) > \alpha$;

(9) $\mu(A) < +\infty$, *then* $(S)\int_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$.

Remark 2. *Let* $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$, *then* $F(\alpha)$ *is the distribution associated to* f *on* A. *According to (5) and (6) of Proposition 1, we have*

$$F(\alpha) = \alpha \Rightarrow (S)\int_A f d\mu = \alpha \tag{3}$$

Thus, from a numerical point of view, the Sugeno integral canbe calculated as the solution of the equation $F(\alpha) = \alpha$.

Definition 3. *In reference* [28], *let* $(s, m) \in (0, 1]^2$ *be a pair of real numbers. A function* $f : I \subseteq R_+ \to R$ *is said to be (s,m)-convex in the second sense if*

$$f(\lambda x + m(1-\lambda)y) \leq \lambda^s f(x) + m(1-\lambda)^s f(y) \tag{4}$$

holds for all $(x, y) \in I$ *and* $\lambda \in [0, 1]$.

Remark 3. *Some interesting and important inequalities for (s,m)-convex functions in the second sense functions can be found in [28–32]. Note that, if (s,m) = (1,1), then one obtains the definition of an ordinary convex function. If (s,m) = (s,1), then one obtains the definition of an s-convex function in the second sense. Denote by* $K^2_{s,m}$ *the set of all (s,m)-convex functions in the second sense.*

Lemma 1. *Let* $x \geq 0, y \geq 0$, *then the inequality*

$$(x+y)^\theta \leq x^\theta + y^\theta \tag{5}$$

holds for $\theta \in (0, 1]$.

Proof. Obviously, Inequality (5) is true when $\theta = 1$.

When $\theta \in (0, 1)$, we let

$$f(t) = 1 + t^\theta - (1+t)^\theta, \text{for } t \geq 0 \tag{6}$$

The derivative of $f(t)$ is

$$f'(t) = \theta t^{\theta-1} - \theta(1+t)^{\theta-1}, t \geq 0 \tag{7}$$

Since $\theta \in (0, 1), \theta - 1 < 0$; therefore, $f'(t) \geq 0, t \geq 0$. Thus, $f(t)$ is an increasing function, and we have $f(t) \geq f(0) = 0$, for all $t \geq 0$. That is,

$$(1+t)^\theta \leq 1 + t^\theta, \text{for } t \geq 0 \tag{8}$$

Obviously, if $y = 0$, then Inequality (5) is true. If $y > 0$, let $t = \frac{x}{y}$ in (8); thus, we have

$$(1+\frac{x}{y})^\theta \leq 1 + (\frac{x}{y})^\theta, \text{for } x \geq 0.$$

Thus, $(x+y)^\theta \le x^\theta + y^\theta$. This completes the proof.

Remark 4. *According to Lemma 1, we can easily prove that $f(x) = ax^\theta$ is a (s,1)-convex function in the second sense ($K^2_{s,1}$) for $a > 0, \theta \in (0,1]$ and $s \in [0,\theta]$.*

Lemma 2. *Let $x \in [0,1]$, then the inequality*

$$(1-x)^s \le 2^{1-s} - x^s \tag{9}$$

holds for $s \in (0,1]$.

Proof. Let $f(x) = x^s + (1-x)^s$ for $x, s \in [0,1]$. Obviously, $f(x)$ is increasing on the interval [0,1/2] and decreasing on the interval [1/2,1]. Thus, $f(x) \le f(1/2) = 2^{1-s}$, for all $x \in [0,1]$. This completes the proof.

3. Sandor Type Inequalities for the Sugeno Integral Based on the(s,m)-Convex in the Second Sense

Sandor Type inequality, established by Sandor in [23], provides estimates of the mean value of a nonnegative and convex function $f : [a,b] \to R$ with the following inequality:

$$\frac{1}{b-a}\int_a^b f^2(x)dx \le \frac{1}{3}[f^2(a) + f(a)f(b) + f^2(b)] \tag{10}$$

Unfortunately, the following example shows that the Sandor type inequality for the Sugeno integral based on (s,m)-convex functions in the second senseis not valid.

Example 1. *Consider $X = [0,1]$ and let μ be the Lebesgue measure on X. If we take the function $f(x) = x$, then, by Remark 3, we know that $f(x) \in K^2_{s,m}$. Calculate the Sugeno integral $(S)\int_0^1 f^2 d\mu$ as Remark 2, and we obtain the following:*

$$(S)\int_0^1 f^2 d\mu = \frac{3-\sqrt{5}}{2} \approx 0.382.$$

On the other hand,

$$\frac{1}{3}[f^2(0) + f(0)f(1) + f^2(1)] \approx 0.3333.$$

This proves that the Sandor type inequality is not satisfied for the Sugeno integral based on (s,m)-convex functions in the second sense.

In this section, we will establish new Sandor type inequalities for the Sugeno integral based on (s,m)-convex functions in the second sense.

Theorem 1. *Let $(s,m) \in (0,1]^2$ and $f : [a,b] \to [0,\infty)$ is a (s,m)-convex function in the second sense, such that $f(b) > mf(a)$, and let μ be the Lebesgue measure on R. Thus,*

$$(S)\int_a^b f^2 d\mu \le \min\{\beta, b-a\} \tag{11}$$

where β is the positive real solution of the equation

$$(b-ma)\left[1 - \left(\frac{\sqrt{\beta} - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{1/s}\right] = \beta \tag{12}$$

Proof. As $f(x) \in K_{s,m}^2$ for $x \in [a,b]$, we have

$$\begin{aligned} f(x) &= f(m \cdot (1 - \tfrac{x-ma}{b-ma})a + (\tfrac{x-ma}{b-ma})b) \\ &\le m(1 - \tfrac{x-ma}{b-ma})^s f(a) + (\tfrac{x-ma}{b-ma})^s f(b) \end{aligned} \tag{13}$$

According to Lemma 2, we have

$$(1 - \frac{x-a}{b-a})^s \le 2^{1-s} - (\frac{x-a}{b-a})^s \tag{14}$$

Thus,

$$f(x) \le m2^{1-s} f(a) + (\frac{x-ma}{b-ma})^s (f(b) - mf(a)) = g(x) \tag{15}$$

By (15) and the non-negative assumption of function $f(x)$, we have $f^2(x) \le g^2(x)$ for all $x \in [a,b]$. Thus, by (3) of Proposition 1 and Definition 2, we have

$$(S)\int_a^b f^2(x)d\mu \le (S)\int_a^b g^2(x)d\mu = \bigvee_{\beta \ge 0} [\beta \wedge \mu([a,b] \cap \{g^2 \ge \beta\})] \tag{16}$$

In order to calculate the integral $(S)\int_a^b g^2(x)d\mu$, we consider the distribution function F associated to $g^2(x)$ on $[a,b]$ which is given by

$$F(\beta) = \mu([a,b] \cap \left\{g^2 \ge \beta\right\}).$$

That is,

$$F(\beta) = \mu([a,b] \cap \{x | m2^{1-s} f(a) + (\frac{x-ma}{b-ma})^s (f(b) - mf(a)) \ge \sqrt{\beta}\}).$$

When $f(b) > mf(a)$, then

$$\begin{aligned} F(\beta) &= \mu([a,b] \cap \{x | x \ge ma + (b-ma)\left(\tfrac{\sqrt{\beta} - m2^{1-s} f(a)}{f(b) - mf(a)}\right)^{1/s}\}) \\ &= (b-ma)\left[1 - \left(\tfrac{\sqrt{\beta} - m2^{1-s} f(a)}{f(b) - mf(a)}\right)^{1/s}\right] \end{aligned}.$$

Let $F(\beta) = \beta$. Thus,

$$(b-ma)\left[1 - \left(\frac{\sqrt{\beta} - m2^{1-s} f(a)}{f(b) - mf(a)}\right)^{1/s}\right] = \beta \tag{17}$$

A straightforward calculus gives us that the positive real solution of the above equation is equal to $(S)\int_a^b g^2(x)d\mu$.

By (1) of Proposition 1 and (16), we obtain

$$(S)\int_a^b f^2(x)d\mu \le (S)\int_a^b g^2(x)d\mu = \min\{\beta, b-a\}.$$

This completes the proof.

Remark 5. *Let $s = 1$, $m = 1$, $(s, m) \in (0, 1]^2$ and $f : [a, b] \to [0, \infty)$ is a convex function, such that $f(b) > f(a)$. Let μ be the Lebesgue measure on R. Thus, one obtains the following:*

$$(S)\int_a^b f^2 d\mu \leq \min\{\beta, b-a\} \tag{18}$$

where β is the positive real solution of the equation

$$(b-a)\left(1 - \frac{\sqrt{\beta} - f(a)}{f(b) - f(a)}\right) = \beta \tag{19}$$

Equation (19) can be solved as

$$\beta = \frac{(2Af(b) + A^2) - \sqrt{4A^3 f(b) + A^4}}{2} \tag{20}$$

where $A = (b-a)/(f(b) - f(a))$. This result is one of the main results of Caballero and Sadarangani [25].

Theorem 2. *Let $(s, m) \in (0, 1]^2$ and $f : [a, b] \to [0, \infty)$ is an(s,m)-convex function in the second sense, such that $f(b) < mf(a)$. Let μ be the Lebesgue measure on R. Thus,*

$$(S)\int_a^b f^2 d\mu \leq \min\{\beta, b-a\} \tag{21}$$

where β is the positive real solution of the equation

$$(b-ma)\left(\frac{\sqrt{\beta} - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{1/s} = \beta \tag{22}$$

Proof. Similar to the proof of Theorem 1, we consider the function,

$$g(x) = m2^{1-s}f(a) + (\frac{x-ma}{b-ma})^s (f(b) - mf(a)) \tag{23}$$

In this case, the distribution function F is associated to $g^2(x)$ on $[a,b]$, which is given by

$$F(\beta) = \mu([a, b] \cap \left\{g^2 \geq \beta\right\}).$$

That is,

$$\begin{aligned} F(\beta) &= \mu([a, b] \cap \{g^2 \geq \beta\}) \\ &= \mu([a, b] \cap \{x | m2^{1-s}f(a) + (\tfrac{x-ma}{b-ma})^s (f(b) - mf(a)) \geq \sqrt{\beta}\}) \end{aligned} \tag{24}$$

When $f(b) < mf(a)$,

$$\begin{aligned} F(\beta) &= \mu([a, b] \cap \{x | x \leq ma + (b-ma)\left(\tfrac{\sqrt{\beta} - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{1/s}\}) \\ &= (b-ma)\left(\tfrac{\sqrt{\beta} - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{1/s} \end{aligned} \tag{25}$$

Let $F(\beta) = \beta$. Thus,

$$(b-ma)\left(\frac{\sqrt{\beta} - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{1/s} = \beta \tag{26}$$

By (1) of Proposition 1 and (16), we obtain

$$(S)\int_a^b f^2(x)d\mu \le (S)\int_a^b g^2(x)d\mu = \min\{\beta, b-a\} \tag{27}$$

This completes the proof.

Remark 6. *Let $s = 1$, $m = 1$, and $(s,m) \in (0,1]^2$, and $f : [a,b] \to [0,\infty)$ is a convex function, such that $f(b) < f(a)$. Let μ be the Lebesgue measure on R. Thus, one obtains the following*

$$(S)\int_a^b f^2 d\mu \le \min\{\beta, b-a\} \tag{28}$$

where β is the positive real solution of the equation

$$(b-a)\left(\frac{\sqrt{\beta}-f(a)}{f(b)-f(a)}\right) = \beta \tag{29}$$

Equation (29) can be written as

$$b-a+\frac{b-a}{f(b)-f(a)}\left(\sqrt{\beta}-f(b)\right) = \beta \tag{30}$$

Thus, the positive solution of Equation (30) is

$$\beta = \frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2} \tag{31}$$

where $A = (b-a)/(f(b)-f(a))$. This result is another main result of Caballero and Sadarangani [25].

Remark 7. *For the case $f(b) = mf(a)$, we have $g(x) = m2^{1-s}f(a)$ according to Equation (15). Thus, by (2) of Proposition 1,*

$$(S)\int_a^b f^2(x)d\mu \le (S)\int_a^b g^2(x)d\mu = \min\left\{m2^{1-s}f(a), b-a\right\} \tag{32}$$

Moreover, for convex function $f(x)$, i.e., $s = m = 1$, we have $(S)\int_a^b f^2(x)d\mu \le \min\{f(a), b-a\}$, which is in agreement with Caballero and Sadarangani [25].

Example 2. *Let μ be the Lebesgue measure on $X = [0,1]$. If we consider the function $f(x) = \sqrt{x}$, then, by Remark 3, we know that $f(x) \in K^2_{s,1}$ for $s \in (0,1/2]$. Here, we let $s = 1/2$, so $f(1) > mf(0)$. By Theorem 1, through solving the equation $(1-0)\left[1-\left(\frac{\sqrt{\beta}-\sqrt{2}f(0)}{f(1)-f(0)}\right)^2\right] = \beta$, we obtain $\beta = 0.5$. Thus,*

$$(S)\int_0^1 f^2 d\mu \le \min\{\beta, 1-0\} = 0.5.$$

A straightforward calculation shows that $(S)\int_0^1 f^2 d\mu = 0.5$. This also implies that the inequality can get a good estimate of $(S)\int_0^1 f^2 d\mu$.

Example 3. *Let μ be the Lebesgue measure on $X = [1,2]$. If we consider the function $f(x) = \frac{1}{x}$, then we can see that $f(x)$ is a convex function. That is, $f(x) \in K^2_{s,m}$ where $s = m = 1$. Therefore, $f(2) < mf(1)$,* *and, by Theorem 2, through solving the equation* $(2-1)\left(\frac{\sqrt{\beta}-f(1)}{f(2)-f(1)}\right)=\beta$, *we obtain* $\beta = 4-2\sqrt{3} = 0.5359$. *Hence, we get the following estimate:*

$$(S)\int_1^2 f^2 d\mu \le \min\{\beta, 2-1\} = 0.5359.$$

Example 4. *Let* μ *be the Lebesgue measure on* $X=[-1,1]$. *If we consider the function* $f(x)=|x|$, *then we can see that* $f(x)$ *is a convex function. That is,* $f(x)\in K^2_{s,m}$ *where s = m = 1. Therefore,* $f(1)=mf(-1)$, *and, by Remark 5, we obtain the following estimate*:

$$(S)\int_{-1}^1 f^2 d\mu \le \min\{f(-1), 1-(-1)\} = 1.$$

Example 5. *Let* μ *be the Lebesgue measure on* $X=[1,4]$. *If we consider the function* $f(x)=x^{3/2}$, *then we can see that* $f(x)$ *is a convex function. That is* $f(x)\in K^2_{s,m}$ *where s = m = 1. Thus,* $f(4)>mf(1)$. *By Theorem 1, through solving the equation* $(4-m\cdot 1)\left[1-\left(\frac{\sqrt{\beta}-m2^{1-s}f(1)}{f(4)-f(1)}\right)^{1/s}\right]=\beta$, *we obtain* $\beta = 2.7216$.

Therefore,

$$(S)\int_1^4 f^2 d\mu \le \min\{\beta, 4-1\} = 2.7216.$$

In Example 11 of Li et al. [24], they regarded $f(x)=x^{3/2}$ as a general (1/3, 0, 3)-convex function, and they then obtained the result $(S)\int_1^4 f^2 d\mu \le 3$. A straightforward calculation shows that $(S)\int_1^4 f^2 d\mu = 2.6212$. This implies that the Sandor type inequalities constructed in this paper can more accurately estimate the Sugeno integral $(S)\int_a^b f^2 d\mu$ with respect to (s,m)-convex functions in the second sense.

4. Conclusions

In this paper, we establish an upper approximation for the Sugeno fuzzy integral of (s,m)-convex functions in the second sense based on classical Sendor type inequality, which is a useful tool for estimating Sugeno integrals. The results obtained in Caballero and Sadarangani [25] are special cases of this paper. In many applications, assumptions about the convexity of functions have interesting qualitative implications in many fields such as economics, biology, industrial engineering, etc. Fuzzy integrals were introduced by Sugeno in the early 1970s; they have since attracted the attention of many scholars and have been applied in many areas. Thus, the study of the Sugeno fuzzy integral for (s,m)-convex functions in the second sense is an important and interesting topic. In the future, we will study other properties and inequalities related to this convex function under other fuzzy integrals, such as the Choquet integral and the seminormed fuzzy integral.

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References

1. Sugeno, M. Theory of Fuzzy Integrals and Its Applications. Ph.D. Thesis, Tokyo Institute of Technology, Tokyo, Japan, 1975.
2. Rodger, J.A.; George, J.A. Triple bottom line accounting for optimizing natural gas sustainability: A statistical linear programming fuzzy ILOWA optimized sustainment model approach to reducing supply chain global cybersecurity vulnerability through information and communications technology. *J. Clean. Prod.* **2017**, *142*, 1931–1949.
3. Song, J.; Tsang, E.C.C.; Chen, D.; Yang, X. Minimal decision cost reduct in fuzzy decision-theoretic rough set model. *Knowl. -Based Syst.* **2017**, *126*, 104–112. [CrossRef]
4. Ezghari, S.; Zahi, A.; Zenkouar, K. A new nearest neighbor classification method based on fuzzy set theory and aggregation operators. *Expert Syst. Appl.* **2017**, *80*, 58–74. [CrossRef]
5. Chadli, M. An LMI approach to design observer for unknown inputs Takagi-Sugeno fuzzy models. *Asian J. Control* **2010**, *12*, 524–530. [CrossRef]
6. Grabisch, M.; Labreuche, C. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *Ann. Oper. Res.* **2010**, *175*, 247–286. [CrossRef]
7. Yager, R.R.; Alajlan, N. Sugenointegral with possibilisticinputs with application to multi-criteria decision making. *Int. J. Intell. Syst.* **2016**, *31*, 813–826. [CrossRef]
8. Zhang, X.; Zheng, Y. Linguistic quantifiers modeled by interval-valued intuitionistic Sugeno integrals. *J. Intell. Fuzzy Syst.* **2015**, *29*, 583–592. [CrossRef]
9. Li, W.; Feng, Z.; Sun, W.; Zhang, J. Admissibility analysis for Takagi–Sugeno fuzzy singular systems with time delay. *Neurocomputing* **2016**, *205*, 336–340. [CrossRef]
10. Shilkret, N. Maxitive measure and integration. *Indag. Math.* **1971**, *33*, 109–116. [CrossRef]
11. Greco, S.; Matarazzo, B.; Giove, S. The Choquet integral with respect to a level dependent capacity. *Fuzzy Sets Syst.* **2011**, *175*, 1–35. [CrossRef]
12. Mesiar, R.; Mesiarová-Zemánková, A.; Ahmad, K. Level-dependent Sugenointegral. *IEEE Trans. Fuzzy Syst.* **2009**, *17*, 167–172. [CrossRef]
13. Ouyang, Y.; Mesiar, R. On the Chebyshev type inequality for seminormed fuzzy integral. *Appl. Math. Lett.* **2009**, *22*, 1810–1815. [CrossRef]
14. Hu, Y. Chebyshev type inequalities for general fuzzy integrals. *Inf. Sci.* **2014**, *278*, 822–825. [CrossRef]
15. Abbaszadeh, S.; Eshaghi, M. A Hadamard-type inequality for fuzzy integrals based on r-convex functions. *Soft Comput.* **2016**, *20*, 3117–3124. [CrossRef]
16. Agahi, H.; Mesiar, R.; Ouyang, Y. On Stolarsky inequality for Sugeno and Choquet integrals. *Inf. Sci.* **2014**, *266*, 134–139. [CrossRef]
17. Román-Flores, H.; Flores-Franulic, A.; Chalco-Cano, Y. The fuzzy integral for monotone functions. *Appl. Math. Comput.* **2007**, *185*, 492–498. [CrossRef]
18. Chu, Y.; Wang, G.; Zhang, X.; Qiu, S. Generalized convexity and inequalities involving special functions. *J. Math. Anal. Appl.* **2007**, *336*, 768–776. [CrossRef]
19. Gill, P.M.; Pearce, C.E.M.; Pečarić, J. Hadamard'sinequality for r-convex functions. *J. Math. Anal. Appl.* **1997**, *215*, 461–470. [CrossRef]
20. Sarikaya, M.Z.; Kiris, M.E. Some new inequalities of Hermite-Hadamard type for s-convex functions. *Miskolc Math. Notes* **2015**, *16*, 491–501.
21. Muddassar, M.; Bhatti, M.I. Some Generalization of Hadamard's type Inequalities through Differentiability for s-Convex Function and their Applications. *Indian J. Pure Appl. Math.* **2013**, *44*, 131–151. [CrossRef]
22. Micherda, B.; Rajba, T. On some Hermite-Hadamard-Fejér inequalities for (k,h)-convex functions. *Math. Inequal. Appl.* **2012**, *15*, 931–940. [CrossRef]
23. Sándor, J. On the identric and logarithmic means. *Aequ. Math.* **1990**, *40*, 261–270. [CrossRef]
24. Li, D.Q.; Cheng, Y.H.; Wang, X.S. Sandortype inequalities for Sugenointegral with respect to general (α,m,r)-convex functions. *J. Funct. Spaces* **2015**, *2015*, 1–13.
25. Caballero, J.; Sadarangani, K. Sandor's inequality for Sugeno integrals. *Appl. Math. Comput.* **2011**, *218*, 1617–1622. [CrossRef]
26. Yang, X.L.; Song, X.Q.; Lu, W. Sandor's type inequality for fuzzy integrals. *J. Nanjing Univ.* **2015**, *32*, 144–156.

27. Lu, W.; Song, X.Q.; Huang, L.L. Inequalities of Hermite-Hadamard and Sandor for fuzzy integral. *J. Shandong Univ. (Nat. Sci.)* **2016**, *51*, 22–28.
28. Park, J. Some Hadamard's type inequalities for co-ordinated (s,m)-convex mappings in the second sense. *Far East J. Math. Sci.* **2011**, *51*, 205–216.
29. Eftekhari, N. Some remarks on (s,m)-convexity in the second sense. *J. Math. Inequal.* **2014**, *8*, 489–495. [CrossRef]
30. Vivas, M. Féjertype inequalities for (s,m)-convex functions in Second Sense. *Appl. Math. Inf. Sci.* **2016**, *10*, 1689–1696.
31. Du, T.; Li, Y.; Yang, Z. A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions. *Appl. Math. Comput.* **2017**, *293*, 358–369. [CrossRef]
32. Yang, Z.; Li, Y.; Du, T. A generalization of simpson type inequality via differentiable functions using (s,m)-convex functions. *Ital. J. Pure Appl. Math.* **2015**, *35*, 327–338.
33. Wang, Z.; Klir, G. *Fuzzy Measure Theory*; Plenum: New York, NY, USA, 1992.