ON SUPER \((a, d)\)-EDGE ANTIMAGIC TOTAL LABELING OF CERTAIN FAMILIES OF GRAPHS

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Abstract

A \((p, q)\)-graph \(G\) is \((a, d)\)-edge antimagic total if there exists a bijection \(f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}\) such that the edge weights \(\Lambda(uv) = f(u) + f(uv) + f(v)\), \(uv \in E(G)\) form an arithmetic progression with first term \(a\) and common difference \(d\). It is said to be a super \((a, d)\)-edge antimagic total if the vertex labels are \(\{1, 2, \ldots, p\}\) and the edge labels are \(\{p + 1, p + 2, \ldots, p + q\}\). In this paper, we study the super \((a, d)\)-edge antimagic total labeling of special classes of graphs derived from copies of generalized ladder, fan, generalized prism and web graph.

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1. Introduction

By a graph \(G\) we mean a finite, undirected, connected graph without any loops or multiple edges. Let \(V(G)\) and \(E(G)\) be the set of vertices and edges of a graph \(G\), respectively. The order and size of a graph \(G\) is denoted as \(p = |V(G)|\) and \(q = |E(G)|\) respectively. For general graph theoretic notions we refer Harary [6].
By a labeling we mean a one-to-one mapping that carries the set of graph elements onto a set of numbers (usually positive or non-negative integers), called labels. There are several types of labelings and a detailed survey of many of them can be found in the dynamic survey of graph labeling by Gallian [5].

Kotzig and Rosa [9] introduced the concept of magic labeling. They define an edge magic total labeling of a \((p,q)\)-graph \(G\) as a bijection \(f\) from \(V(G) \cup E(G)\) to the set \(\{1,2,\ldots,p+q\}\) such that for each edge \(uv \in E(G)\), the edge weight \(f(u)+f(uv)+f(v)\) is a constant.

Enomoto et al. [3] defined a super edge magic labeling as an edge magic total labeling such that the vertex labels are \(\{1,2,\ldots,p\}\) and edge labels are \(\{p+1,p+2,\ldots,p+q\}\). They have proved that if a graph with \(p\) vertices and \(q\) edges is super edge magic then, \(q \leq 2p - 3\). They also conjectured that every tree is super edge magic.

As a natural extension of the notion of edge magic total labeling, Hartsfield and Ringel [7] introduced the concept of an antimagic labeling and they defined an antimagic labeling of a \((p,q)\)-graph \(G\) as a bijection \(f\) from \(E(G)\) to the set \(\{1,2,\ldots,q\}\) such that the sums of label of the edges incident with each vertex \(v \in V(G)\) are distinct.

Simanjuntak et al. [10] defined an \((a,d)\)-edge antimagic total labeling as a one to one mapping \(f\) from \(V(G) \cup E(G)\) to \(\{1,2,\ldots,p+q\}\) such that the set of edge weight \(\{f(u)+f(uv)+f(v) : uv \in E(G)\}\) is equal to \(\{a,a+d,a+2d,\ldots,a+(q-1)d\}\) for any two integers \(a > 0\) and \(d \geq 0\).

An \((a,d)\)-edge antimagic total labeling of a \((p,q)\)-graph \(G\) is said to be super \((a,d)\)-edge antimagic total if the vertex labels are \(\{1,2,\ldots,p\}\) and the edge labels are \(\{p+1,p+2,\ldots,p+q\}\). The super \((a,0)\)-edge antimagic total labeling is usually called as super edge magic in the literature (see [3, 4]).

An \((a,d)\)-edge antimagic vertex labeling of a \((p,q)\)-graph \(G\) is defined as a one to one mapping \(f\) from \(V(G)\) to the set \(\{1,2,\ldots,p\}\) such that the set of edge weight \(\{f(u)+f(v) : uv \in E(G)\}\) is equal to \(\{a,a+d,a+2d,\ldots,a+(q-1)d\}\) for any two integers \(a > 0\) and \(d \geq 0\).

In [2] Bača et al. proved that if a \((p,q)\)-graph \(G\) has an \((a,d)\)-edge antimagic vertex labeling then \(d(q-1) \leq 2p - 1 - a \leq 2p - 4\).

Also in [1] Bača and Barrientos proved the following: if a graph with \(q\) edges and \(q+1\) vertices has an \(\alpha\)-labeling, then it has an \((a,1)\)-edge antimagic vertex labeling. A tree has \((3,2)\)-edge antimagic vertex labeling if and only if it has an \(\alpha\)-labeling and the number of vertices in its two partite set differ by at most 1. If a tree with at least two vertices has a super \((a,d)\)-edge antimagic total labeling, then \(d\) is at most 3. If a graph has an \((a,1)\)-edge antimagic vertex labeling, then it also has a super \((a_1,0)\)-edge antimagic total labeling and a super \((a_2,2)\)-edge antimagic total labeling.

In [12] Sugeng et al. studied the super \((a,d)\)-edge antimagic total properties
of ladders, generalized prisms and antiprisms.

We make use of the following lemmas for our further discussion.

**Lemma 1.** If a \((p, q)\)-graph \(G\) is super \((a, d)\)-edge antimagic total, then \(d \leq \frac{2p+q-5}{q-1}\).

**Lemma 2.** If a \((p, q)\)-graph \(G\) has an \((a, 1)\)-edge antimagic vertex labeling and odd number of edges, then it has a super \((a', 1)\)-edge antimagic total labeling, where \(a' = a + p + \frac{q-1}{2}\).

**Lemma 3.** If a \((p, q)\)-graph \(G\) has an \((a, d)\)-edge antimagic vertex labeling, then \(G\) has a super \((a', d')\)-edge antimagic total labeling, where \(a' = a + p + 1\) and \(d' = d + 1\) or \(a' = a + p + q\) and \(d' = d - 1\).


In this paper, we study the super \((a, d)\)-edge antimagic total labeling of special classes of graphs derived from copies of generalized ladder, fan, generalized prism and web graph.

2. A Graph Derived from Copies of Generalized Ladder

Let \((u_{i,1}, u_{i,2}, \ldots, u_{i,n}, v_{i,1}, v_{i,2}, \ldots, v_{i,n}), 1 \leq i \leq t\), be a collection of \(t\) disjoint copies of the generalized ladder \(L_n\), \(n \geq 2\), such that \(u_{i,j}\) is adjacent to \(u_{i,j+1}\), \(v_{i,j+1}\) and \(v_{i,j}\) is adjacent to \(v_{i,j+1}\) for \(1 \leq j \leq n - 1\) and \(u_{i,j}\) is adjacent to \(v_{i,j}\) for \(1 \leq j \leq n\). We denote the graph obtained by joining \(u_{i,n}\) to \(u_{i+1,1}, u_{i+1,2}, v_{i+1,1}\), \(1 \leq i \leq t - 1\), as \(L_n^{(t)}\). Clearly, the vertex set \(V\) and the edge set \(E\) of the graph \(L_n^{(t)}\) are given by

\[V(L_n^{(t)}) = \{u_{i,j}, v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq n\}\] and \(E(L_n^{(t)}) = E_1 \cup E_2 \cup E_3\) where

\[E_1 = \{u_{i,j}u_{i,j+1}, v_{i,j}v_{i,j+1}, u_{i,j}v_{i,j+1} : 1 \leq i \leq t, 1 \leq j \leq n - 1\},\]

\[E_2 = \{u_{i,j}v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq n\},\]

\[E_3 = \{u_{i,n}u_{i+1,1}, u_{i,n}u_{i+1,2}, u_{i,n}v_{i+1,1} : 1 \leq i \leq t - 1\}.\]

It is easy to see that for \(L_n^{(t)}\), we have \(p = 2nt\) and \(q = 4nt - 3\).

**Lemma 4.** The graph \(L_n^{(t)}, n, t \geq 2\) has an \((a, 1)\)-edge antimagic vertex labeling.

**Proof.** Let us define a bijection \(f_1 : V(L_n^{(t)}) \to \{1, 2, \ldots, 2nt\}\) as follows:

\[f_1(u_{i,j}) = 2(i-1)n + 2j - 1\] if \(1 \leq i \leq t\) and \(1 \leq j \leq n\),

\[f_1(v_{i,j}) = 2(i-1)n + 2j\] if \(1 \leq i \leq t\) and \(1 \leq j \leq n\).

By direct computation, we observe that the edge weights of all the edges of \(L_n^{(t)}\), constitute an arithmetic sequence \(\{3, 4, \ldots, 4nt - 1\}\). Thus \(f_1\) is an \((3, 1)\)-edge antimagic vertex labeling of \(L_n^{(t)}\). \(\blacksquare\)
Theorem 5. The graph $L^{(t)}_n$, $n, t \geq 2$, has a super $(a, d)$-edge antimagic total labeling if and only if $d \in \{0, 1, 2\}$.

Proof. If the graph $L^{(t)}_n$, $n, t \geq 2$, is super $(a, d)$-edge antimagic total, then by Lemma 1, we get $d \leq 2$.

Conversely, by Lemma 4 and Lemma 3, we see that the graph $L^{(t)}_n$, $n, t \geq 2$, has a super $(6nt, 0)$-edge antimagic total labeling and a super $(2nt + 4, 2)$-edge antimagic total labeling.

Also by Lemma 2, we conclude that the graph $L^{(t)}_n$, $n, t \geq 2$, has a super $(4nt + 2, 1)$-edge antimagic total labeling, since $q = 4nt - 3$, which is odd for all $n$ and $t$.

![Figure 1. $(a, 1)$-edge antimagic vertex labeling of $L^{(3)}_4$.](image)

3. A Graph Derived from Copies of Fan Graph

Let $(u_i, v_i, v_{i,1}, v_{i,2}, \ldots, v_{i,m})$, $1 \leq i \leq t$, be a collection of $t$ disjoint copies of the fan graph $F_{m,2}$, $m \geq 2$, such that $u_i$ is adjacent to $w_i$ and $v_{i,j}$ is adjacent to both $u_i$ and $w_i$ for $1 \leq j \leq m$. We denote the graph $[8]$ obtained by joining $v_{i,m}$ to $u_{i+1}, v_{i+1,1}, v_{i+1,2}$, $1 \leq i \leq t - 1$, as $F^{(t)}_{m,2}$. Clearly, the vertex set $V$ and the edge set $E$ of the graph $F^{(t)}_{m,2}$ are given by

\begin{align*}
V(F^{(t)}_{m,2}) &= \{u_i, w_i, v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq m\} \\
E(F^{(t)}_{m,2}) &= \{u_iw_i, uw_{i,j}, w_iv_{i,j} : 1 \leq i \leq t, 1 \leq j \leq m\} \\
&\cup \{v_{i,m}v_{i+1,1}, v_{i,m}v_{i+1,2} : 1 \leq i \leq t - 1\}.
\end{align*}

It is easy to see that for $F^{(t)}_{m,2}$, we have $p = (m + 2)t$ and $q = (m + 2)2t - 3$.

Lemma 6. The graph $F^{(t)}_{m,2}$, $m, t \geq 2$, has an $(a, 1)$-edge antimagic vertex labeling.

Proof. Let us define a bijection $f_2 : V(F^{(t)}_{m,2}) \to \{1, 2, \ldots, (m + 2)t\}$ as follows:

\begin{align*}
f_2(u_i) &= (i - 1)(m + 2) + 1 \quad \text{if } 1 \leq i \leq t, \\
f_2(w_i) &= (m + 2)i \quad \text{if } 1 \leq i \leq t, \\
f_2(v_{i,j}) &= f_2(u_i) + j \quad \text{if } 1 \leq i \leq t \text{ and } 1 \leq j \leq m.
\end{align*}
By direct computation, we observe that the edge weights of all the edges of $F^{(t)}_{m,2}$ constitute an arithmetic sequence $\{3, 4, \ldots, 2t(m+2) - 1\}$. Thus $f_2$ is an $(3,1)$-edge antimagic vertex labeling of $F^{(t)}_{m,2}$.

Theorem 7. The graph $F^{(t)}_{m,2}$, $m, t \geq 2$, has a super $(a, d)$-edge antimagic total labeling if and only if $d \in \{0, 1, 2\}$.

Proof. If the graph $F^{(t)}_{m,2}$, $m, t \geq 2$, is super $(a, d)$-edge antimagic total, then by Lemma 1, we get $d \leq 2$.

Conversely, by Lemmas 3 and 6, we see that the graph $F^{(t)}_{m,2}$, $m, t \geq 2$, has a super $((m+2)3t, 0)$-edge antimagic total labeling and a super $((m+2)t + 4, 2)$-edge antimagic total labeling.

Also by Lemma 2, we conclude that the graph $F^{(t)}_{m,2}$, $m, t \geq 2$, has a super $((m+2)^2t + 2, 1)$-edge antimagic total labeling, since $q = (m+2)2t - 3$, which is odd for all $m$ and $t$.

4. A Graph Derived from Copies of Generalized Prism

Let $(v^{(k)}_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n)$, $1 \leq k \leq t$, be a collection of $t$ disjoint copies of the generalized prism $C_m \times P_n$, $m \geq 3$, $n \geq 2$, such that $v^{(k)}_{i,j}$ is adjacent to $v^{(k)}_{i+1,j}$ for $1 \leq i \leq m-1$, $1 \leq j \leq n$, $v^{(k)}_{m,j}$ is adjacent to $v^{(k)}_{1,j}$ for $1 \leq j \leq n$ and $v^{(k)}_{i,j}$ is adjacent to $v^{(k)}_{i,j+1}$ for $1 \leq i \leq m$, $1 \leq j \leq n-1$. We denote the graph obtained by joining $v^{(k)}_{m,n}$ to $v^{(k+1)}_{1,1}$ if $n$ is odd or $v^{(k)}_{1,n}$ to $v^{(k+1)}_{1,1}$ if $n$ is even for $1 \leq i \leq m$, $1 \leq k \leq t-1$ as $(C_m \times P_n)^{(t)}$. Clearly, the vertex set $V$ and the edge set $E$ of the graph $(C_m \times P_n)^{(t)}$ are given by $V((C_m \times P_n)^{(t)}) = \{v^{(k)}_{i,j} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E((C_m \times P_n)^{(t)}) = \{v^{(k)}_{i,j}v^{(k+1)}_{i,j} : 1 \leq i \leq m, 1 \leq k \leq t-2\}$. The graph derived from copies of generalized prism is shown in Figure 2.
\[ m, 1 \leq j \leq n \text{ and } E((C_m \times P_n)_{(t)}) = E_1 \cup E_2 \cup E_3 \text{ where} \]
\[
E_1 = \{ e^{(k)}_{i,j} v^1_{i,j,1} : 1 \leq k \leq t, 1 \leq i \leq m-1, 1 \leq j \leq n \} \\
\quad \cup \{ e^{(k)}_{m,i,j} v^1_{i,j,1} : 1 \leq k \leq t, 1 \leq j \leq n \}, \\
E_2 = \{ e^{(k)}_{i,j} v^2_{i,j,1} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n-1 \}, \\
E_3 = \{ e^{(k)}_{i,j} v^3_{i,j,1} : 1 \leq k \leq t-1, 1 \leq i \leq m \} \\
\quad \cup \{ e^{(k)}_{i,j} v^{(k+1)}_{i,j,1} : 1 \leq k \leq t-1, 1 \leq i \leq m \}.
\]

It is easy to see that for \((C_m \times P_n)_{(t)}\), we have \(p = mnt\) and \(q = m(2nt - 1)\).

**Lemma 8.** For odd \(m, m \geq 3\) and \(n, t \geq 2\), the graph \((C_m \times P_n)_{(t)}\) has an \((a,1)\)-edge antimagic vertex labeling.

**Proof.** Let us define a bijection \(f_3 : V((C_m \times P_n)_{(t)}) \to \{1, 2, \ldots, mnt\}\) as follows.

If \(j \) is odd and \(1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq t\), then
\[
f_3(v^k_{i,j}) = \begin{cases}
(k-1)mn + (j-1)m + \frac{i+1}{2} & \text{if } i \text{ is odd}, \\
(k-1)mn + (j-1)m + \frac{m+i+1}{2} & \text{if } i \text{ is even}.
\end{cases}
\]

If \(j \) is even and \(1 \leq i \leq m, 2 \leq j \leq n, 1 \leq k \leq t\), then
\[
f_3(v^k_{i,j}) = \begin{cases}
(k-1)mn + (j-1)m + \frac{m+i}{2} & \text{if } i \text{ is odd}, \\
(k-1)mn + (j-1)m + \frac{i}{2} & \text{if } i \text{ is even}.
\end{cases}
\]

By direct computation, we observe that the edge weights of all the edges of \((C_m \times P_n)_{(t)}\) constitute an arithmetic sequence \(\{\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+4mnt-3}{2}\}\).

Clearly \(\frac{m+3}{2}\) is an integer only when \(m\) is odd. Thus \(f_3\) is an \((\frac{m+3}{2}, 1)\)-edge antimagic vertex labeling of \((C_m \times P_n)_{(t)}\), for odd \(m\).

**Theorem 9.** For odd \(m, m \geq 3\) and \(n, t \geq 2\), the graph \((C_m \times P_n)_{(t)}\) has a super \((a,d)\)-edge antimagic total labeling if and only if \(d \in \{0, 1, 2\}\).

**Proof.** If the graph \((C_m \times P_n)_{(t)}\), \(m \geq 3\) and \(n, t \geq 2\), is super \((a,d)\)-edge antimagic total, then by Lemma 1 we get
\[
d \leq \frac{2p+q-5}{q-1} = \frac{2mnt+m(2nt-1)-5}{mnt-m-1} = 2 + \frac{m-3}{2mnt-m-1}.
\]

Since \(2mnt - m - 1 > 0\), for \(m \geq 3, n, t \geq 2\), it follows that \(\frac{m-3}{2mnt-m-1} < 1\) and hence \(d < 3\).

Conversely, by Lemma 8 and Lemma 3, we obtain that for odd \(m\), the graph \((C_m \times P_n)_{(t)}\), \(m \geq 3, n, t \geq 2\), is both super \(\left(\frac{m+3}{2} + p + q, 0\right)\)-edge antimagic total and super \(\left(\frac{m+3}{2} + p + 1, 2\right)\)-edge antimagic total.

Also by Lemma 2, we conclude that the graph \((C_m \times P_n)_{(t)}\), \(m \geq 3, n, t \geq 2\), has a super \(\left(\frac{m+3}{2} + p + \frac{q+1}{2}, 1\right)\)-edge antimagic total labeling, since \(q = m(2nt - 1)\), which is odd for odd \(m\).
5. A Graph Derived from Copies of Generalized Web Graph

Let \(v_{i,j}^{(k)}, 1 \leq i \leq m, 1 \leq j \leq n + 1\), \(1 \leq k \leq t\), be a collection of \(t\) disjoint copies of the generalized web graph \(W(m, n)\), \(m \geq 3, n \geq 2\), such that \(v_{i,j}^{(k)}\) is adjacent to \(v_{i+1,j}^{(k)}\) for \(1 \leq i \leq m - 1, 1 \leq j \leq n, v_{m,j}^{(k)}\) is adjacent to \(v_{i,j}^{(k)}\) for \(1 \leq j \leq n\) and \(v_{i,j}^{(k)}\) is adjacent to \(v_{i,j+1}^{(k)}\) for \(1 \leq i \leq m, 1 \leq j \leq n\). We denote the graph obtained by joining \(v_{i,j}^{(k)}\) to \(v_{i+1,j}^{(k+1)}\) and \(v_{i,j}^{(k+1)}\) for \(1 \leq i \leq m, 1 \leq k \leq t - 1\) as \((W(m, n))^{(t)}\).

Clearly, the vertex set \(V\) and the edge set \(E\) of the graph \((W(m, n))^{(t)}\) are given by \(V((W(m, n))^{(t)}) = \{v_{i,j}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n + 1\}\) and \(E((W(m, n))^{(t)}) = E_1 \cup E_2 \cup E_3\) where

\[
E_1 = \{v_{i,j}^{(k)}, v_{i+1,j}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m - 1, 1 \leq j \leq n\}
\cup \{v_{m,j}^{(k)} : 1 \leq k \leq t, 1 \leq j \leq n\},
\]

\[
E_2 = \{v_{i,j}^{(k)}, v_{i,j+1}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n\},
\]

\[
E_3 = \{v_{i,j}^{(k)}, v_{i,j}^{(k+1)}, v_{i,j}^{(k+1)} : 1 \leq k \leq t - 1, 1 \leq i \leq m\}.
\]

It is easy to see that for \((W(m, n))^{(t)}\), we have \(p = mt(n+1)\) and \(q = 2m(nt+t-1)\).

**Lemma 10.** For odd \(m, m \geq 3, n, t \geq 2\), the graph \((W(m, n))^{(t)}\) has an \((a, 1)\)-edge antimagic vertex labeling.

**Proof.** Let us define a bijection \(f_4 : V((W(m, n))^{(t)}) \rightarrow \{1, 2, \ldots, mt(n+1)\}\) as follows:

**Case (i):** \(n\) is even.

If \(j\) is odd and \(1 \leq i \leq m, 1 \leq j \leq n + 1, 1 \leq k \leq t\), then

\[
f_4(v_{i,j}^{(k)}) = \left\{
\begin{array}{ll}
(k-1)(mn+m) + (j-1)m + \frac{i+1}{2} & \text{if \(i\) is odd,} \\
(k-1)(mn+m) + (j-1)m + \frac{m+i+1}{2} & \text{if \(i\) is even.}
\end{array}
\right.
\]

If \(j\) is even and \(1 \leq i \leq m, 2 \leq j \leq n, 1 \leq k \leq t\), then
Figure 4. (a, 1)-edge antimagic vertex labeling of \((W(3, 3))^{(2)}\).

\[
\begin{align*}
f_4(v_{i,j}) &= \begin{cases} 
  (k - 1)(mn + m) + (j - 1)m + \frac{m+i+1}{2} & \text{if } i \text{ is odd}, \\
  (k - 1)(mn + m) + (j - 1)m + \frac{i}{2} & \text{if } i \text{ is even}.
\end{cases}
\end{align*}
\]

Case (ii): \(n\) is odd.

If \(j\) is odd and \(1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq t\), then
\[
f_4(v_{i,j}) = \begin{cases} 
  (k - 1)(mn + m) + (j - 1)m + \frac{m+i+1}{2} & \text{if } i \text{ is odd}, \\
  (k - 1)(mn + m) + (j - 1)m + \frac{i}{2} & \text{if } i \text{ is even}.
\end{cases}
\]

If \(j\) is even and \(1 \leq i \leq m, 2 \leq j \leq n + 1, 1 \leq k \leq t\), then
\[
f_4(v_{i,j}) = \begin{cases} 
  (k - 1)(mn + m) + (j - 1)m + \frac{i+1}{2} & \text{if } i \text{ is odd}, \\
  (k - 1)(mn + m) + (j - 1)m + \frac{m+i+1}{2} & \text{if } i \text{ is even}.
\end{cases}
\]

In both the cases, we observe that under the bijection \(f_4\), the edge weights of all the edges of \((W(m, n))^{(t)}\) constitute an arithmetic sequence \(\{\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+4mn+4m(t-1)+1}{2}\}\). Clearly \(\frac{m+3}{2}\) is an integer only when \(m\) is odd. Hence the vertex labeling \(f_4\) is an \((\frac{m+3}{2}, 1)\)-edge antimagic vertex labeling of \((W(m, n))^{(t)}\), for odd \(m\).

**Theorem 11.** For odd \(m, m \geq 3, n, t \geq 2\) and \(d \in \{0, 2\}\), the graph \((W(m, n))^{(t)}\), has a super \((a, d)\)-edge antimagic total labeling.

**Proof.** By Lemmas 3 and 10, we see that for odd \(m\), the graph \((W(m, n))^{(t)}\), \(m \geq 3, n, t \geq 2\) has a super \((\frac{m+3}{2} + p + q, 0)\)-edge antimagic total labeling and a super \((\frac{m+3}{2} + p + 1, 2)\)-edge antimagic total labeling.

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