CONTROL OF NON-MINIMUM PHASE SYSTEMS USING EXTENDED HIGH-GAIN OBSERVERS

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ABSTRACT

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Control schemes that achieve stabilization and output regulation in the case of non-minimum phase nonlinear systems are presented. These techniques utilize continuously implemented sliding mode control, and an extended high-gain observer to generate estimates of the output and its first $\rho - 1$ derivatives ($\rho$ is the relative degree of the system), in addition to a signal that renders the zero dynamics of the given plant observable. The results include exponential stability of the origin of the output feedback system in the stabilization problem, exponential stability of the origin of the error system in the regulation problem, and asymptotic recovery of the performance of the state feedback designs in both instances.

The stabilization and regulation problems pose unique challenges in the case of non-minimum phase nonlinear systems. This work provides a design methodology that can be incorporated into any robust control scheme. The design process conforms to the nonlinear separation principle, and comprises the following three steps.

1) Augment a dynamic stabilizing compensator to the system and moreover, design it such that the resultant system is minimum phase and relative degree one with respect to a virtual output to be constructed using the state estimates.

2) Use any robust control technique such as sliding mode control, saturated high-gain feedback, Lyapunov redesign, etc. to design a static state feedback control law that stabilizes the relative degree one minimum phase system obtained in the previous step.

3) Design an extended high-gain observer (EHGO) to estimate the system output
and its first $\rho - 1$ derivatives, and additionally, the coupling signal that is required to solve the auxiliary design problem stemming from step 1). Replace all the states and the coupling signal in the state feedback control design with their estimates, and if necessary, saturate the resulting control law outside a compact set of interest, or saturate the estimates obtained from the EHGO.

Simulation examples are provided—in particular, the stabilization algorithm is applied to the Translational Oscillator with a Rotating Actuator (TORA) nonlinear benchmark system.
DEDICATION

To my mother Zeenath, who is Inspiration personified.
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Chapter 1

Introduction

Engineers have always concerned themselves with applying humanity’s collective knowledge toward harnessing natural resources and phenomena in order to solve or to address commonly encountered technological challenges, and for the betterment of human society at large. Control engineering, with its interdisciplinary roots, is uniquely positioned among all of the engineering disciplines in that it facilitates the manipulation and control of large and complex physical systems, provided the relevant system models and associated control solutions are sufficiently accurate in some sense, not to mention tractable and amenable to economically and technologically feasible implementation methods.

1.1 Background and Preliminaries

Control theory primarily addresses the problem of designing control algorithms for physical systems that can be modeled using techniques derived largely—although not exclusively—from classical physics and the theory of ordinary differential equations. We can encompass a very broad range of deterministic (as opposed to stochastic)
control problems by considering systems modeled by the equations

\[ \dot{x} = f(t, x, u, w, \theta), \quad (1.1a) \]
\[ y = h(t, x, u, w, \theta), \quad (1.1b) \]

where \( x \in D \subset \mathbb{R}^n \) is the system state, \( u \in \mathbb{R}^p \) is a control input to the system that we wish to design, \( w \in \mathbb{R}^d \) is a vector containing disturbance and reference signals, \( \theta \in \Theta \subset \mathbb{R}^p \) is a vector of constant unknown system parameters, \( y \in \mathbb{R}^m \) given by the mapping \( h : [0, \infty) \times D \times \mathbb{R}^p \times \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^m \) is the system output, and \( t \in [0, \infty) \) is a special scalar parameter—usually interpreted as time—that characterizes the trajectories that comprise the solution set of (1.1), provided one exists. A system that lacks an explicit dependence on time in the functions \( f(\cdot) \) and \( h(\cdot) \) is said to be autonomous or time-invariant, while a system like (1.1) above is said to be non-autonomous or time-varying. Finally, the mapping \( f : [0, \infty) \times D \times \mathbb{R}^p \times \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^n \) is a vector-valued function that describes the dynamic behavior of the states \( x \) of the physical system (1.1) or plant, in engineering parlance.

Generally speaking, the control task is to design the input \( u \) of the system (1.1) to, on the one hand, stabilize the latter in the absence of disturbances and/or reference signals \( w \), or optionally, to also reject the effect of these disturbances, and/or to track any reference signals that happen to be contained in \( w \). The control input could be a static function of the output measurements \( y \) and possibly also the reference signals, but it is often the case that it is necessary to design an additional dynamic system—broadly referred to as a compensator—to generate the desired signal \( u \) that is capable of achieving the control task; mathematically, this implies that the augmented system formed by the interconnection of this aforementioned dynamic compensator and the plant (1.1) has to be stable in some sense. A significant class of control problems admits solutions that require the control input \( u \)—also known as a control law—to
be a function of the plant states $x$, possibly in addition to the states of a dynamic compensator (if one is required). As can be seen from (1.1), the only signal that is available to the control designer is the output measurement $y$, so in order to implement a control law that needs to be a function of the state vector $x$, it is often necessary to construct a dynamical system called a state estimator or observer that is capable of reconstructing the states from the output measurements. Thus, observer-based output feedback control designs can be considered to fall under the broader class of control schemes in which the input $u$ is generated by combining output measurements $y$ with the states of a dynamic compensator.

The advantage in the case of a certain class of systems that allow for observer-based control schemes is the idea of the separation principle—in essence, for this class of systems, it may be possible to separate the observer and control designs into two distinct sub-problems. In the first step, the control law $u$ is designed as a function of the states $x$ and if necessary, the states of a dynamic compensator. Such a law is referred to as a state feedback control law. In the next step, an observer is designed using the output $y$ and our knowledge of the plant model in order to generate estimates of the states $x$. In the final stage, the results of the previous steps are combined by substituting the estimates of $x$ into the control law, thus yielding an output feedback control law that is capable of controlling the plant. This type of solution is often simpler or easier to develop and to implement than the alternative of attempting to design a control scheme using only the output $y$, and without the benefit of the state estimates—indeed, in the absence of the latter, the control problem could very well be intractable. In the sequel, we shall revisit the idea of the separation principle for a certain type of observer-based nonlinear control system, and discuss some of the differing interpretations of this concept.

If both $f(t, x, 0, 0, \theta)$ and $h(t, x, 0, 0, \theta)$ happen to be linear in $x$, then (1.1) is said to be a linear system. Similarly, if $u$ is linear in $x$, and the compensator subsystem
that generates \( u \)—if one is present—is also linear in its state variables, then the interconnection of the compensator with the plant (1.1), called the closed-loop system, will likewise be linear in the resulting augmented state vector, which basically comprises the states of the plant as well as the compensator. The theory of linear systems, which is essentially an amalgamation of the theory of linear differential equations and results from complex analysis and frequency domain functional analysis, has been thoroughly researched for a period spanning most of the past two centuries. In practice, a significant majority of control designs tend to rely upon results from linear system theory. Many systems can be modeled adequately using linear differential equations. Even in the event that nonlinearities or model uncertainties are present, the theory of robust linear control has successfully addressed many types of problems in which systems can be modeled as being essentially linear, but incorporating certain types of structural model uncertainties. Thus, linear system theory has historically been very successful in addressing many issues pertaining to linear and quasi-linear systems, and robust control problems. Linear systems and control will be revisited periodically in the discussion to follow.

Despite the ubiquity of linear control methods in engineering applications, it is often the case that nonlinearities are inherent in certain physical systems such as robots or physical systems encountered in the study of the mechanics of rigid bodies, flexible structures, etc., while other systems contain components that exhibit severe nonlinearities, either by happenstance or design, that diverge very quickly from linear approximations. One of the most common examples of the latter type of nonlinearity is saturation, which simulates physical limitations on the so-called control effort (effectively, bounds on the magnitude and/or time rate of change of the control signal \( u \)). For instance, the input voltage to a motor—or practically any electrical, electromechanical or electromagnetic device, for that matter—if increased beyond some critical value, will often cause actuator saturation. Other examples
include, but aren’t limited to, hysteresis, dead bands, backlash nonlinearities, switching or relay action, etc. Thus, control schemes that take into account these kinds of nonlinear phenomena might be able to produce superior results compared to linear control methods—examples where this has proven true include electromechanical systems, robots, aerospace applications, digital and analog electronic devices, smart material-based or other types of systems with significant model nonlinearities, etc.

Nonlinear control has been a very active area of research for the past four decades. Prior to the 1970s, most results in nonlinear control addressed challenges pertaining to rigid body mechanics, nonlinear circuit and electronic elements, aeronautics and aerospace applications, and these problems were largely analyzed using techniques such as the describing functions method and phase plane analysis, or were posed and solved in an optimal control framework [4]. Research into this field gathered pace during the seventies, and the year 1978 saw the publication of the first major textbook on nonlinear system theory by Vidyasagar [42]. Since then, many other books on nonlinear systems and control have been published, serving to collect all the major results and design techniques up to that point—of particular note are the books by Isidori [19], first published in 1985, Khalil [29] (first edition: 1992) and Sastry [38] (1999). In addition to the aforementioned books that provide a broad coverage of Lyapunov stability theory (which forms the backbone of stability analysis for nonlinear systems), standard nonlinear control design techniques, and elements of differential geometry needed for the exposition of the control theoretic results contained therein, many other books and monographs have been published on specific topics within the purview of nonlinear systems, such as robust nonlinear control, adaptive control, regulation, etc. Some of these books will be mentioned while discussing these particular topics.

In contrast, linear system theory and control had been thoroughly researched, and many major results had been published by the 1970s. The globally linear nature
of linear spaces naturally led to global results, unlike the case of nonlinear control, where local or semiglobal results are much more common. A semiglobal result (such as stabilization) is one that can be achieved on a compact set of finite size containing the desired equilibrium set, but the former can be made arbitrary large in certain cases by appropriate choice of some design parameter such as controller gain, for instance. A watershed moment in modern control theory occurred circa 1960, with the publication of a pair of seminal papers by Kalman, the first of which introduced the concepts of controllability and observability [24]. Indeed, the juncture at which these papers were published is often regarded as the dividing line between classical and modern control theory. Over the next couple of decades, considerable advances were made in the development of robust control schemes for linear systems, culminating in the publication of a paper on $H_\infty$ control by Zames in 1981 [44].

1.2 Some Definitions

We take this opportunity to define a few terms before we discuss the types of problems that are tackled in this work. To this end, let us consider the following autonomous system

$$\dot{x} = f(x,u),\quad (1.2)$$

where $f : D \times \mathbb{R} \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $(D \times \mathbb{R}) \subset (\mathbb{R}^n \times \mathbb{R})$ into $\mathbb{R}^n$. Suppose there exists a state feedback control law

$$u = \Gamma(x),\quad (1.3)$$

where $\Gamma : D \to \mathbb{R}$. Also, suppose that $x^*$ is an equilibrium point of the closed-loop system $(1.2)$–$(1.3)$; consequently, we have $f(x^*,\Gamma(x^*)) = 0$. We state the definitions of the stability of the equilibrium point $x^*$ (or the lack thereof) and some related
concepts below.

The following definition is taken from [29].

Definition 1.1. The equilibrium point \( x = x^* \) of (1.2)–(1.3) is

- stable if, for every \( \varepsilon > 0 \), there exists a \( \delta = \delta(\varepsilon) > 0 \) such that
  \[
  \|x(0) - x^*\| < \delta \Rightarrow \|x(t) - x^*\| < \varepsilon, \quad \forall t \geq 0
  \]

- unstable, if it is not stable

- asymptotically stable, if it is stable and \( \delta \) can be chosen such that
  \[
  \|x(0) - x^*\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = x^*
  \]

Let \( \phi(t; x) \) be a solution of (1.2)–(1.3) that starts at initial state \( x \) at time \( t = 0 \). When the equilibrium point \( x = x^* \) of (1.2)–(1.3) is asymptotically stable, its region of attraction is defined as the set of all points \( x \) such that \( \phi(t; x) \) is defined for all \( t \geq 0 \) and \( \lim_{t \to \infty} \phi(t; x) = x^* \). If it is the case that for any initial state \( x \), the trajectory \( \phi(t; x) \) approaches \( x^* \) as \( t \to \infty \) no matter how large \( \|x - x^*\| \) gets, then the region of attraction is the whole space \( \mathbb{R}^n \). If an asymptotically stable equilibrium point \( x^* \) possesses this aforementioned property, it is said to be globally asymptotically stable.

For the purposes of the next definition, we suppose that the unforced version of system (1.2), i.e.
\[
\dot{x} = f(x, 0), \quad (1.4)
\]
has an asymptotically stable equilibrium point at the origin \( x = 0 \).

The following definition is adapted from [29].

Definition 1.2. The system (1.2) is said to be regionally input-to-state stable if there exist a class \( \mathcal{KL} \) function \( \beta \) and a class \( \mathcal{K} \) function \( \gamma \) such that, for any initial state
$x(0) \in \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}^n$ is a compact set, and for any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq 0$ and satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma \left( \sup_{0 \leq \tau \leq t} \|u(\tau)\| \right). \quad (1.5)$$

Inequality (1.5) guarantees that for any bounded input $u(t)$, the state $x(t)$ will be bounded. Furthermore, as $t$ increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq 0} \|u(t)\|$. It can also be shown that if $u(t) \to 0$ as $t \to \infty$, then $\lim_{t \to \infty} x(t) = 0$. Since, with $u(t) \equiv 0$, (1.5) reduces to

$$\|x(t)\| \leq \beta(\|x(0)\|, t),$$

regional input-to-state stability implies that the origin of the unforced system (1.4) is asymptotically stable. Moreover, if inequality (1.5) holds for any initial state $x(0) \in \mathbb{R}^n$, then we drop the modifier “regionally” from the above definition and state that the system (1.2) is simply input-to-state-stable, which in turn implies that the origin of (1.4) is globally asymptotically stable.

### 1.3 Stabilization of Non-Minimum Phase Systems

The stabilization problem for non-minimum phase systems (to be defined shortly) forms the crux of this work. In order to introduce the concepts and terminology associated with this problem, we shall consider single-input, single-output (SISO—i.e. $u \in \mathbb{R}$ and $y \in \mathbb{R}$) autonomous systems that have the following structure.

$$\dot{x}_p = f(x_p) + g(x_p)u, \quad (1.6a)$$

$$y = h(x_p). \quad (1.6b)$$
All the other variables apart from $u$ and $y$ belong to the domains defined above for the system (1.1), and the mappings $f : D \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$, and $h : D \to \mathbb{R}$ are assumed to be sufficiently smooth for all $x_p \in D$. Moreover, $f(0) = 0$.

In the special case of linear time-invariant systems, we can take $f(x_p) = Ax_p$, $g = B$, and $h(x_p) = Cx_p$, which gives

$$
\dot{x}_p = Ax_p + Bu, \quad (1.7a)
$$

$$
y = Cx_p, \quad (1.7b)
$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$ and $C \in \mathbb{R}^{1 \times n}$. The system equations (1.7) are referred to as the state-space realization for a linear system with input $u$ and output $y$. The input-to-output map for (1.7) can also be represented in transfer function form as

$$
H(s) = \frac{Y(s)}{U(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}, \quad (1.8)
$$

where $m < n$ and $b_m \neq 0$, and the polynomials $Y(s)$ and $U(s)$ are the Laplace transforms of the time domain signals $y(t)$ and $u(t)$, respectively. The difference between the degrees of the numerator and denominator polynomials above, $n - m$, is called the relative degree of the linear system described by $H(s)$, or equivalently, by (1.7). Given an input-output map for a linear system such as (1.8), several non-unique, albeit equivalent state-space realizations of $Y(s)/U(s)$ exist in the time domain. Two of the most common state-space realizations of transfer functions are the controllable canonical form and the observable canonical form. The controllable canonical representation of (1.8) is given by

$$
\dot{\tilde{x}}_p = A_c \tilde{x}_p + B_c u, \quad (1.9a)
$$

$$
y = C_c \tilde{x}_p, \quad (1.9b)
$$
where the triple \{A_c, B_c, C_c\} is given by

\[
A_c = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1}
\end{pmatrix}, \quad B_c = \begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix},
\]

\[
C_c = \begin{pmatrix}
b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0
\end{pmatrix},
\]

where the coefficients \(a_i\) and \(b_j\), \(0 \leq i \leq n - 1\) and \(0 \leq j \leq m\), are the same as the ones that appear in (1.8). We note that

\[
y^{(i)} = \begin{cases}
C_c A_c i \tilde{x}_p, & 1 \leq i \leq n - m - 1, \\
C_c A_c^{n-m} \tilde{x}_p + C_c A_c^{n-m-1} B_c u, & i = n - m,
\end{cases}
\]

and consequently, the relative degree \(\rho = n - m\) can be interpreted as the minimum number of times the output \(y\) has to be differentiated in order for the control input \(u\) to show up in the resulting expression. If all the roots of the numerator polynomial of \(H(s)\) are in the open left half plane, then (1.7) is said to be minimum phase, or non-minimum phase otherwise.

The above interpretation of relative degree in terms of successive derivatives of the output is also applicable to a certain class of nonlinear systems in the form of (1.6). Indeed, the derivative of the output is

\[
\dot{y} = L_f h(x_p) + L_g h(x_p)u,
\]

where

\[
L_f h(x_p) \triangleq \frac{\partial h(x_p)}{\partial x_p} f(x_p)
\]
is called the *Lie derivative* of \( h \) with respect to \( f \), and can be interpreted as the total time derivative of \( h \) along the trajectories of \( f \). Now, if \( L_g h(x_p) = 0 \), then \( y \) will be independent of \( u \), in which case we could continue calculating successive derivatives of \( y \) until \( u \) finally appears, and the result would be as follows.

\[
y^{(i)} = \begin{cases} 
L^i_f h(x_p), & 1 \leq i \leq \rho - 1, \\
L^\rho_f h(x_p) + L_g L_{f}^{\rho-1} h(x_p)u, & i = \rho,
\end{cases}
\]

for some positive integer \( \rho \), in analogous fashion to the linear case. It is apparent from the above equation that the state feedback control

\[
u = \frac{v - L^\rho_f h(x_p)}{L_g L_{f}^{\rho-1} h(x_p)},
\]

which can be implemented if \( f, g \) and \( h \) are known precisely, will reduce the input-output map to

\[
y^{(\rho)} = v,
\]

which is a chain of \( \rho \) integrators, and hence this number is referred to as the *relative degree* of the system (1.1). We note that the system will have a well defined relative degree in a region \( D_0 \subset D \), only if \( L_g L_{f}^{\rho-1} h(x_p) \neq 0 \), for all \( x_p \in D_0 \).

The above procedure of taking successive derivatives of the output until the input appears suggests that we could define a transformation \( T : D_0 \rightarrow D_0 \) as [29]

\[
x_t = T(x_p) \triangleq \begin{pmatrix} z \\ x \end{pmatrix} \triangleq \begin{pmatrix} \varphi(x_p) \\ \pi(x_p) \end{pmatrix} = \begin{pmatrix} \varphi_1(x_p) & \cdots & \varphi_{n-\rho}(x_p) \\ h(x_p) & \cdots & L_{f}^{\rho-1} h(x_p) \end{pmatrix}^\top, \quad (1.10)
\]
where \( \varphi_1 \) to \( \varphi_{n-\rho} \) satisfy the partial differential equations

\[
\frac{\partial \varphi_i}{\partial x_p} g(x_p) = 0, \text{ for } 1 \leq i \leq n - \rho, \forall x_p \in D_0.
\]

The functions \( \varphi_1 \) to \( \varphi_{n-\rho} \) can be shown to exist at least locally, and moreover, can be chosen so as to make \( \varphi(x) \) a diffeomorphism on \( D \) [29, Theorem 13.1]. A diffeomorphism is a continuously differentiable map that also has a continuously differentiable inverse.

The change of variables (1.10) thus transforms (1.6) into

\[
\dot{z} = f_0(z, x), \tag{1.12a}
\]

\[
\dot{x} = Ax + B[b_0(z, x) + a_0(z, x)u], \tag{1.12b}
\]

\[
y = Cx, \tag{1.12c}
\]

where \( x \in \mathbb{R}^\rho, z \in \mathbb{R}^{n-\rho}, (A, B, C) \) is a chain of \( \rho \) integrators in canonical form, and

\[
f_0(z, x) = \left. \frac{\partial \varphi}{\partial x_p} f(x_p) \right|_{x_p=T^{-1}(x_t)},
\]

\[
b_0(z, x) = \left. L^{\rho} f h(x_p) \right|_{x_p=T^{-1}(x_t)},
\]

\[
a_0(z, x) = \left. L_0 g L^{\rho-1} f h(x_p) \right|_{x_p=T^{-1}(x_t)}.
\]

Equation (1.12) is called the normal form representation of the system (1.6). In this form, the system comprises an external part given by the \( \dot{x} \)-equation in (1.12), and an internal part given by the \( \dot{z} \)-equation. If the functions \( a_0(\cdot) \) and \( b_0(\cdot) \) are known, then the external part of (1.12) can linearized by the state feedback control \( u = [v - b_0(z, x)]/a_0(z, x) \), but we should also note that the internal part will be made unobservable from the output \( y \) by this same control law. Setting \( x = 0 \) in the
internal dynamics of (1.12) gives

\[ \dot{z} = f_0(z, 0), \quad (1.13) \]

which is referred to as the zero dynamics of the system (1.6). The system is said to be minimum phase if the equilibrium point of (1.13) is asymptotically stable, and non-minimum phase otherwise. In this work, we are interested in designing output feedback stabilizers and regulators for systems that are potentially non-minimum phase.

There are several applications that give rise to system models that can be expressed in the form of (1.12), and these have been documented and analyzed extensively in the literature. Common sources of such models are mechanical and electromechanical systems in which the outputs comprise position or displacement variables, but not their derivatives (velocities, accelerations, and so forth)—typical examples of such models include induction motors [30], an inverted-pendulum-on-a-cart system [29], and the translational oscillator with a rotating actuator (TORA) benchmark system [6]. The latter two examples happen to be non-minimum phase, as well—the open loop TORA system has a pair of conjugate poles on the imaginary axis, while the inverted pendulum system has a pole in the right half of the complex plane.

Concerning non-minimum phase systems, it should be noted that historically, no real distinction has been necessary between the aforementioned type of plant or a minimum phase plant in developing control design techniques for linear systems, as long as the model is assumed to be completely known. The only difference of note between the two types of systems is the well known fact that right half plane zeros tend to reduce the gain and phase margins of the closed loop system, thus imposing constraints upon the performance measures that can be achieved by the design. For
instance, [17] sets forth trade-offs that are inherent in the controller designs for plants with right half plane zeros and poles, and the limitations imposed on the sensitivity and complementary sensitivity functions of the closed loop system. Reference [45] extends the so-called loop transfer recovery (LTR) procedure to non-minimum phase linear plants, and shows that when applying LTR to a non-minimum phase system, there is a trade-off between the feedback properties of the state feedback loop and the ability to recover these properties in the output feedback design.

Non-minimum phase systems pose a much greater design challenge when the plant dynamics are nonlinear. Indeed, it is fairly common in the literature pertaining to nonlinear systems to assume that the plant under consideration is minimum phase—specifically, that the zero dynamics are either input-to-state stable or globally asymptotically stable. A few papers on the topic of output feedback control of non-minimum phase systems have been published in the past decade, however. Specifically, the papers [25, 34, 21, 36] deal with the case of stabilization of an equilibrium point. Reference [21] in particular introduced a high-gain feedback based design tool for a very large class of systems, possibly non-minimum phase, in the normal form. The extension of the aforementioned technique to render it independent of any particular control scheme and to show that the resulting design is robust to uncertainties in the control coefficient (the signal $a_0(\cdot)$ in (1.12)) and in other system parameters forms the crux of this dissertation. This extension is possible by utilizing a so-called extended high-gain observer (EHGO) to estimate the signal $b_0(\cdot)$ in (1.12), which renders the zero dynamics observable if this same signal is viewed as the output of a reduced order auxiliary system that incorporates the zero dynamics of (1.12). In accordance with Isidori’s design tool [21], if a stabilizing controller is known for the auxiliary system, then it is possible to construct an output feedback controller that stabilizes (1.12). While Isidori used a high-gain feedback design, the EHGO-based output feedback controller in our work may incorporate any robust control scheme.
such as sliding mode control (SMC), saturated high-gain feedback, etc. The stabilization problem is tackled in Chapter 2.

1.4 Regulation of Non-Minimum Phase Systems

The next chapter of this dissertation focuses on the problem of output feedback regulation of non-minimum phase systems. The regulation problem is a special case of the output tracking and disturbance rejection problems. In the latter case, the control design is required to make the output of the system match an external reference signal and to reject unknown disturbance signals. The objective is the same in the regulation problem, but the external signals are assumed to be sinusoidal with known frequencies or constant—this allows for the incorporation of an internal model into the system, which is capable of asymptotically generating the reference signal to be tracked and the disturbances to be rejected. As is the case with unstable zero dynamics, there are some unique challenges when designing regulators for nonlinear systems as opposed to the linear case.

The regulation problem for linear systems was solved in the 1970s. As mentioned earlier, no distinction is necessary between minimum and non-minimum phase plants in the design of output feedback regulators for linear plants; moreover, it is possible for the internal model to have poles in the right half complex plane, unlike in the case of nonlinear systems, whose internal model poles are confined to the imaginary axis. The key papers on output feedback regulation of linear systems are by Francis and Wonham [15], who introduced the internal model principle, and by Davison [12].

Generally speaking, the presence of nonlinearities in a system model results in the generation of higher order harmonics of any sinusoidal signals that enter into the system. Consequently, the internal model must reproduce these higher order harmonics if we are to successfully solve the regulation problem—this issue is explained in
further detail in [26]. Nevertheless, regulation theory is fairly well developed even in the case of nonlinear systems, and the major results have been collected in standard reference works and textbooks such as [20, 8, 29, 18], to name a few. On the other hand, just as with the stabilization problem, there are fewer results in the specific case of non-minimum phase systems. This is not surprising since the inclusion of a marginally stable internal model contributes additional marginally stable dynamics to the overall system, which specifically manifest themselves as critically stable zero dynamics (in addition to others that may come from the plant) that are not observable from the output tracking error. Some recent papers dealing with unstable zero dynamics are [32, 10], and they represent extensions of the design tool of [21] to the regulation problem. All these papers utilize a high gain feedback controller design and do not consider uncertainty in the control coefficient. In Chapter 3 of this dissertation, we seek to extend the idea of [21] in analogous fashion to Chapter 2 by utilizing an extended high-gain observer, which allows for control designs other than high gain feedback, and we also consider uncertain control coefficients.

In this work, the regulation problem is tackled in Chapter 3 for single-input, single-output systems of the form [27]

\[
\begin{align*}
\dot{x_p} &= f(x_p, \theta) + g(x_p, \theta)u + \varphi(x_p, d, \theta), \\
y &= h(x_p, \theta) + \gamma(d, \theta) - y_r,
\end{align*}
\]

(1.14) where \(x_p \in \mathbb{R}^n\) is the system state, \(u \in \mathbb{R}\) is the input, \(y \in \mathbb{R}\) is the measured output, \(d \in \mathbb{R}^p\) is a time varying disturbance input and \(y_r \in \mathbb{R}\) is a time varying reference signal, with both \(y_r\) and \(d\) having known models, and \(\theta\) is a vector of uncertain constant parameters that belongs to a compact set \(\Theta \subset \mathbb{R}^l\). The functions \(f, g, h, \varphi\) and \(\gamma\) all depend continuously on \(\theta\), and for all \(\theta \in \Theta\), they are sufficiently smooth in \(x_p\) and \(d\) for all \(x_p \in D_\theta\), an open connected subset of \(\mathbb{R}^n\) that could depend on \(\theta\),
and all $d$ in a compact set of interest. Moreover, $\varphi(x, 0, \theta) = 0$ and $\gamma(0, \theta) = 0$, for all $\theta \in \Theta$ and $x_p \in D_\theta$. The control input $u$ is to be designed so as to make the output $y$ go to zero.

The disturbance-free system (1.14), with $d = 0$, is assumed to have a uniform relative degree $\rho \leq n$ for all $\theta \in \Theta$ and $x_p \in D_\theta$, that is,

$$L_g h(x_p, \theta) = L_g L_f h(x_p, \theta) = \cdots = L_g L_f^{\rho-2} h(x_p, \theta) = 0,$$

and

$$|L_g L_f^{\rho-1} h(x_p, \theta)| \geq \kappa > 0,$$

where $\kappa$ is independent of $\theta$. Furthermore, we assume that there exists a diffeomorphism

$$T : D_\theta \times \Theta \to D_\theta,$$

(1.15a)

$$T : x_p \mapsto \begin{pmatrix} z \\ x \end{pmatrix}$$

(1.15b)

that transforms (1.14), with $d = 0$, into the normal form

$$\dot{z} = f_0(z, x, \theta),$$

(1.16a)

$$\dot{x}_i = x_{i+1}, \quad 1 \leq i \leq \rho - 1,$$

(1.16b)

$$\dot{x}_\rho = b_0(z, x, \theta) + a_0(z, x, \theta) u,$$

(1.16c)

$$y = x_1 - y_r.$$  

(1.16d)

We note that (1.16) will be identical to the normal form equations (1.12) above in the absence of uncertainties $\theta$. Systems transformable into the normal form (1.16) are studied in Chapter 2, and therefore the results therein are applicable to any
disturbance-free system of the form (1.14) (i.e. \( d = 0 \) and \( y_r = 0 \) in the stabilization problem).

When \( d \neq 0 \), we assume that (1.15) transforms the disturbance-driven system (1.14) into the form

\[
\dot{z} = f_d(z, x_1, \ldots, x_{m+1}, d, \theta), \\
\dot{x}_i = x_{i+1} + \psi_i(x_1, \ldots, x_i, d, \theta), \quad 1 \leq i \leq m - 1, \ m > 1, \\
\dot{x}_i = x_{i+1} + \psi_i(x_1, \ldots, x_i, d, \theta), \quad 1 < m \leq i \leq \rho - 1, \\
\dot{x}_\rho = b_0(z, x, \theta) + a_0(z, x, \theta)u + \psi_\rho(z, x, d, \theta), \\
y = x_1 + \gamma(d, \theta) - y_r,
\]

where \( 1 \leq m \leq \rho - 1 \), and the functions \( \psi_i \) vanish at \( d = 0 \).

Let \( \rho_0 \) be the disturbance relative degree, that is, the disturbance appears for the first time in \( y(\rho_0) \), and set \( \rho^* = \rho - \rho_0 \). Also, let

\[
\mathcal{D}(t) \triangleq \begin{pmatrix} d(t) \\ \vdots \\ d(\rho^*(t)) \end{pmatrix} \in D_d, \quad \mathcal{Y} \triangleq \begin{pmatrix} y_r(t) \\ \vdots \\ y_{r(\rho)}(t) \end{pmatrix} \in Y,
\]

where \( D_d \in \mathbb{R}^{(\rho^*+1)p} \) and \( Y \in \mathbb{R}^{\rho+1} \) are compact sets. We assume that \( \mathcal{D}(t) \) and \( \mathcal{Y}(t) \) are generated by the exosystem

\[
\dot{w} = S_0 w, \\
\begin{pmatrix} \mathcal{D} \\ \mathcal{Y} \end{pmatrix} = \Gamma_0 w,
\]

where \( S_0 \in \mathbb{R}^{nw \times nw} \) has distinct eigenvalues on the imaginary axis, and \( w(t) \) belongs
to a compact set $W \subset \mathbb{R}^{nw}$. We now define the functions $\bar{x}_1(w, \theta)$ through $\bar{x}_m(w, \theta)$ as

$$
\bar{x}_1 = yr - \gamma(d, \theta),
$$

$$
\bar{x}_{i+1} = \frac{\partial \bar{x}_i}{\partial w} S_0 w - \psi_i(\bar{x}_1, \ldots, \bar{x}_i, d, \theta), \quad 1 \leq i \leq m - 1.
$$

Next, we assume there exists a unique mapping $\bar{z}(w, \theta)$ that solves the partial differential equation

$$
\frac{\partial \bar{z}}{\partial w} S_0 w = f_d(\bar{z}, \bar{x}_1, \ldots, \bar{x}_m, (\partial \bar{x}_m / \partial w) S_0 w - \psi_m(\bar{z}, \bar{x}_1, \ldots, \bar{x}_m, d, \theta), d, \theta), \quad (1.19)
$$

for all $w \in W$. Using $\bar{z}(w, \theta)$, we now define $\bar{x}_{m+1}(w, \theta)$ to $\bar{x}_\rho(w, \theta)$ as

$$
\bar{x}_{i+1} = \frac{\partial \bar{x}_i}{\partial w} S_0 w - \psi_i(\bar{z}, \bar{x}_1, \ldots, \bar{x}_i, d, \theta), \quad m \leq i \leq \rho - 1.
$$

The steady-state zero-error manifold is given by $\{z = \bar{z}(w, \theta), x = \bar{x}(w, \theta)\}$. The steady-state value of the control input on this manifold is given by

$$
\chi(w, \theta) = \frac{1}{a_0(\bar{z}, \bar{x}, \theta)} [(\partial \bar{x}_\rho / \partial w) S_0 w - b_0(\bar{z}, \bar{x}, \theta) - \psi_\rho(\bar{z}, \bar{x}, d, \theta)]. \quad (1.20)
$$

Suppose there exists a set of real numbers $c_0, \ldots, c_{q-1}$, independent of $\theta$, such that $\chi(w, \theta)$ satisfies the identity

$$
L_s^q \chi = c_0 \chi + c_1 L_s \chi + \cdots + c_{q-1} L_s^{q-1} \chi,
$$

for all $(w, \theta) \in W \times \Theta$, where $L_s \chi = (\partial \chi / \partial w) S_0 w$, and the characteristic polynomial equation

$$
\lambda^q - c_{q-1} \lambda^{q-1} - \cdots - c_1 \lambda - c_0 = 0
$$
has distinct roots on the imaginary axis.

By defining

\[ S = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & \cdots & c_{q-2} & c_{q-1}
\end{pmatrix}, \quad \tau = \begin{pmatrix}
\chi \\
L_s \chi \\
L_s^2 \chi \\
\vdots \\
L_s^{q-2} \chi \\
L_s^{q-1} \chi
\end{pmatrix}, \]

\[ \Gamma = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}_{1 \times q}, \]

it can be shown that \( \chi(w, \theta) \) is generated by the internal model

\[
\frac{\partial \tau(w, \theta)}{\partial w} S_0 w = S \tau(w, \theta),
\]

\[
\chi(w, \theta) = \Gamma \tau(w, \theta).
\]

In order to convert the regulation problem to one of stabilizing an equilibrium point, we apply the change of variables

\[
\eta = z - \bar{z}(w, \theta), \quad (1.21a)
\]

\[
e_i = y^{(i-1)}, \quad 1 \leq i \leq \rho. \quad (1.21b)
\]

By a series of calculations, it can be verified that

\[
e_1 = x_1 - \bar{x}_1(w, \theta),
\]

\[
e_i = x_i - \bar{x}_i(w, \theta) - \vartheta_i(e_1, \ldots, e_{i-1}, w, \theta), \quad 2 \leq i \leq m,
\]

\[
e_i = x_i - \bar{x}_i(w, \theta) - \vartheta_i(\eta, e_1, \ldots, e_{i-1}, w, \theta), \quad m + 1 \leq i \leq \rho,
\]
where the functions $\theta_i$ vanish at $(\eta, e) = (0, 0)$. With the change of variables (1.21), we obtain the following representation for system (1.17) in the new variables.

\[
\dot{\eta} = f(\eta, e, w, \theta),
\]

\[
\dot{e}_i = e_{i+1}, \quad 1 \leq i \leq \rho - 1,
\]

\[
\dot{e}_\rho = b(\eta, e, w, \theta) + a(\eta, e, w, \theta)u,
\]

\[
y = e_1.
\]

The functions $f(\cdot), a(\cdot)$ and $b(\cdot)$ satisfy

\[
f(0, 0, w, \theta) = 0,
\]

\[
a(0, 0, w, \theta) = a_0(\bar{z}(w, \theta), \bar{x}(w, \theta), \theta)
\]

\[
b(0, 0, w, \theta) = -\chi(w, \theta)a_0(\bar{z}(w, \theta), \bar{x}(w, \theta), \theta).
\]

In these new variables, the zero-error manifold is given by $\{\eta = 0, e = 0\}$.

The above procedure shows how a system of the form (1.14) can be converted to error coordinates, with a view toward obtaining a solution to the regulation problem. In Chapter 3, we consider a special case of the system (1.14), and we begin the analysis and the design of the regulator for that system with normal form equations that are a special case of (1.17). Specifically, the normal form equations assumed for the system studied in Chapter 3 do not have any perturbation terms in the chain of integrators, unlike equations (1.17) above.
1.5 Extended High-Gain Observers in Nonlinear Control

A common feature of all the designs presented in this work is the utilization of the extended high-gain observer (EHGO) to estimate the output and its derivatives up to the $\rho^{th}$ order, which is the relative degree of the system, plus one additional signal, namely $b_0(\cdot)$ in (1.12) or a perturbed version of this signal, which renders the zero dynamics observable if this function is viewed as a *virtual output* of the system. The EHGO was also utilized by Freidovich and Khalil [16] as a disturbance observer, and the analyses in this work share some similarities with this paper. During the analysis of both the stabilization and regulation designs, it is shown that the output feedback system has exactly the same structure as the more general high-gain observer based closed loop systems studied by Atassi and Khalil [2, 3], and consequently, the EHGO-based designs in the subsequent chapters conform to the separation principle of Atassi and Khalil, in which the output feedback design exhibits the following three properties: a) recovery of the stability of the state feedback design, b) asymptotic recovery of the region of attraction of the state feedback case, and c) closeness of trajectories of the output and state feedback designs.

An important consideration to take into account when designing control schemes that utilize high-gain observers is the so-called *peaking phenomenon* that occurs when the high-gain parameter $\varepsilon > 0$, which is required to be small, is made sufficiently small [14, 29]. On the one hand, it can be shown by means of singular perturbation analysis [29] that reducing $\varepsilon$ has the effect of reducing or even eliminating the effect of uncertainties or disturbances in the system, but on the other hand, the peaking phenomenon will inevitably manifest itself as $\varepsilon$ is made progressively smaller. This phenomenon is essentially caused by an impulse-like response in the observer states that typically gets transmitted to the plant via the control input, which is often a
function of the state estimates obtained from the observer. Consequently, this results in the somewhat counter-intuitive behavior of the output feedback system relative to the state feedback system, whereby the trajectories of the former start to deviate further away from those of the latter as $\varepsilon$ is reduced and the peaking phenomenon becomes more pronounced.

It is possible to mitigate the above-mentioned effect of the peaking phenomenon by saturating the control, and possibly also the state estimates themselves, if necessary, outside a compact set of interest. This has the effect of creating a buffer that protects the plant from peaking. Notwithstanding the saturation of the output feedback control, the performance of the state feedback control can still be recovered by judicious choices of the high-gain observer parameter $\varepsilon$ and the saturation level for the control. The reason this can be achieved is as follows: a small enough $\varepsilon$ ensures that there is a separation of time scales between the states of the observer, which will exhibit fast responses, and those of the plant, which comprise the slow dynamics of the overall system. Once saturation has been incorporated into the output feedback design, the control will saturate during the peaking period. However, since the peaking period shrinks to zero as $\varepsilon$ tends to zero, for sufficiently small $\varepsilon$, the peaking period becomes so small that the states of the plant remain close to their initial values, even as the observer states peak rapidly and then quickly approach the state trajectories of the plant. Hence, subsequent to the peaking period, the state estimation errors become $O(\varepsilon)$ and the output feedback control becomes close to the state feedback control. Therefore, when the output feedback control is either globally bounded or made to saturate outside a compact set, the trajectories of the resulting closed-loop system asymptotically approach those of the state feedback system as $\varepsilon$ tends to zero. Indeed, it was proved by Atassi and Khalil [2, 3] that this technique of combining high-gain observers with globally bounded state feedback designs, which was originally proposed by Esfandiari and Khalil [14], 1) leads to a separation prin-
ciple that is independent of the state feedback design, 2) facilitates the recovery of the region of attraction and the trajectories achieved under state feedback, and 3) is robust to certain model uncertainties.

1.6 Sliding Mode Control

Another common thread in this work is the use of continuously-implemented sliding mode control. Firstly, sliding mode control is a scheme in which a virtual signal, typically denoted $s$, is defined as a function of the system states, and possibly compensator states, if the latter exists. The control law basically comprises the product of an envelope function and a switching term $\text{sgn}(s)$. Switching occurs about $s = 0$, and one objective of sliding mode control is to drive the system such that $s = 0$ within a finite amount of time and for all time thereafter; consequently, the set of all system states satisfying $s = 0$ is referred to as the sliding manifold. Continuously-implemented sliding mode control is essentially an approximation of sliding mode control, whereby the switching term $\text{sgn}(s)$ is approximated by $\text{sat}(s/\mu)$, where $\mu > 0$ is a design parameter that is typically chosen to be small. When $s > \mu$, the signum and saturation functions both evaluate to $\pm 1$, and hence the difference between the two arises when $s \leq \mu$—the set of all system states that satisfy this inequality is referred to as the boundary layer. Using this terminology, inside the boundary layer, the continuous sliding mode control acts as high-gain feedback. The saturation function approximates the switching achieved by the signum function about the origin if $\mu$, which controls the slope of the linear region of the saturation function, is chosen to be small enough. Indeed, as $\mu \to 0$, the saturation function approaches the signum function.

While sliding mode control is a popular control scheme for nonlinear systems due to its robustness properties, it tends to suffer from a phenomenon known as chattering,
especially in most practical applications. In an ideal system, the trajectories under sliding mode control are expected to start sliding along the manifold $s = 0$ soon as this manifold is reached. However, in reality, there will be a delay between the time the sign of $s$ changes and the time the control reacts to this change and effects the switch. During this delay period, the trajectory passes from a region, say $s > 0$, through the manifold $s = 0$ and beyond, so that now, $s < 0$. When the control eventually switches, the trajectory reverses direction and heads back toward the manifold, but delays in the control switching will ensure that it will overshoot and pass through to the other side of the manifold yet again. This process repeats several times over a rather short period of time and results in very high frequency oscillations in the immediate vicinity of the sliding manifold, and this is what is referred to as chattering. Chattering results in persistent oscillations and low control accuracy, high heat losses in electrical circuits, and considerable wear and tear of mechanical components such as relays, motors, etc. It may excite unmodeled high-frequency dynamics, which can not only degrade the performance of the system, but may also lead to instability [29].

Continuously-implemented sliding mode control can eliminate chattering because no switching occurs inside the boundary layer—instead, the system will operate under linear feedback within this region. The aforementioned fact necessitates the design of this type of control scheme in two parts, one outside the boundary layer, where we typically seek to make this control identical to sliding mode control, and the other component is the design of the linear feedback control inside the boundary layer. This scheme generally does not recover the equilibrium point of sliding mode control, but instead achieves ultimate boundedness of the trajectories with an ultimate bound proportional to the size of the boundary layer thickness parameter $\mu$. Consequently, this design can guarantee that the trajectories will reach a small set containing the desired equilibrium point, but what happens inside that set is problem dependent. As $\mu$ is made smaller, the ultimate bound decreases, as well, but pushing $\mu$ too small
will result in chattering, which this technique is often chosen to avoid in the first place.

There are certain instances in which the origin of the closed-loop system can be made asymptotically stable under continuously-implemented sliding mode control, provided the parameter $\mu$ is chosen small enough. This is the case for the stabilization and regulation problems studied in Chapters 2 and 3, respectively.

\section*{1.7 Organization}

This dissertation is organized as follows. Chapter 2 presents an output feedback controller design to stabilize the origin of a non-minimum phase system. Chapter 3 deals with the regulation problem for non-minimum phase systems, which is solved by introducing a servocompensator driven by the tracking error and then designing a stabilizing compensator to stabilize the resultant system, in a manner that is analogous to the technique of Davison [12] for linear systems. While each chapter ends with a simulation example, it is always helpful to demonstrate the designs on a non-trivial application example. Consequently, in Chapter 4, the stabilization design of Chapter 2 is implemented on the translational oscillator with a rotating actuator (TORA) benchmark system, and simulation results are provided. Finally, Chapter 5 contains concluding remarks and a discussion of possible future directions of this work.
Chapter 2

Stabilization of Non-Minimum Phase Systems

2.1 Introduction

The problem of stabilizing nonlinear systems via output feedback has drawn much attention in recent years. On the basis of certain assumptions made about the system structure, various control schemes have achieved global and semi-global results. A common assumption in many of these results, however, calls for the zero dynamics of the system under consideration to be either input-to-state stable, or globally asymptotically stable. While stabilization results abound for various classes of minimum phase nonlinear systems, there have been a few notable developments in the case of non-minimum phase systems in recent times, as well. Some examples include the work by Karagiannis et al.[25], which uses a reduced order observer in conjunction with backstepping and the small-gain theorem; Marino and Tomei [34], which provides a stability result on a class of systems that is required to be minimum phase with respect to some linear combination of its states, though it may be non-minimum phase with respect to its output; and also a result by Isidori [21] that guarantees
robust, semiglobal practical stabilization of nonlinear systems in the normal form, which could be non-minimum phase. The work [21] is particularly noteworthy because it provides a simple and very useful design tool for a broad class of nonlinear systems. The basic idea of an auxiliary system derived from the original system to help solve the stabilization problem for the latter forms the basis of this chapter, and was set forth in [21].

In this chapter we arrive at the auxiliary system of [21] from a different perspective. We employ an extended high gain observer (EHGO) as a virtual sensor to estimate a key signal that renders the zero dynamics of a non-minimum phase system observable, thus facilitating the design of a stabilizing compensator. This signal is used to drive a dynamic compensator, which is designed with the objective of making the resulting augmented system, comprising the plant and the compensator, a relative-degree-one minimum-phase system with respect to a virtual output. The stabilization of the augmented system is then pursued by a separation approach that builds on the work of Atassi and Khalil [2]. The design of the compensator is equivalent to the stabilization of the auxiliary system of [21]. The relationship between the stabilization of the auxiliary system and the creation of a relative-degree-one minimum-phase system was exploited in the proof of [21], but the perspective adopted in this chapter opens up new avenues that can be explored further in problems beyond stabilization, such as regulation, for example. It also opens the door for different approaches to the design of the stabilizing controller. While in this chapter we present a continuously-implemented sliding mode controller to stabilize the augmented system, other robust control techniques could also be used. In previous work, the EHGO was used as a disturbance observer [16]. The analysis of the output feedback system in the current paper has some similarities with [16].

While the techniques of both this chapter and [21] rely on the stabilization of the same auxiliary system, the controller designs are different. The paper [21] uses
high-gain feedback to implement the output feedback control in such a way that the closed-loop system can be represented as the feedback connection of an asymptotically stable system with a memoryless high-gain system. In this work, we do not use high-gain feedback. Instead, an EHGO is used to estimate the signals that are needed to implement the state feedback controller. This difference between the two controllers could be significant in some applications because high-gain feedback, by its nature, would cause a large spike in the control signal in the initial transient, which will not be observed with the proposed controller. It is worth mentioning that both controllers use high-gain observers to estimate derivatives of the output from the output measurement. The dimension of the extended observer used in the current paper is higher by one, however.

There are two main technical differences between the results presented in this chapter and those of [21]. First, our design allows for uncertainty in the control coefficient, which was not addressed in [21]. Handling this uncertainty dominates our analysis in Section 2.3. Second, we prove exponential stability of the closed-loop system, under the assumption of exponential stability of the closed-loop auxiliary system. The paper [21] proves only practical stabilization, but it requires the less stringent condition of asymptotic stability of the closed-loop auxiliary system.

### 2.2 Problem Statement

A single-input, single-output system with relative degree $\rho$, under a suitable diffeomorphism, can be expressed in the following normal form [19].

\[
\dot{\eta} = f(\eta, \xi, \theta), \quad (2.1)
\]
\[
\dot{\xi}_i = \xi_{i+1}, \quad 1 \leq i \leq \rho - 1, \quad (2.2)
\]
\[
\dot{\xi}_\rho = b(\eta, \xi, \theta) + a(\eta, \xi, \theta)u, \quad (2.3)
\]
where \( \eta \in D_\eta \subset \mathbb{R}^{n-\rho}, \xi \in D_\xi \subset \mathbb{R}^\rho \), with \( D_\eta \) and \( D_\xi \) being domains containing \( \eta = 0 \) and \( \xi = 0 \) respectively, \( y \) is a measured output, \( \theta \in \Theta \subset \mathbb{R}^p \) is a vector of unknown constant parameters and \( \Theta \) is a compact set.

**Assumption 2.1.** The functions \( a(\cdot) \) and \( b(\cdot) \) are continuous in \( \theta \) and continuously differentiable with respect to \( (\eta, \xi) \), with locally Lipschitz derivatives. The function \( f(\cdot) \) is continuous in \( \theta \) and locally Lipschitz in \( (\eta, \xi) \). In addition, \( a(\eta, \xi, \theta) \neq 0 \), \( b(0, 0, \theta) = 0 \) and \( f(0, 0, \theta) = 0 \), \( \forall \theta \in \Theta \).

In this paper, the objective is to find a robust, stabilizing output feedback controller for systems of the form (2.1)–(2.3), which could potentially include systems with unstable zero dynamics. To this end, we begin by first considering the state feedback case. We shall eventually utilize an extended high gain observer in the output feedback design, and hence for the purposes of the state feedback design, we can assume that the signals \( \xi_1, \ldots, \xi_{\rho-1} \) and

\[
 b(\eta, \xi, \theta) + \Delta a(\eta, \xi, \theta) u,
\]

where \( \Delta a(\eta, \xi, \theta) = a(\eta, \xi, \theta) - a_n(\xi) \) and \( a_n(\xi) \) is a known nominal model of \( a(\eta, \xi, \theta) \), are all available for feedback. During the analysis of the output feedback system, it will be shown that the extended high gain observer recovers the signal given by (2.5) in addition to the output and its first \( \rho - 1 \) derivatives. The expression in (2.5) is based on the anticipation that with the extended high-gain observer, we will be able to estimate this quantity arbitrarily closely. Roughly speaking, this can be seen from (2.3) as follows. If the derivative \( \dot{\xi}_\rho = y^{(\rho)} \) and \( a(\cdot) \) were available, we could calculate \( b \) from the expression \( b = \dot{\xi}_\rho - au \). If only an estimate \( a_n(\cdot) \) were available for the uncertain term \( a(\cdot) \), we would use \( b_n = \dot{\xi}_\rho - a_n u \), which results in the error
\[ b_n - b = (a - a_n)u = \Delta_a u. \]

By requiring \( a_n \) to depend only on \( \xi \), we have set up the problem such that the variables that need to be estimated are the derivatives of the output \( y \) up to the \( \rho \)th order, which will be the role of the extended high-gain observer. Note that if \( a(\cdot) \) were a known function of \( \xi \), we could take \( a_n = a \).

We require the following assumption about \( a(\cdot) \) and \( a_n(\cdot) \).

**Assumption 2.2.** Suppose that \( a(\eta, \xi, \theta)/a_n(\xi) \geq k_0 > 0 \), for all \( (\eta, \xi) \in D_\eta \times D_\xi \) and \( \theta \in \Theta \). Moreover, \( a_n(\xi) \) is locally Lipschitz in \( \xi \) over \( D_\xi \) and globally bounded in \( \xi \).

### 2.3 Controller Design for the State Feedback System

In the case of minimum phase systems, the use of high-gain observers allows us to reduce the problem of stabilizing a system with relative degree higher than one to one of stabilizing a relative degree one system. This idea is explicated in [33]. For instance, we could define a virtual output \( s = \xi_\rho + \sum_{i=1}^{\rho-1} k_i \xi_i \), where the constants \( k_i \) are chosen such that the system with output \( s \) is minimum phase. Then, a static stabilizing control law utilizing the virtual output \( s \) could be designed using standard techniques, and this design could then be implemented by replacing the virtual output \( s \) with its estimate from a high-gain observer. This definition of the virtual output will not work if the system (2.1)–(2.4) is non-minimum phase. However, with an extended high-gain observer, we will have access to the states \( \xi_1 \) to \( \xi_\rho \) in addition to the signal \( b(\eta, \xi, \theta) + \Delta_a(\eta, \xi, \theta)u \), which can all be used in the state feedback design.

We augment the plant (2.1)–(2.4) with the dynamic compensator

\[ \dot{\phi} = L(\phi, \xi_1, \ldots, \xi_{\rho-1}) + M(\phi, \xi_1, \ldots, \xi_{\rho-1})[b(\eta, \xi, \theta) + \Delta_a(\eta, \xi, \theta)u], \quad (2.6) \]
\[ s = \xi_\rho - N(\phi, \xi_1, \ldots, \xi_{\rho-1}), \]  

(2.7)

where \( \phi \in \mathbb{R}^r \). The augmented system will have relative degree one with respect to the virtual output \( s \) if

\[ 1 - \frac{\partial N}{\partial \phi} M \frac{\Delta a}{a} \geq k^\dagger > 0, \]

(2.8)

where \( k^\dagger \) is some real constant. Our task now is to design \( L(\cdot), M(\cdot), \) and \( N(\cdot) \) such that the augmented system is minimum phase. It turns out that this task is equivalent to the stabilization of the auxiliary system of [21], as we shall see in the development to follow.

With the change of variables \( \xi_\rho \mapsto s \), the augmented system is given by

\[ \dot{\eta} = f(\eta, \xi_1, \ldots, \xi_{\rho-1}, s + N(\phi, \xi_1, \ldots, \xi_{\rho-1}), \theta), \]  

(2.9a)

\[ \dot{\xi}_1 = \xi_2, \]  

(2.9b)

\[ \vdots \]  

(2.9c)

\[ \dot{\xi}_{\rho-1} = s + N(\phi, \xi_1, \ldots, \xi_{\rho-1}), \]  

(2.9d)

\[ \dot{\phi} = M(\phi, \xi_1, \ldots, \xi_{\rho-1}) \left[ b(\eta, \xi_1, \ldots, \xi_{\rho-1}, s + N(\cdot, \theta)) \right. \]

\[ + \left. \Delta a(\eta, \xi_1, \ldots, \xi_{\rho-1}, s + N(\cdot, \theta)) \right] + L(\phi, \xi_1, \ldots, \xi_{\rho-1}), \]  

(2.9e)

\[ \dot{s} = b(\eta, \xi_1, \ldots, \xi_{\rho-1}, s + N(\cdot, \theta)) - \frac{\partial N}{\partial \phi} \left[ L(\cdot) + M(\cdot)(b(\cdot) + \Delta a(\cdot)u) \right] \]

\[ + a(\cdot)u - \frac{\partial N}{\partial \xi_1} \xi_2 - \ldots - \frac{\partial N}{\partial \xi_{\rho-2}} \xi_{\rho-1} - \frac{\partial N}{\partial \xi_{\rho-1}}[s + N(\cdot)]. \]  

(2.9f)

We now set

\[ x_a = \left( \eta^\top \quad \xi_1 \quad \ldots \quad \xi_{\rho-2} \quad \xi_{\rho-1} \right)^\top, \]

\[ f_b(x_a, \phi, s, \theta) = b - \frac{\partial N}{\partial \phi}(L + Mb) - \frac{\partial N}{\partial \xi_{\rho-1}}(s + N(\cdot)) - \frac{\partial N}{\partial \xi_{\rho-2}} \xi_{\rho-1} - \ldots - \frac{\partial N}{\partial \xi_1} \xi_2, \]

\[ g_b(x_a, \phi, s, \theta) = a - \frac{\partial N}{\partial \phi} M \Delta a, \]
Due to Assumption 2.2 and (2.8), we have

$$g_b/a_n \geq k^\top k_0 > 0.$$  \hspace{1cm} (2.10)

The system equations can be rewritten in a compact form as follows.

$$\dot{x}_a = f_a(x_a, \phi, s, \theta),$$ \hspace{1cm} (2.11)

$$\dot{\phi} = L(x_a, \phi) + M(x_a, \phi)[b(x_a, \phi, s, \theta) + \Delta_a(x_a, \phi, s, \theta)u],$$ \hspace{1cm} (2.12)

$$\dot{s} = f_b(x_a, \phi, s, \theta) + g_b(x_a, \phi, s, \theta)u.$$ \hspace{1cm} (2.13)

With $s$ viewed as the output, this system has relative degree one. The internal dynamics are given by (2.11), (2.12). However, the system is not in the normal form due to the term $M \Delta_a u$ in (2.12). We thus apply a change of variables that eliminates the control input from (2.12).

Let $T: \phi \mapsto q$ such that

$$\frac{\partial T}{\partial \phi} M \Delta_a + \frac{\partial T}{\partial s} g_b = 0 \hspace{1cm} (2.14)$$

and $\left( x_a^\top \hspace{1cm} (T(x_a, \phi, s, \theta))^\top \hspace{1cm} s \right)^\top$ is a diffeomorphism of $\left( x_a^\top \hspace{1cm} \phi^\top \hspace{1cm} s \right)^\top$. When $\Delta_a = 0$, we have $T = \phi$. Hence, for sufficiently small $|\Delta_a|$, it can be argued that a function $T$ exists [11]. Let $\phi = P(x_a, q, s, \theta)$ be the inverse transformation. Due to
the continuous dependence of the solution of (2.14) on $\Delta a$ [11], we can write $T$ and $P$ as

$$T = \phi + \Phi(x_a, \phi, s, \theta),$$

$$P = q + Q(x_a, q, s, \theta),$$

where $\Phi$ and $Q$ are of the order of $|\Delta a|$. The transformed system equations due to (2.14) are given by

$$\dot{x}_a = f_a(x_a, q + Q(x_a, q, s, \theta), s, \theta),$$

$$\dot{q} = L(x_a, q + Q(x_a, q, s, \theta)) + M(x_a, q + Q(x_a, q, s, \theta))b(x_a, q + Q(x_a, q, s, \theta), s, \theta)$$

$$+ \frac{\partial \Phi}{\partial x_a} f_a(x_a, q + Q(x_a, q, s, \theta), s, \theta) + \frac{\partial \Phi}{\partial \phi} [L(\cdot) + M(\cdot)b(\cdot)]$$

$$+ \frac{\partial \Phi}{\partial s} f_b(x_a, q + Q(x_a, q, s, \theta), s, \theta),$$

$$\dot{s} = g_b(x_a, q + Q(x_a, q, s, \theta), s, \theta)u + f_b(x_a, q + Q(x_a, q, s, \theta), s, \theta).$$

When $\Delta a = 0$, (2.11)–(2.13) simplify to

$$\dot{x}_a = f_a(x_a, \phi, s, \theta),$$

$$\dot{\phi} = L(x_a, \phi) + M(x_a, \phi)b(x_a, \phi, s, \theta),$$

$$\dot{s} = f_b(x_a, \phi, s, \theta) + a(x_a, \phi, s, \theta)u.$$

Let

$$x_s \triangleq \begin{pmatrix} x_a^\top & q^\top \end{pmatrix}^\top,$$

and

$$F(x_s, s, \theta) = \begin{pmatrix} f_a(x_a, q + Q(x_a, q, s, \theta), s, \theta) \\ L + Mb + \frac{\partial \Phi}{\partial x_a} f_a + \frac{\partial \Phi}{\partial \phi} (L + Mb) + \frac{\partial \Phi}{\partial s} f_b \end{pmatrix}.$$ 

Since $x_s$ includes the states $(\xi_1, \ldots, \xi_{\rho-1}, q)$, we write $L(\cdot)$, $M(\cdot)$ and $N(\cdot)$ as $L(x_s)$,

---

1 See Section A.2 for details regarding the existence of solutions to equation (2.14), and continuous dependence of the former on the parameters of the latter.
$M(x_s)$ and $N(x_s)$. The internal dynamics (2.15)–(2.16) can now be written as

$$\dot{x}_s = F(x_s, s, \theta).$$

(2.20)

Our objective is to design the compensator \{L, M, N\} such that the internal dynamics (2.20) are input to state stable with $s$ viewed as an input. The control law based on continuously-implemented sliding mode control is taken to be

$$u = -\frac{\beta(\xi, \phi)}{a_n(\xi)} \text{sat} \left(\frac{s}{\mu}\right),$$

(2.21)

where $\beta(\cdot)$ is to be chosen. The closed-loop state feedback system is given by (2.13), (2.20), (2.21). We now make the following assumptions.

**Assumption 2.3.** Let $D \subset \mathbb{R}^{n+r}$ be a domain containing the origin of the system (2.13), (2.20), (2.21) such that $(\eta, \xi, \phi) \in D_{\eta} \times D_{\xi} \times D_{\phi} \Rightarrow (x_s, s) \in D$. Moreover, suppose there exists a continuously differentiable Lyapunov function $V(x_s, \theta)$ such that

$$\alpha_1(\|x_s\|) \leq V(x_s, \theta) \leq \alpha_2(\|x_s\|),$$

(2.22)

$$\frac{\partial V}{\partial x_s} F(x_s, s, \theta) \leq -\alpha_3(\|x_s\|), \quad \forall \|x_s\| \geq \gamma(|s|),$$

(2.23)

for all $(x_s, s) \in D \subset \mathbb{R}^{n+r}$, where $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ and $\gamma(\cdot)$ are class $\mathcal{K}$ functions.

**Assumption 2.4.**

1) $L(x_s), M(x_s)$ and $\beta(\xi, \phi)$ are locally Lipschitz functions in their arguments over the domain of interest, and $L(0) = 0$.

2) $L(x_s), M(x_s)$ and $\beta(\xi, \phi)$ are globally bounded functions of $\xi$.

3) With $s = 0$, the origin $x_s = 0$ of (2.20) is locally exponentially stable.
Remark 2.1.

1) The set $D$ is well-defined for any sufficiently smooth $N(\cdot)$ in the definition of the variable $s = \xi - N(\cdot)$.

2) Inequality (2.23) is equivalent to regional input-to-state stability of the system (2.20) with $s$ viewed as an input.

Let us now consider the set

$$\Omega \triangleq \{ V(x_s, \theta) \leq c_0 \} \times \{|s| \leq c\}, \quad (2.24)$$

where $c > \mu$, $c_0 \geq \alpha_2(\gamma(c))$, and $c, c_0$ are chosen such that $\Omega$ is a compact subset of $D$. As in the development preceding [29, Theorem 14.1 (§14.1.2)], we show that $\Omega$ is a positively invariant set for the system (2.20), and can thus serve as an estimate of the region of attraction.

We now have

$$s\dot{s} = s \left[ f_b(x_a, \phi, s, \theta) - \frac{g_b(x_a, \phi, s, \theta)}{a_n(\xi)} \beta(\xi, \phi) \text{sat} \left( \frac{s}{\mu} \right) \right].$$

$\beta(\cdot)$ is chosen such that

$$\beta(\xi, \phi) \geq \beta_0 + \left| \frac{a_n(\xi)f_b(x_a, \phi, s, \theta)}{g_b(x_a, \phi, s, \theta)} \right|, \quad \beta_0 > 0. \quad (2.25)$$

Hence, for $s \geq \mu$, we have

$$s\dot{s} = sf_b - (g_b/a_n)\beta |s| \leq (g_b/a_n)(|a_nf_b/g_b| - \beta) |s|$$

$$\leq -(g_b/a_n)\beta_0 |s| \leq -k^\dagger k_0 \beta_0 |s|.$$ 

Hence, whenever $|s(0)| > \mu$, $|s(t)|$ will decrease until it reaches the set $\{|s| \leq \mu\}$ in finite time and will remain inside thereafter.
We would now like to study the behavior of the remaining states given by (2.20). In order to proceed with the analysis, we utilize Assumption 2.3, which calls for the existence of a Lyapunov function for the internal dynamics. The derivative of $V$ along the trajectories of (2.20) is given by

$$
\dot{V} = \frac{\partial V}{\partial x_s} F(x_s, s, \theta).
$$

Thus, on the boundary $V = c_0$, we have

$$
\dot{V} = \frac{\partial V}{\partial x_s} F(x_s, s, \theta) \leq -\alpha_3(\|x_s\|) \leq -\alpha_3(\frac{1}{2}(c_0)).
$$

Hence, $\Omega$ is a positively invariant set for the system (2.20).

As a consequence of the preceding analysis, all trajectories starting in $\Omega$, stay in $\Omega$ and reach the set $\Omega \cap \{|s| \leq \mu\}$ in finite time. Inside this set, we have

$$
\dot{V} \leq -\alpha_3(\|x_s\|), \quad \forall \|x_s\| \geq \gamma(\mu).
$$

The above analysis shows that the trajectories enter the positively invariant set

$$
\Omega_\mu = \{V(x_s, \theta) \leq \alpha_2(\gamma(\mu))\} \times \{|s| \leq \mu\}
$$

in finite time. Now, inside the set $\Omega_\mu$, the closed-loop system is given by the following equations in the singularly perturbed form.

$$
\dot{x}_a = f_a, \quad (2.26a)
$$

$$
\dot{q} = L + Mb + \frac{\partial \Phi}{\partial x_a} f_a + \frac{\partial \Phi}{\partial \phi} (L + Mb) + \frac{\partial \Phi}{\partial s} f_b, \quad (2.26b)
$$

$$
\mu \dot{s} = \mu f_b - \frac{g_b}{a_n} \beta s. \quad (2.26c)
$$
We can use standard singular perturbation arguments [29, Chapter 11] to show that the origin of (2.26) is exponentially stable for sufficiently small $\mu$, with a region of attraction independent of $\mu$—the details are provided in the Appendix. Hence, for sufficiently small $\mu$, all trajectories in $\Omega_\mu$ converge to the origin. We summarize this result in the following proposition.

**Proposition 2.1.** Consider the closed-loop system given by (2.13), (2.20), (2.21). Let Assumptions 2.1 through 2.4 hold. Then, there exists a positive constant $\mu^*$ such that the origin of (2.13), (2.20), (2.21) is exponentially stable for all $0 < \mu < \mu^*$. Moreover, $\Omega$, given by (2.24), is an estimate of the region of attraction.

**Proof.** See Section A.3 for details of the proof. \qed

**Remark 2.2.**

1) In [21], the closed-loop auxiliary system (which corresponds to the internal dynamics (2.20) in our paper, with $s = 0$ and $\Delta_a = 0$) was assumed to be asymptotically stable, and consequently, Theorem 1 of that paper establishes practical stabilization for the state feedback system. This is generally the result in the event that the state feedback system fails to satisfy certain interconnection conditions. If assumptions are made such that these additional conditions are satisfied, then the state feedback system could be shown to be asymptotically stable as well [29].

2) Assumption 2.3 requires the compensator $\{L, M, N\}$ to be designed such that (2.20) is input to state stable with $s$ viewed as the input. This problem is identical to the auxiliary design problem of [21] when $\Delta_a = 0$. In [21], the interconnection of the auxiliary system with its dynamic compensator was assumed to be globally asymptotically stable, with a Lyapunov function independent of the unknown system parameters. The fulfillment of this assumption is still an open problem, but as shown in [21, Proposition 2], in the case of linear systems, the stabilizability
and detectability of the original system is a sufficient condition for the existence of a stabilizing controller for the auxiliary system.

3) We assumed local exponential stability of the origin of the internal dynamics (2.20) in Assumption 2.4, and this in conjunction with the regularity conditions given by Assumptions 2.1 and 2.4 allowed for us to prove exponential stability of the origin of the closed-loop state feedback system in Theorem 2.1.

4) If all the assumptions hold globally, then we could take \( D = \mathbb{R}^{n+r} \), and with the appropriate choice of controller parameters, the positively invariant set \( \Omega \subset D \) that depends on these parameters can also be made arbitrarily large, and thus our design allows for semi-global stabilization of the state feedback system.

2.4 Output Feedback Control Using an Extended High-Gain Observer

2.4.1 Output Feedback Controller Design

The design of the output feedback controller in this paper closely follows the work by Freidovich and Khalil [16]. It relies upon the estimates of the states and of \( b(\eta, \xi, \theta) \) that are obtained by using an extended high-gain observer for the system (2.1)–(2.3), which is taken as

\[
\dot{\hat{\xi}}_i = \dot{\hat{\xi}}_{i+1} + \left( \frac{\alpha_i}{\varepsilon^i} \right) (\xi_1 - \dot{\hat{\xi}}_1), \quad 1 \leq i \leq \rho - 1, \tag{2.27}
\]

\[
\dot{\hat{\xi}}_\rho = \dot{\sigma} + a_n(\hat{\xi}) u + \left( \frac{\alpha_\rho}{\varepsilon^\rho} \right) (\xi_1 - \dot{\hat{\xi}}_1), \tag{2.28}
\]

\[
\dot{\hat{\sigma}} = \left( \frac{\alpha_{\rho+1}}{\varepsilon^\rho+1} \right) (\xi_1 - \dot{\hat{\xi}}_1), \tag{2.29}
\]
where \( \varepsilon \) is a positive constant to be specified, and the positive constants \( \alpha_i \) are chosen such that the roots of \( \lambda^{\rho+1} + \alpha_1 \lambda^{\rho} + \ldots + \alpha_{\rho-1} \lambda + \alpha_{\rho} = 0 \) are in the open left-half plane. It is apparent from (2.27)–(2.29) that the \( \hat{\xi}_1, \ldots, \hat{\xi}_\rho \) are used to estimate the output and its first \( \rho - 1 \) derivatives, while \( \hat{\sigma} \) is intended to provide an estimate for \( b(\cdot) \). With the aid of these estimates, the output feedback controller for the original system can be taken as

\[
\dot{\phi} = L(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) + M(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) \hat{\sigma},
\]

(2.30)

\[
l(\hat{\xi}, \phi) = \frac{\beta(\hat{\xi}, \phi)}{a_n(\hat{\xi})} \text{sat} \left( \frac{\hat{\xi}_{\rho} - N(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1})}{\mu} \right),
\]

(2.31)

\[
u = K \text{sat}(l(\hat{\xi}, \phi)/K),
\]

(2.32)

where

\[
K > \max_{(x_s, s) \in \Omega} \left| \frac{\beta(\xi, \phi)}{a_n(\xi)} \right|.
\]

(2.33)

The control is saturated at \( \pm K \) outside the compact set of interest in order to protect the system from peaking during the observer’s transient response.\(^2\)

The closed loop system under output feedback can now be expressed as

\[
\dot{\eta} = f(\eta, \xi, \theta),
\]

(2.34)

\[
\dot{\xi} = A \xi + B \left[ b(\eta, \xi, \theta) + a(\eta, \xi, \theta) K \text{sat}(l(\hat{\xi}, \phi)/K) \right],
\]

(2.35)

\[
\dot{\phi} = L(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) + M(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) \hat{\sigma},
\]

(2.36)

\[
\dot{\hat{\xi}} = A \hat{\xi} + B \left[ \hat{\sigma} + a_n(\hat{\xi}) K \text{sat}(l(\hat{\xi}, \phi)/K) \right] + H(\varepsilon)(y - C \hat{\xi}),
\]

(2.37)

\[
\dot{\hat{\sigma}} = \left( \alpha_{\rho+1}/\varepsilon^{\rho+1} \right) (y - C \hat{\xi}),
\]

(2.38)

\[
y = C \xi,
\]

(2.39)

\(^{\text{2}}\text{See [14] for the role played by saturation in overcoming the effect of peaking.}\)
where

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
1 & 0 & \cdots & 0
\end{pmatrix},
\]

\[
H(\varepsilon) = \begin{pmatrix}
(\alpha_1/\varepsilon) \\
(\alpha_2/\varepsilon^2) \\
\cdots \\
(\alpha_\rho/\varepsilon^\rho)
\end{pmatrix}^\top.
\]

We now introduce a change of variables [16]

\[
\nu_i \triangleq (\xi_i - \hat{\xi}_i)/\varepsilon^{i+1}, \quad 1 \leq i \leq \rho,
\]

\[
\nu_{\rho+1} \triangleq \Delta_a(\varepsilon, \eta, \xi, \nu, \theta)Kg_\varepsilon(l(\xi, \phi)/K) + b(\eta, \xi, \theta) - \hat{\sigma},
\]

where \(\Delta_a(\varepsilon, \eta, \xi, \nu, \theta) = a(\eta, \xi, \theta) - a_n(\hat{\xi})\), and \(g_\varepsilon(\cdot)\) is an odd function defined by [16]

\[
g_\varepsilon(y) = \begin{cases}
y, & 0 \leq y \leq 1, \\
y + (y - 1)/\varepsilon - 0.5(y^2 - 1)/\varepsilon, & 1 < y < 1 + \varepsilon, \\
1 + 0.5\varepsilon, & y \geq 1 + \varepsilon.
\end{cases}
\]

The function \(g_\varepsilon(\cdot)\) is nondecreasing, continuously differentiable with a locally Lipschitz derivative, bounded uniformly in \(\varepsilon\) on any bounded interval of \(\varepsilon\), and satisfies

\[
|g_\varepsilon'| \leq 1 \quad \text{and} \quad |g_\varepsilon(y) - \text{sat}(y)| \leq \varepsilon/2
\]

for all \(y \in \mathbb{R}\). The closed loop system can now be expressed in terms of (2.34)–(2.36),
\[ (2.39) \]

\[ \varepsilon \dot{\nu} = \varepsilon [\bar{B}_1 \Delta_1(\xi, \eta, \nu, \phi, \theta, \varepsilon) + \bar{B}_2 \Delta_2(\xi, \eta, \nu, \phi, \theta, \varepsilon)] + \Lambda \nu, \tag{2.42} \]

where

\[
\Lambda = \begin{pmatrix}
-\alpha_1 & 1 & 0 & \cdots & 0 \\
-\alpha_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_\rho & 0 & 0 & \cdots & 1 \\
-\alpha_{\rho+1} & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \bar{B}_1 = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad \bar{B}_2 = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \Delta_2(\xi, \eta, \nu, \phi, \theta, \varepsilon) = \Delta_0/\varepsilon,
\]

\[
\Delta_0(\xi, \eta, \nu, \phi, \theta, \varepsilon) =
\]

\[
a(\eta, \xi, \theta)K[g_\varepsilon(l(\hat{\xi}, \phi)/K) - g_\varepsilon(l(\xi, \phi)/K)]
\]

\[
+ a_n(\xi)Kg_\varepsilon(l(\xi, \phi)/K) - a_n(\hat{\xi})Kg_\varepsilon(l(\xi, \phi)/K)
\]

\[
+ [a(\eta, \xi, \theta) - a_n(\hat{\xi})]K[sat(l(\hat{\xi}, \phi)/K) - g_\varepsilon(l(\hat{\xi}, \phi)/K)],
\]

\[
\Delta_1(\xi, \eta, \nu, \phi, \theta, \varepsilon) =
\]

\[
b + \hat{\Delta}_aKg_\varepsilon(l(\xi, \phi)/K) + \Delta_2g_\varepsilon(l(\xi, \phi)/K)\left\{ \frac{\partial l(\xi, \phi)}{\partial \xi} \hat{\xi} + \frac{\partial l(\xi, \phi)}{\partial \phi} \left[ L(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) + \left( \Delta_2Kg_\varepsilon\left( \frac{l(\xi, \phi)}{K} \right) \right) + b - \nu_{\rho+1} \right] M(\phi, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) \right\}.
\]

We note that the functions \( \Delta_1 \) and \( \Delta_2 \) are locally Lipschitz in their arguments and bounded from above by affine-in-\( ||\eta|| \) functions, uniformly in \( \varepsilon \). This is because \( a_n(\cdot) \) and \( g_\varepsilon(\cdot) \) are continuously differentiable with locally Lipschitz derivatives and globally bounded. Moreover, we can use (2.40) and the definition of \( g_\varepsilon \) to show that \( \Delta_0/\varepsilon \) is
locally Lipschitz [16]. The slow dynamics can be written as

\[
\dot{x}_o = \begin{pmatrix} \dot{x}_s \\ \dot{s} \end{pmatrix} = f_r(x_o, \nu, \varepsilon, \theta),
\]

(2.43)

where \( x_o = \begin{pmatrix} x_s^T \\ s \end{pmatrix}^T \),

\[
f_r(x_o, \nu, \varepsilon, \theta) = \begin{pmatrix}
\phi(\eta, \xi, \theta) \\
\xi_2 \\
\vdots \\
\xi_\rho \\
f_1(\xi, \eta, \nu, \phi, \theta, \varepsilon) \\
b(\cdot) - \frac{\partial N}{\partial x_s} F(x_s, s, \theta) + a(\cdot) K \text{sat}(l(\hat{\xi}, \phi)/K)
\end{pmatrix},
\]

\[
f_1(\cdot) = M(\cdot)[b + \Delta_a(\varepsilon, \eta, \xi, \nu, \theta) K \varepsilon \text{sat}(l(\xi, \phi)/K)] + L(\cdot) - M(\cdot) \nu_{\rho+1} + F \quad \text{are defined as in (2.18), (2.19), and \( \hat{x}_s = \begin{pmatrix} \eta^T \\
\hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1} \\
q^T \end{pmatrix}\). Then, the reduced system is obtained by setting \( \varepsilon = 0 \) in (2.43), which yields

\[
\dot{x}_o = f_r(x_o, 0, 0, \theta) = \begin{pmatrix} F(x_s, s, \theta) \\
f_2(\xi, \eta, \nu, \phi, \theta, \varepsilon)
\end{pmatrix},
\]

(2.44)

where \( f_2(\xi, \eta, \nu, \phi, \theta, \varepsilon) = b(\eta, \xi, \theta) - \frac{\partial N}{\partial x_s} F(x_s, s, \theta) + a(\eta, \xi, \theta) K \text{sat}(l(\xi, \phi)/K) \). We note that in the set \( \Omega \) given by (2.24), the reduced system (2.44) is identical to the closed-loop state feedback system (2.13), (2.20), (2.21), which we know to have an exponentially stable equilibrium point at its origin.

\(^3\)See Section A.1 for details.
The dynamics of the fast variable $\nu$ are given by the boundary layer system

$$\frac{\partial \nu}{\partial \tau} = \Lambda \nu, \quad \tau \triangleq \frac{t}{\varepsilon},$$

which we note is exponentially stable.

In the subsection below, we state some results pertaining to the recovery of the performance of the system (2.13), (2.20) in the case of output feedback with sufficiently small $\varepsilon$. These results follow directly from the work of Atassi and Khalil [2] because equations (2.34)–(2.36) and (2.42) are precisely of the same form as equations (10)–(13) in [2]. It should also be noted that while the structure of system (2.34)–(2.36), (2.42) is very similar to that of the closed-loop system studied by Freidovich and Khalil [16], a key difference between this particular work and the former is the fact that both the nonlinear terms in the fast dynamics (2.42) are $O(\varepsilon)$ in this work, while there is one term in [16] that is not so. The reason for such a term to appear in [16] was the fact that the estimate $\hat{\sigma}$ was used in the control law, whereas in this work, this quantity is not utilized directly in the control. We can see this by examining (2.32) and (2.36)—$\hat{\sigma}$ appears in the compensator dynamics, but $u$ is not explicitly dependent upon it.

### 2.4.2 Exponential Stability and Trajectory Convergence

Let $(x_0(t, \varepsilon), \nu(t, \varepsilon))$ denote the trajectory of the system (2.42), (2.43) starting from $(x_0(0), \nu(0))$. Also, let $x_T(t)$ be the solution of (2.44) starting from $x_0(0)$. We know that the system (2.13), (2.20), (2.21) has an exponentially stable equilibrium point, so let us suppose that $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be a compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be a compact subset of $\mathbb{R}^{\rho+1}$. The recovery of exponential stability and the fact that $x_0(t, \varepsilon)$ converges to $x_T(t)$ as $\varepsilon \to 0$, uniformly in $t$, for all $t \geq 0$, is established by the following theorem [2, Theorems 1, 2, 3 and 5], provided $x_0(0) \in \mathcal{S}$
and \(\begin{pmatrix} \hat{\xi}(0)^\top & \hat{\sigma}(0) \end{pmatrix}^\top \in \mathcal{Q}\).

**Theorem 2.1.** Let Assumptions 2.1 through 2.4 hold, and suppose the vector field

\[ f_T(x_o, 0, 0, \theta) \]

is continuously differentiable around the origin. Moreover, assume \(x_o(0) \in \mathcal{S}\) and \(\begin{pmatrix} \hat{\xi}(0)^\top & \hat{\sigma}(0) \end{pmatrix}^\top \in \mathcal{Q}\). Then,

1) there exists \(\varepsilon^{*}_1 > 0\) such that, for every \(0 < \varepsilon \leq \varepsilon^{*}_1\), the origin of system (2.43), (2.42) is exponentially stable, and \(\mathcal{S} \times \mathcal{Q}\) is included in the region of attraction.

2) given any \(\delta > 0\), there exists \(\varepsilon^{*}_2 > 0\) such that, for every \(0 < \varepsilon \leq \varepsilon^{*}_2\), we have

\[
\|x_o(t, \varepsilon) - x_r(t)\| \leq \delta, \quad \forall t \geq 0.
\]

**Remark 2.3.** Theorem 1 of [2] establishes boundedness of \((x_o, \nu)\) for all trajectories starting in \(\mathcal{S} \times \mathcal{Q}\). Theorem 2 of [2] shows that the trajectories are ultimately bounded, and the ultimate bound can be made arbitrarily small by choosing \(\varepsilon\) small enough. Finally, Theorem 5 of [2] establishes that the origin is exponentially stable with a region of attraction that is independent of \(\varepsilon\). The combination of the three results yields the first bullet of the foregoing theorem. The second bullet follows from Theorem 3 of [2].

### 2.5 Example

We illustrate the design procedure presented in this chapter by using a second order system as an example. This system has the following structure.

\[
\dot{x}_1 = \tan x_1 + x_2,
\]

(2.47)
\[
\dot{x}_2 = x_1 + u, \quad (2.48) \\
y = x_2. \quad (2.49)
\]

We note that this system has a relative degree of one, and is already in the normal form. We can thus write the auxiliary system as

\[
\dot{x}_1 = \tan x_1 + u_a, \quad (2.50) \\
y_a = x_1. \quad (2.51)
\]

The design of the stabilizing controller is now carried out by first considering the auxiliary system and proceeding in a step-by-step fashion, in accordance with the procedure detailed in Sections 2.3 and 2.4.

### 2.5.1 Design of the Stabilizing Controller

We begin by noting that we could linearize the closed-loop auxiliary system equation with the choice of the input \( u_a = -\tan y_a - y_a \), resulting in a stable system. However, the design procedure requires that \( y_a \) serve as an input to a dynamic controller, and that the control input \( u_a \) not be driven directly by \( y_a \)—a static output feedback law is therefore unsuitable. Hence, a dynamic compensator for the auxiliary system (2.50), (2.51) is designed using a low-pass filter, and feedback linearization is in effect achieved by utilizing the filter’s output. This gives the controller

\[
\dot{\phi} = (y_a - \phi)/\varepsilon_a, \quad (2.52) \\
u_a = -\tan \phi - \phi. \quad (2.53)
\]

A choice of controller parameter \( \varepsilon_a = 0.1 \) yields closed-loop auxiliary system with an exponentially stable equilibrium point at the origin. Now, based on our knowledge
of the stabilizing dynamic controller (2.52), (2.53) for the auxiliary system (2.50), (2.51), we choose our sliding manifold for the partial state feedback system, and the resulting controller is given by

\[ s = x_2 + \phi + \tan \phi, \]  
(2.54) \[ \dot{\phi} = (x_1 - \phi)/\varepsilon a, \]  
(2.55) \[ u = -\frac{\beta}{a_n} \text{sat} \left( \frac{s}{\mu} \right), \]  
(2.56)

where the constant \( \beta \) satisfies the inequality (2.25), and \( a_n \) is a nominal value of the control coefficient \( a = 1 \) in the original plant.

To stabilize the system using only output feedback, we utilize the following extended high-gain observer.

\[ \dot{\hat{x}}_2 = \hat{\sigma} + a_n u + (\alpha_1/\varepsilon)(y - \hat{x}_2), \]  
(2.57) \[ \dot{\hat{\sigma}} = (\alpha_2/\varepsilon^2)(y - \hat{x}_2). \]  
(2.58)

Hence, the extended-high-gain-observer-based output feedback controller is given by

\[ \dot{\phi} = \frac{0.6 \text{sat} (\frac{\hat{\sigma} - \phi}{0.6\varepsilon a})}{\text{sat} \left( \frac{y + \phi + \tan \phi}{\mu} \right).} \]  
(2.59) \[ u = -\frac{\beta}{a_n} \text{sat} \left( \frac{s}{\mu} \right). \]  
(2.60)

The expression on the right-hand side of (2.59) was saturated at \( \pm 0.6 \), and these values were determined by means of simulation. This was done because Theorem 2.1 (Assumption 2.4) requires that the functions \( L(\cdot) \) and \( M(\cdot) \) be globally bounded.

In the following subsection, the above design is compared via simulation with a control design performed according to the high-gain feedback method of Isidori [21]. It is based on the same controller (2.52)–(2.53) for the auxiliary system as the one
designed above. Thus, the output feedback controller for the full system is given by

\[
\dot{\phi} = \left[ k(y + \tan \phi + \phi) - \phi \right]/\varepsilon a, \tag{2.61}
\]

\[
u = \left\{ \left( 2 + \tan^2 \phi \right)[\phi - k(y + \tan \phi + \phi)]/\varepsilon a - k(y + \tan \phi + \phi) \right\}/a_n. \tag{2.62}
\]

We note that since the controller is driven only by the output, an observer is not required in this method.
2.5.2 Numerical Simulations

The step-by-step tuning procedure leading to performance recovery is illustrated by Figures 2.1 and 2.2. The constant $\beta$ was chosen to be 8. The compensator parameter $\varepsilon_a$ was fixed at 0.1 and the width of the boundary layer was reduced in the partial state feedback system from $\mu = 1$ down to 0.1, and Figure 2.1 shows the recovery of the auxiliary system performance. Next, $\mu$ was fixed at 0.1 and the extended high-gain observer parameter $\varepsilon$ was reduced from $\varepsilon = 10^{-2}$ to $10^{-3}$, and Figure 2.2 shows the recovery of the performance of the state feedback system.

Next, in order to compare the performance of the above design with that of Isidori’s, the design parameters for each method were chosen such that the initial
condition satisfies

$$(x_1(0), y(0), \phi(0)) \in I = [-0.5, 0.5] \times [-0.1, 0.1] \times \{0\}.$$ 

To this end, the high-gain control parameter that appears in Isidori’s design was chosen to be $k = 12$. Figure 2.3 and Figure 2.4 show how the performance compares between Isidori’s method and the output feedback controller. Upon comparing the control efforts in Figure 2.4, it is apparent that the initial transient response of Isidori’s design causes $u$ to start out at a value of about 25, while in the EHGO-based SMC design, it starts out at 8. Subsequently, however, the overshoot and maximum rate of change are about the same in each case. This fact is used as the basis for performance
comparison between the two designs. It is apparent from Figures 2.3 through 2.6 that the design procedure of this paper was able to provide better transient response than the high-gain feedback approach. Figure 2.5 shows a comparison between the two methods when the nominal value of the control coefficient is $a_n = 0.86$, while in Figure 2.6, it is $a_n = 1.04$. The EHGO-based SMC design is able to tolerate a slightly wider range of uncertainties, and Figures 2.7 and 2.8 show the simulation results for $a_n = 0.76$ and $a_n = 1.05$, respectively. We note that the settling time in Figure 2.7 is much longer than in all the other cases.
Figure 2.5: A comparison between Isidori’s (dash-dot) and the EHGO-based (solid) design for $a_n = 0.86$.

2.6 Conclusion

A robust, stabilizing output feedback controller for systems in the normal form, which could potentially include unstable zero dynamics, was presented. The control scheme adopted herein incorporated continuously-implemented sliding mode control—chosen for its robustness properties as well as its ability to prescribe or constrain the motion of trajectories in the sliding phase—and an extended high gain observer to estimate one of the unknown functions. Stabilization in the case of an unknown control coefficient and uncertain constant parameters was shown for the state as well as output feedback cases.

The control design presented in this chapter is an alternative to the design of [21].
A main contribution of this chapter is that it shows how a non-minimum phase plant can be stabilized despite uncertainty in the control coefficient $a(\cdot)$, and it demonstrates the potential of the EHGO acting as a virtual sensor that can be used to estimate signals whose knowledge is necessary for the successful realization of the aforementioned control task. Although sliding mode control was used in this work, the controller design is introduced in Section 2.3 in such a way that the technique can be extended to other control schemes quite easily, and the EHGO-as-a-virtual-sensor concept can be applied to a broader class of problems, as well. The paper by Isidori [21] assumes asymptotic stability of the origin of the auxiliary system, and it proves practical stabilization of the output feedback system by means of a dynamic high-gain feedback based controller. In this work, the auxiliary system is assumed to be locally
exponentially stable. With the aid of some regularity assumptions, it was shown that an extended high-gain observer based dynamic, continuously-implemented sliding mode controller is able to exponentially stabilize the output feedback system in the presence of uncertainty in the control coefficient \( a(\cdot) \).

The tuning of the design parameters \( \mu \) and \( \varepsilon \) to achieve a desired performance merits comment. The separation result of Atassi and Khalil [2] was invoked so as to perform the state feedback design first, which employed a continuously implemented sliding mode control instead of employing it “as-is” to address the issue of chattering. Consequently, a key component in the state feedback design is the tuning of the boundary layer parameter \( \mu \) to recover the properties of “true” sliding mode control—however, if \( \mu \) is made too small, the issue of chattering arises yet again,
so we are limited in terms of being unable to make this parameter arbitrarily small [29]. Similarly, in the case of the output feedback design, we would like to make the extended high-gain observer parameter $\varepsilon$ as small as possible in order to recover the performance of the state feedback system. However, in most practical applications, we are limited by bandwidth constraints, and so $\varepsilon$ cannot be made arbitrarily small. Moreover, the presence of measurement noise results in the need for a trade-off between state feedback performance recovery and robustness to model uncertainties on the one hand, and the potential for the exacerbation of the effect of measurement noise on the other. Specifically, measurement noise imposes a lower bound on $\varepsilon$ that is $O(\varphi^{1/(\rho+1)})$, where $\varphi$ is an upper bound on the amplitude of the measurement noise, in accordance with the results of Ahrens and Khalil [1, 5].
Chapter 3

Output Regulation of Non-Minimum Phase Systems

3.1 Introduction

A lot of research has been conducted in the past couple of decades on the problem of output regulation of nonlinear systems; references [9, 40, 26, 39] are but a tiny cross-section of the standard literature on this topic. Most of the available results have in fact been collected in well known books on nonlinear control, such as [18, 19]. Despite this wealth of literature, one area of nonlinear control is still very much an open avenue of research in many respects, due mainly to the challenging nature of the problem—and this is the problem of robust output feedback regulation of non-minimum phase systems. A major breakthrough was achieved on the stabilization problem by the pioneering work of [21]. This work was later followed up by [32, 10], in which the same technique was extended and applied to the problem of nonlinear output regulation of non-minimum phase systems.

Both of the aforementioned approaches utilize high-gain feedback techniques. The results contained therein do not consider uncertain control coefficients, and this par-
ticular issue is addressed in this chapter. This work is an extension, in analogous fashion, of the stabilization problem investigated in Chapter 2. The approach utilizes an extended high-gain observer (EHGO) to estimate a “virtual output” in addition to the system output (or tracking error in the case of regulation) and its first $\rho$ derivatives (this number being the relative degree of the system), in conjunction with a continuous implementation of sliding mode control. The key idea of [21, 32] is retained in both these approaches, and consequently, the results take advantage of the knowledge of a dynamic controller for the auxiliary subsystem associated with the original problem, and the definition of the latter in [36] and in this work is identical to the original work of [21, 32, 10], respectively.

Despite the aforementioned similarities between this work and prior work by [21, 32, 10], we should note that our approach to the output feedback design is different. The papers [21, 32, 10] use high-gain feedback to implement the output feedback control in such a way that the closed-loop system can be represented as the feedback connection of an asymptotically stable system with a memoryless high-gain system. In this work, we do not use high-gain feedback. Instead, an EHGO is used to estimate all of the signals that are needed to implement the state feedback controller, thus allowing for us to obtain results in keeping with the spirit of the Separation Principle. Consequently, our approach does not confine us to any specific control design method such as high-gain feedback, sliding mode control (SMC), Lyapunov redesign, etc., although continuously implemented SMC was chosen for this work. Moreover, high-gain feedback, by its nature, would cause a large spike in the control signal in the initial transient, which will not be observed with the proposed controller. Another technical difference between the results of the current work and those of [32, 10] is that our design allows for uncertainty in the control coefficient, which was not addressed in the former works. Handling this uncertainty dominates our analysis in Section 3.2.2.

A preliminary version of this chapter appears in [35]. The main difference between
the aforementioned paper and this work is that we use the standard servocompensator driven by the tracking error herein, whereas in [35], the servo is driven by the virtual output obtained from the extended high-gain observer. The approach adopted in this chapter simplifies some of the technical details in the analysis of the state feedback system, as well as the design of the stabilizing controller for the auxiliary system. The example used in this chapter is a linearized pendulum model, which is also different from the second order system utilized in [35].

This chapter is organized as follows—in Section 3.2, the problem statement and some preliminary assumptions are introduced. In the main Section 3.2.2 that follows, the control design is carried out initially for the state feedback case, in which the output and its first \( \rho \) derivatives are assumed to be available, along with a virtual output that renders the zero dynamics of the system observable. The state feedback regulation result is summarized, and this is followed by the output feedback design that utilizes an extended high-gain observer to estimate the states corresponding to the chain of integrators in the normal form, as well as an estimate of the aforementioned virtual output, albeit potentially perturbed due to any uncertainty in the control coefficient. The section is concluded by stating the main result of this chapter. Appropriate assumptions are made in this section as required. Section 3.3 illustrates the result using a third order system as an example, and some numerical results are presented, along with a comparison of the technique of this chapter and that of [32] via simulation of this example using both approaches. Section 3.4 contains some concluding remarks summarizing the results.
3.2 Problem Description and Assumptions

3.2.1 Background and Preliminaries

In this chapter, we consider a single-input, single-output system that is in the normal form and has a well-defined relative degree, and that could potentially include unstable zero dynamics. The system model takes the form

\[
\begin{align*}
\dot{z} &= f_0(z, x, w, \theta), \\
\dot{x} &= Ax + B[b_0(z, x, w, \theta) + a_0(z, x, w, \theta)u], \\
y &= Cx - q(w, \theta),
\end{align*}
\]

where \((z, x) \in D_z \times D_x \subset \mathbb{R}^n\) is the system state, \(u \in \mathbb{R}\) is the input and \(y \in \mathbb{R}\) the output, \(\theta \in \Theta \subset \mathbb{R}^p\) is a vector of constant parameters, where \(\Theta\) is a compact set, \(q(w, \theta)\) is a reference signal, the control coefficient \(a_0(z, x, w, \theta) \neq 0\), and the disturbance \(w : [0, \infty) \to \mathbb{R}^d\) is generated by the following neutrally stable exosystem.

\[
\dot{w} = S_0 w.
\]

Moreover, the matrices \(A \in \mathbb{R}^{\rho \times \rho}, B \in \mathbb{R}^{\rho \times 1}\) and \(C \in \mathbb{R}^{1 \times \rho}\) have the following structure.

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix},
\]

and \(C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}\). The goal in this chapter is to design a control algorithm that regulates the output \(y\) to zero.
Let $\Omega_s \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^d$ be given compact sets. Then, bearing in mind the aforementioned goal of output regulation, we shall require the following assumptions.

**Assumption 3.1.** Let

$$
\bar{x}(w, \theta) \triangleq \begin{pmatrix} q(w, \theta) & L_sq(w, \theta) & \cdots & L_s^{\rho-1} q(w, \theta) \end{pmatrix}^\top,
$$

where $L_sq \triangleq (\partial q/\partial w)S_0 w$. There exists a unique mapping $\bar{z}(w, \theta)$ that solves the partial differential equation

$$
\frac{\partial \bar{z}(w, \theta)}{\partial w}S_0 w = f_0(\bar{z}(w, \theta), \bar{x}(w, \theta), w, \theta),
$$

for all $w \in W$.

Under this assumption, the system (3.1) has a steady-state zero-error manifold given by $\{ (z, x) | z = \bar{z}(w, \theta), x = \bar{x}(w, \theta) \}$. The steady-state value of the control signal $u$ on this manifold is given by

$$
c(w, \theta) = \frac{L_s^{\rho} q(w, \theta) - b_0(\bar{z}(w, \theta), \bar{x}(w, \theta), w, \theta)}{a_0(\bar{z}(w, \theta), \bar{x}(w, \theta), w, \theta)}.
$$

**Assumption 3.2.** There exist real constants $d_0, \ldots, d_{m-1}$, all independent of $\theta$, such that $c(w, \theta)$ satisfies the identity

$$
L_s^m c = d_0 c + d_1 L_s c + \cdots + d_{m-1} L_s^{m-1} c,
$$

for all $(w, \theta) \in W \times \Theta$, and the characteristic polynomial

$$
\lambda^m - d_{m-1} \lambda^{m-1} - \cdots - d_0
$$

has distinct zeros on the imaginary axis.
Assumption 3.2 is motivated by the nonlinear counterpart to the well known internal model principle from regulation theory [12]. When the system is nonlinear, its controller must reproduce, in addition to the trajectories generated by the exosystem, any higher order nonlinear deformations of the aforementioned trajectories that may be present (see, for example, [26]).

Let us now define

\[
S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ d_0 & d_1 & d_2 & \cdots & d_{m-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} c \\ L_sc \\ \vdots \\ L_{s}^{m-2}c \\ L_{s}^{m-1}c \end{pmatrix},
\]

and \( \Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{1 \times m} \). It can be shown that \( c(w, \theta) \) is generated by the internal model

\[
\frac{\partial \tau(w, \theta)}{\partial w} S_0 w = S \tau(w, \theta), \tag{3.4a}
\]

\[
c(w, \theta) = \Gamma \tau(w, \theta), \tag{3.4b}
\]

where the pair \((S, \Gamma)\) is observable. Suppose we choose matrices \( J \) and \( K_1 \) such that \((S, J)\) is controllable and \( F = S - JK_1 \) is Hurwitz. It can be shown that there exists a unique matrix \( X \) such that [8]

\[
SX = XS, \quad \text{and} \quad -K_1 X = \Gamma.
\]

Hence, it is easy to verify that \( c(w, \theta) \) can also be generated by

\[
L_s(X \tau(w, \theta)) = SX \tau(w, \theta), \tag{3.5a}
\]

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\begin{equation}
c(w, \theta) = -K_1 X \tau(w, \theta).
\end{equation}

We will have occasion to use (3.5) when designing the servocompensator.

We solve the output regulation problem by converting it to a stabilization problem. To this end, we define and apply the change of variables \( \eta \triangleq z - \bar{z}(w, \theta) \) and \( e \triangleq x - \bar{x}(w, \theta) \), and also define \( \xi \triangleq \begin{pmatrix} e_1 & \cdots & e_{\rho-1} \end{pmatrix}^\top \). The system equations (3.1) are then transformed into

\begin{align}
\dot{\eta} &= f(\eta, \xi, e_{\rho}, w, \theta), \quad (3.6a) \\
\dot{\xi} &= P \xi + Q e_{\rho}, \quad (3.6b) \\
\dot{e}_{\rho} &= b(\eta, \xi, e_{\rho}, w, \theta) + a(\eta, \xi, e_{\rho}, w, \theta)[u - c(w, \theta)], \quad (3.6c) \\
y &= R \xi = \xi_1 = e_1, \quad (3.6d)
\end{align}

where the control coefficient \( a(\eta, \xi, e_{\rho}, w, \theta) = a_0(z, x, w, \theta) \) and the functions \( f(\cdot) \) and \( b(\cdot) \) are such that, for all \( (w, \theta) \in W \times \Theta, f(0, 0, 0, w, \theta) = 0 \) and \( b(0, 0, 0, w, \theta) = 0 \). The triple \((P, Q, R)\) has the same structure as \((A, B, C)\) in (3.1), except that \( P \in \mathbb{R}^{(\rho-1) \times (\rho-1)}, Q \in \mathbb{R}^{(\rho-1) \times 1} \) and \( R \in \mathbb{R}^{1 \times (\rho-1)} \), respectively. The zero-error manifold in terms of the new variables now becomes

\[ \{(\eta, e) \mid \eta = 0, e = 0\}. \]

Next, we augment the above system with the servocompensator

\[ \dot{\zeta} = S \zeta + J \xi_1, \quad (3.7) \]

where \( \zeta \in \mathbb{R}^m \).

When we examine the error system (3.6), we notice that in order for regulation to
occur, i.e., for \( y(t) \to 0 \), the control \( u(t) \) would have to asymptotically converge to a steady state given by \( c(w, \theta) \). This is obviously not implementable due to \( w, \theta \) being unknown. Nevertheless, this problem has been solved in the case of minimum phase systems: \([18, 28, 39, 37]\). However, the techniques presented in the aforementioned papers assume that the zero dynamics of the system are either input to state stable or globally asymptotically stable, and would hence fail in the case of non-minimum phase systems. The challenge of stabilizing non-minimum phase systems was tackled by Isidori, leading to a design tool that can be applied to a large class of systems in which the coupling term \( b(\cdot) \) between the internal and external dynamics of a system in the normal form is exploited in the design of the dynamic output feedback stabilizer \([21]\). This tool was extended to the regulation problem by Marconi et al.\([32]\). Regulation is more complicated when compared to the stabilization problem, since it not only involves the stabilization of the error system, but also the asymptotic reconstruction of the steady-state control \( c(w, \theta) \). As a matter of fact, the steady state control has to be reconstructed by adding a marginally stable internal model unit to the compensator. Thus, the regulation problem clearly poses a greater challenge than the stabilization case.

In the next subsection we present the state feedback control design approach that addresses the above-mentioned challenges.

### 3.2.2 State Feedback Regulator Design

As in the stabilization case, the state feedback design for the system (3.6), (3.7) is carried out by utilizing virtual signals \( \xi_1, \ldots, \xi_{p-1} \) and

\[
\sigma = b(\cdot) - a_n(\cdot)K_1\zeta - a(\cdot)c(\cdot) + \Delta a(\eta, \xi, e, w, \theta)u, \tag{3.8}
\]

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where $\Delta a(\eta, \xi, e, \rho, w, \theta) = a(\eta, \xi, e, \rho, w, \theta) - a_n(\xi, e)$. These signals can be estimated by introducing the following extended-high gain observer (EHGO).

\[
\dot{\hat{\xi}} = P\hat{\xi} + Q\hat{e}_\rho + H(\varepsilon)(y - \hat{\xi}_1), \\
\dot{\hat{e}}_\rho = \dot{\sigma} + a_n(\hat{\xi}, \hat{e}_\rho)(u + K_1\zeta) + (\dot{\alpha}_\rho/\varepsilon)\varepsilon^{\rho-1}(y - \hat{\xi}_1), \\
\dot{\dot{\sigma}} = (\dot{\alpha}_{\rho+1}/\varepsilon^{\rho+1})(y - \hat{\xi}_1),
\]

where

\[
H(\varepsilon) = \left(\hat{\alpha}_1/\varepsilon, \hat{\alpha}_2/\varepsilon^2, \ldots, \hat{\alpha}_{\rho-1}/\varepsilon^{\rho-1}\right)^\top,
\]

and $a_n(\xi, e)$ is a nominal function used in the subsequent state feedback control design in place of $a(\eta, \xi, e, \rho, w, \theta)$. In the above observer equations, we have followed the standard notational convention of labeling the estimates of the virtual signals with the hatted versions of the corresponding symbols. It can be shown [16] using high gain observer theory [3] in conjunction with singular perturbation analysis [31] that the signals $\hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}, \hat{e}_\rho, \hat{\sigma}$ become arbitrarily close to $\xi_1, \ldots, \xi_{\rho-1}, e_\rho$ and $\sigma$, respectively. In the case of minimum phase systems, the regulation problem for a plant with well-defined relative degree can always be reduced to one of regulating a relative degree one system by utilizing a high-gain observer. For example, a virtual output could be defined as $s = e_\rho + \sum_{i=1}^{\rho-1} k_i\xi_i$, with the constants $k_i$ chosen so as to render the system minimum phase with respect to the output $s$. Then, a static control law utilizing the virtual output $s$ could be designed in accordance with standard techniques, and this design could be implemented by replacing $s$ with its estimate from a high-gain observer. This technique will not work if the system is non-minimum phase, however.

Since we are considering non-minimum phase systems in this work, we introduce, in analogous fashion to the stabilization case of [36], the dynamic compensator given
by
\[ \dot{\phi} = L(\xi, \phi, \zeta) + M(\xi, \phi, \zeta)\sigma. \] (3.9)

Next, we define a new virtual output
\[ s = e\rho - N(\xi, \phi, \zeta). \] (3.10)

Our objective is to design the triple \{L, M, N\} such that the augmented system (3.6), (3.7), (3.9), (3.10) with \(s\) taken as its output is minimum phase. In order to continue with the analysis and bearing in mind the goal of reducing the regulation problem to one of stabilization, we introduce the change of variable \(s \mapsto s, \zeta \mapsto \psi \triangleq \zeta - X\tau(w, \theta)\) and \(u \mapsto v \triangleq u + K_1\zeta = u + K_1\psi - c(w, \theta)\) (due to (3.5)) to obtain the following equations describing the augmented system.

\[
\begin{align*}
\dot{\eta} &= f(\eta, \xi, s + N(\xi, \phi, \psi + X\tau), w, \theta), \quad \text{(3.11a)} \\
\dot{\psi} &= S\psi + J\xi, \quad \text{(3.11b)} \\
\dot{\zeta} &= P\xi + Q[s + N(\xi, \phi, \psi + X\tau)], \quad \text{(3.11c)} \\
\dot{s} &= b(\cdot) - a(\cdot)K_1\psi + a(\cdot)v - \frac{\partial N}{\partial \zeta}(S\psi + J\xi) - \frac{\partial N}{\partial \zeta}SX\tau \\
&\quad - \frac{\partial N}{\partial \xi}[P\xi + Q(s + N(\cdot))] - \frac{\partial N}{\partial \phi}[L + M(b - aK_1\psi + \Delta a(\cdot)v)], \quad \text{(3.11d)} \\
\dot{\phi} &= L(\cdot) + M(\cdot)[b(\cdot) - a(\cdot)K_1\psi + \Delta a(\cdot)v]. \quad \text{(3.11f)}
\end{align*}
\]

We note that if \(\rho = 1\), then the \(\dot{\xi}\)-equation in (3.11) will not exist, and \(\xi_1\) would be replaced by \(s + N(\cdot)\). Now, let

\[ f_s(\eta, \xi, s, \phi, \psi, w, \theta) = b(\cdot) - a(\cdot)K_1\psi - \frac{\partial N}{\partial \xi}[P\xi + Q(s + N(\cdot))] - \frac{\partial N}{\partial \zeta}SX\tau \]
\[ \quad - \frac{\partial N}{\partial \phi}[L(\cdot) + M(\cdot)[b(\cdot) - a(\cdot)K_1\psi]] - \frac{\partial N}{\partial \zeta}(S\psi + J\xi_1), \]

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\[
gs(\eta, \xi, s, \phi, \psi, w, \theta) = a(\cdot) - \frac{\partial N}{\partial \phi} M(\cdot) \Delta a(\cdot),
\]
\[
f_{\phi}(\eta, \xi, s, \phi, \psi, w, \theta) = L(\cdot) + M(\cdot)[b(\cdot) - a(\cdot)K_1 \psi].
\]

The system (3.11) will have a relative degree of one with respect to \(s\) if
\[
\frac{g_s(\eta, \xi, s, \phi, \psi, w, \theta)}{a(\eta, \xi, s, \phi, \psi, w, \theta)} \geq k^\dagger > 0,
\]
for some positive real constant \(k^\dagger\). When \(\Delta a \neq 0\), we transform (3.11) into the
normal form by the change of variable \(T : \phi \mapsto q\) that satisfies the partial differential
equation
\[
\frac{\partial T}{\partial s} g_s + \frac{\partial T}{\partial \phi} M \Delta a = 0.
\]
Due to (3.13), we note that the vector \(\begin{pmatrix} \eta^T & \xi^T & s & q^T & \psi^T \end{pmatrix}^T\) is a diffeomorphism
of \(\begin{pmatrix} \eta^T & \xi^T & s & \phi^T & \psi^T \end{pmatrix}^T\). When \(\Delta a = 0\), we can take \(T = \phi\), and the system
(3.11) will be in the normal form. Since a solution exists for the \(\Delta a = 0\) case, we
can argue that for sufficiently small \(|\Delta a|\), (3.13) still has a solution \(T\) [11]. If we let
\(\phi = T^{-1}(\cdot)\) be the inverse transformation, then due to the continuous dependence of
the solution of (3.13) on \(\Delta a\) [11], we can write \(T\) and \(T^{-1}\) as
\[
T = \phi + \Phi(\eta, \xi, s, \phi, \psi, w, \theta),
\]
\[
T^{-1} = q + \varphi(\eta, \xi, s, q, \psi, w, \theta),
\]
where \(\Phi\) and \(\varphi\) are of the order of \(O(|\Delta a|)\). The transformed system equations are
now given by
\[
l(\eta, \xi, s + N(\xi, q + \varphi(\cdot), \psi + X \tau, w, \theta), w, \theta),
\]
\[
\dot{\psi} = S \psi + J \xi_1,
\]
\[(3.14)\]
\[ \dot{\xi} = P\xi + Q[s + N(\cdot)], \quad (3.14c) \]
\[ \dot{s} = f_s(\eta, \xi, s, q + \varphi(\cdot), \psi, w, \theta) + g_s(\cdot)v, \quad (3.14d) \]
\[ \dot{q} = \left(1 + \frac{\partial \Phi}{\partial \phi}\right) f_\phi + \frac{\partial \Phi}{\partial \xi} [P\xi + Q(s + N)] \]
\[ + \frac{\partial \Phi}{\partial \eta} f + \frac{\partial \Phi}{\partial s} f_s + \frac{\partial \Phi}{\partial w} S_0w + \frac{\partial \Phi}{\partial \psi} (S\psi + J_1). \quad (3.14f) \]

With \( s \) viewed as the output, and due to (3.12), this is a relative degree one system in the normal form. The zero dynamics shall have trajectories confined to the set \( \{(\eta, \xi, s, q, \psi)|s = 0\} \), and are given by

\[ \dot{\eta} = f(\eta, \xi, N(\xi, q + \varphi, \psi + X\tau, w, \theta), w, \theta), \quad (3.15a) \]
\[ \dot{\psi} = S\psi + J_1, \quad (3.15b) \]
\[ \dot{\xi} = P\xi + QN(\cdot), \quad (3.15c) \]
\[ \dot{q} = \left(1 + \frac{\partial \Phi}{\partial \phi}\right) f_\phi + \frac{\partial \Phi}{\partial \eta} f + \frac{\partial \Phi}{\partial \xi} (P\xi + QN) + \frac{\partial \Phi}{\partial \psi} (S\psi + J_1) + \frac{\partial \Phi}{\partial w} S_0w. \quad (3.15d) \]

When \( \Delta_\alpha = 0 \), (3.15) reduces to

\[ \dot{\eta} = f(\eta, \xi, N(\xi, q, \psi + X\tau, w, \theta), w, \theta), \quad (3.16a) \]
\[ \dot{\psi} = S\psi + J_1, \quad (3.16b) \]
\[ \dot{\xi} = P\xi + QN(\cdot), \quad (3.16c) \]
\[ \dot{q} = L(\cdot) + M(\cdot)[b(\cdot) - a(\cdot)K_1\psi]. \quad (3.16d) \]

This can be viewed as a feedback interconnection between the extended auxiliary plant of [32]:

\[ \dot{\eta} = f(\eta, \xi, u_\alpha, w, \theta), \quad (3.17a) \]
\[ \dot{\xi} = P\xi + Qu_\alpha, \quad (3.17b) \]
\[ \dot{\psi} = S\psi + J\xi_1, \quad (3.17c) \]
\[ y_a = b(\cdot) - a(\cdot)K_1\psi, \quad (3.17d) \]

and the auxiliary controller

\[ \dot{\phi} = L(\xi, \phi, \zeta) + M(\xi, \phi, \zeta)y_a, \quad (3.18a) \]
\[ u_a = N(\xi, \phi, \zeta). \quad (3.18b) \]

We note that the system (3.17)–(3.18) is very similar to the extended auxiliary system of [32]—the only difference is in the structure of the servocompensator given by the \( \dot{\psi} \)-equation. In [32], both the \( \dot{\psi} \) and the \( \dot{q} \) equations are driven by \( b(\cdot) - a(\cdot)K_1\psi \), whereas here, the \( \dot{\psi} \) equation is driven by \( \xi_1 \). This fact makes the analysis of the closed-loop system tractable in the case of unknown control coefficient \( a(\cdot) \), whereas in the method of [32], the lack of knowledge of this function would complicate the analysis because the counterpart of the transformation that maps \( \zeta \) to \( \psi \) would depend on \( a(\cdot) \) using this method; moreover, the control coefficient is assumed to be 1 in this latter work.

Next, let \( x_s = \left( \eta^T \quad \psi^T \quad \xi^T \quad q^T \right)^T \) and

\[
F(x_s, s, w, \theta) = \begin{pmatrix}
    f(\eta, \xi, N(\xi, q + \varphi, \psi + X\tau, w, \theta), w, \theta) \\
    S\psi + J\xi_1 \\
    P\xi + Q[s + N(x_s)] \\
    f_1(x_s, s + N(x_s), w, \theta)
\end{pmatrix},
\]

where \( f_1(x_s, s + N(x_s), w, \theta) = \left( 1 + \frac{\partial \Phi}{\partial \phi} \right) f_\phi + \frac{\partial \Phi}{\partial \eta} f + \frac{\partial \Phi}{\partial \psi} (S\psi + J\xi_1) + \frac{\partial \Phi}{\partial \xi} [P\xi + Q(s + N(x_s))] + \frac{\partial \Phi}{\partial w} S_0 w \). With these newly defined variables, (3.14) can be rewritten
as
\[
\begin{align*}
\dot{x}_s &= F(x_s, s, w, \theta), \\
\dot{s} &= f_s(x_s, s + N, w, \theta) + g_s(x_s, s + N, w, \theta)v.
\end{align*}
\tag{3.19}
\tag{3.20}
\]

We note that the system (3.19)–(3.20) is relative degree one with \(s\) viewed as the output. Our goal now is to design the dynamic compensator \(\{L, M, N\}\) such that the resulting augmented system is minimum phase, and the internal dynamics (3.19) are input-to-state stable with \(s\) viewed as an input to the latter. Finally, the stabilizing control input \(v\) for the augmented system can be designed using any viable robust control technique for relative degree one minimum phase systems such as high gain feedback, Lyapunov redesign, etc. We choose continuously implemented sliding mode control (SMC) in this chapter, but it is important to note that we are not confined to any particular technique. SMC offers several advantages such as control saturation at a predetermined level—which mitigates the effects of the peaking phenomenon caused by the high gain observer—trajectory convergence to a region inside the boundary layer in finite time, and high gain feedback inside the aforementioned region. The control input \(v\) is thus designed as
\[
v = -\beta(\xi, e\rho, \phi, \zeta) \frac{a_n(\xi, e\rho)}{a_n(x_s, s + N(x_s))} \text{sat}\left(\frac{s}{\mu}\right),
\tag{3.21}
\]
where \(\mu\) is a constant design parameter that specifies the thickness of the boundary layer, and \(\beta(\cdot)\) is chosen in accordance with Assumption 2.2 and inequality (3.22) below.

**Assumption 3.3.** It is assumed that
\[
a(x_s, s + N(x_s), w, \theta)/a_n(x_s, s + N(x_s)) \geq k_0 > 0,
\]

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for all \((x_s, s) \in D\) and \((w, \theta) \in W \times \Theta\). Moreover, \(a_n(x_s, s + N(x_s))\) is locally Lipschitz in \((x_s, s)\) over the domain of interest, and globally bounded in \(\xi\).

The function \(\beta(\cdot)\) is also chosen to satisfy

\[
\beta(\xi, e_\rho, \phi, \zeta) \geq \beta_0 + \left| \frac{a_n(\xi, e_\rho)f_s(x_s, s + N, w, \theta)}{g_s(x_s, s + N, w, \theta)} \right|,
\]

(3.22)

where \(\beta_0 > 0\). Assumption 2.2 and (3.22) ensure that when \(s \geq \mu\), we have \(s \dot{s} \leq -k^\top k_0 |s|\), and so whenever \(|s(0)| > \mu\), \(|s(t)|\) will decrease until it reaches the set \(\{|s| \leq \mu\}\) in finite time and remain inside thereafter.

The closed-loop state feedback system is given by (3.19), (3.20), (3.21). We require this system to satisfy the following assumptions.

**Assumption 3.4.**

1) Let \(D \subset \mathbb{R}^{m+n+r}\) be a domain containing the origin, and let \((x_s, s) \in D\). Moreover, suppose there exists a continuously differentiable Lyapunov function \(V(x_s, w, \theta)\) such that

\[
\alpha_2(\|x_s\|) \geq V(x_s, w, \theta) \geq \alpha_1(\|x_s\|),
\]

(3.23a)

\[-\alpha_3(\|x_s\|) \geq \frac{\partial V}{\partial x_s} F(x_s, s, w, \theta) + \frac{\partial V}{\partial w} S_0 w, \quad \forall \|x_s\| \geq \gamma(|s|),\]

(3.23b)

for all \((x_s, s) \in D \subset \mathbb{R}^{m+n+r}\), and \((w, \theta) \in W \times \Theta\), where \(\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)\) and \(\gamma(\cdot)\) are class \(K\) functions.

2) With \(s = 0\), the origin \(x_s = 0\) of (3.19) is locally exponentially stable.

**Remark 3.1.**

1. The set \(D\) is well-defined for any sufficiently smooth \(N(\cdot)\) in the definition of the variable \(s = e_\rho - N(\cdot)\).
2. Inequality (3.23b) is equivalent to regional input-to-state stability of (3.19) with 
s viewed as an input.

Assumption 3.5.

1) \( L(x_s), M(x_s) \) and \( \beta(x_s) \) are locally Lipschitz functions in their arguments over 
the domain of interest, and \( L(0) = 0 \).

2) \( L(x_s) \) and \( M(x_s) \) are globally bounded functions of \( \xi \).

Remark 3.2. We require the global boundedness assumption on \( L(x_s) \) and \( M(x_s) \) to protect the system from the peaking phenomenon that may occur in the output 
feedback implementation when the extended high-gain observer is introduced [3, 29]. 
The peaking will be transmitted to the plant states in the closed-loop system, and in 
the absence of the above assumption, there is a possibility of the state trajectories 
having a finite escape time.

Under the above assumptions, we obtain the following result pertaining to the 
stability of this state feedback error system.

Proposition 3.1. Consider the closed loop state feedback error system (3.19), (3.20). 
Let Assumptions 3.1 through 3.5 hold. Then, there exists a positive constant \( \mu^* \) such 
that (3.19), (3.20) has an exponentially stable equilibrium at \( (x_s = 0, s = 0) \) for 
all \( \mu \in (0, \mu^*) \). Moreover, the set \( \Omega \) defined below is an estimate of the region of 
attraction.

\[
\Omega \triangleq \{ V(x_s, w, \theta) \leq c_0 \} \times \{|s| \leq c\}, \quad (3.24)
\]

where \( c > \mu, \ c_0 \geq \alpha_2(\gamma(c)) \), and \( c, c_0 \) are chosen such that \( \Omega \) is a compact subset of 
\( D \).

Proof. See Section A.4 for the proof.

Remark 3.3.
1) Assumption 2.3 requires the compensator \( \{L, M, N\} \) to be designed such that (3.19) is input to state stable with \( s \) viewed as the input. This problem is similar to the auxiliary design problem of [32] when \( \Delta a = 0 \), with the difference being in the structure of the servocompensator employed in each work. In [32], the interconnection of the extended auxiliary system with its dynamic compensator was assumed to be globally asymptotically stable and locally exponentially stable, with a Lyapunov function independent of the unknown system parameters.

2) If all the assumptions hold globally, then we could take \( D = \mathbb{R}^{m+n+r} \), and with the appropriate choice of controller parameter \( \beta(\cdot) \) (see (3.22)), the positively invariant set \( \Omega \subset D \) that depends on this parameter can be made arbitrarily large, and thus our design allows for semi-global stabilization of the state feedback error system.

### 3.2.3 Output Feedback Control Using Extended High-Gain Observers

#### 3.2.3.1 Output Feedback Regulator Design

In order to estimate the states \( e \) of the error system (3.6) and the virtual output \( \sigma \), an extended high-gain observer with the following structure is introduced.

\[
\begin{align*}
\dot{\xi} &= P\dot{\xi} + Q\dot{e}_\rho + H(\varepsilon)(y - \dot{\xi}_1), \\
\dot{e}_\rho &= \dot{\sigma} + a_n(\dot{\xi}, \dot{e}_\rho)(u + K_1z) + (\dot{\alpha}_\rho/\varepsilon^\rho)(y - \dot{\xi}_1), \\
\dot{\sigma} &= (\dot{\alpha}_{\rho+1}/\varepsilon^{\rho+1})(y - \dot{\xi}_1),
\end{align*}
\]

where \( H(\varepsilon) = \left( \dot{\alpha}_1/\varepsilon, \; \dot{\alpha}_2/\varepsilon^2, \; \ldots \; \dot{\alpha}_{\rho}/\varepsilon^{\rho} \right) \top \), and the constants \( \dot{\alpha} \) are chosen so as to make the polynomial \( \lambda^{\rho+1} + \dot{\alpha}_1\lambda^{\rho} + \ldots + \dot{\alpha}_{\rho}\lambda + \dot{\alpha}_{\rho+1} \) Hurwitz. The structure of this observer is very similar to the one used in Chapter 2, with the only
difference being the fact that this observer is driven by the signal \( v = u + K_1 \zeta \) instead of by \( u \). It shall be established during the analysis of the output feedback system to follow that as \( \varepsilon \to 0 \), we shall have \( \hat{\xi} \to \xi \), \( \hat{e}_\rho \to e_\rho \), and moreover, \( \hat{\sigma} \to b - a_n K_1 \zeta - ac + (a - a_n)u = \sigma \).

The state and virtual output estimates obtained from this observer are used to replace the unavailable states and unmeasured signals in the state feedback design, and the extended high-gain observer-based output feedback regulator is given by (3.25) and

\[
\begin{align*}
\dot{\zeta} &= S \zeta + J y, \\
\dot{\phi} &= L(\hat{\xi}, \phi, \zeta) + M(\hat{\xi}, \phi, \zeta) \hat{\sigma}, \\
u &= K \operatorname{sat}(l(\hat{\xi}, \phi)/K) - K_1 \zeta, \\
l(\cdot) &= -\beta(\hat{\xi}, \hat{e}_\rho, \phi, \zeta) \frac{\sigma}{a_n(\hat{\xi}, \hat{e}_\rho)} \operatorname{sat}\left(\frac{\dot{e}_\rho - N(\hat{\xi}, \phi, \zeta)}{\mu}\right),
\end{align*}
\]

where

\[
K > \max_{(x_s, s) \in \Omega} \left| \frac{\beta(\hat{\xi}, \hat{e}_\rho, \phi, \zeta)}{a_n(\hat{\xi}, \hat{e}_\rho)} \right|.
\]

The sliding mode control component \( v \) is saturated at \( \pm K \) outside \( \Omega \) in order to protect the system from peaking during the observer’s transient response.

We now introduce a change of variables as follows.

\[
\begin{align*}
\nu_i &= (\xi_i - \hat{\xi}_i)/\varepsilon^{\rho+1-i}, \quad 1 \leq i \leq \rho - 1, \\
\nu_\rho &= (e_\rho - \hat{e}_\rho)/\varepsilon, \\
\nu_{\rho+1} &= b(\eta, \xi, e_\rho, w, \theta) - a(\eta, \xi, e_\rho, w, \theta) K_1 \psi - \hat{\sigma} \\
&\quad + \Delta a(\eta, \xi, e_\rho, w, \theta) K g_\varepsilon(l(x_s, s)/K),
\end{align*}
\]

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where \( g_\varepsilon \) is an odd function defined by
\[
g_\varepsilon(z) = \begin{cases} 
  z, & 0 \leq z \leq 1, \\
  z + (z - 1)/\varepsilon - 0.5(z^2 - 1)/\varepsilon, & 1 < z < 1 + \varepsilon, \\
  1 + 0.5\varepsilon, & z \geq 1 + \varepsilon,
\end{cases}
\]
and we note that \( |g_\varepsilon'| \leq 1 \) and \( |g_\varepsilon(z) - \text{sat}(z)| \leq \varepsilon/2 \). Moreover, (3.28) is very similar to the change of variables utilized in Chapter 2, except for the additional term involving \( \psi \) in the \( \nu_{\rho + 1} \)-equation.

With the change of variables (3.28), and after applying the transformations \( \zeta \mapsto \psi \), \( e_\rho \mapsto s \) and \( \phi \mapsto q \), the closed-loop output feedback system (3.6), (3.25), (3.26), can be written as
\[
\dot{x}_S = F(x_S, s, w, \theta) + G(x_S, s, \nu, \varepsilon, w, \theta), \quad (3.29a)
\]
\[
\dot{s} = f_s(\cdot) + g_s(\cdot)K\text{ sat}(l(x_s, s)/K) + \varepsilon K_1 J_{\nu_{\rho + 1}} a(x_s, s + N(x_s), w, \theta) + a(x_s, s + N(x_s), w, \theta)K[\text{ sat}(l(x_s, s)/K) - \text{ sat}(l_{\eta}(x_s, s, \nu, \varepsilon)/K)], \quad (3.29b)
\]
and
\[
\varepsilon \dot{\nu} = \tilde{A}_\nu + \varepsilon[\tilde{B}_1 \Delta_1(\eta, \xi, e_\rho, \nu, w, \theta, \varepsilon) + \tilde{B}_2 \Delta_2(\eta, \xi, e_\rho, \nu, w, \theta, \varepsilon)], \quad (3.30)
\]
where
\[
G(x_S, s, \nu, \varepsilon, w, \theta) = \begin{pmatrix}
0 \\
0 \\
0 \\
M(x_S, \nu, \varepsilon) \Delta_{\rho} K g_\varepsilon(l(x_s, s)/K) - \nu_{\rho + 1}
\end{pmatrix},
\]
\[ l(x_s, s) = \frac{-\beta(\xi, e\rho, \phi, \zeta)}{a_n(\xi, e\rho)} \sat\left( \frac{e\rho - N(\xi, \phi, \psi + X\tau)}{\mu} \right), \]

\[ l_n(\cdot) = \frac{-\beta(\hat{\xi}, \hat{e}\rho, \phi, \zeta)}{a(\xi, \hat{e}\rho)} \sat\left( \frac{e\rho - \varepsilon\nu - N(x_s, \nu, \varepsilon)}{\mu} \right) \]

\[ = l(\hat{\xi}, \hat{e}\rho, q, \zeta), \]

\[ \tilde{A} = \begin{pmatrix} -\hat{\alpha}_1 & 1 & 0 & \cdots & 0 \\ -\hat{\alpha}_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\hat{\alpha}_{\rho} & 0 & 0 & \cdots & 1 \\ -\hat{\alpha}_{\rho+1} & 0 & 0 & \cdots & 0 \end{pmatrix}_{(\rho+1) \times (\rho+1)}, \]

\[ \tilde{B}_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{(\rho+1) \times 1}, \quad \tilde{B}_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{(\rho+1) \times 1}, \]

\[ \Delta_2(\eta, \xi, e\rho, \nu, w, \theta, \varepsilon) = \Delta_0(\eta, \xi, e\rho, \nu, w, \theta, \varepsilon)/\varepsilon, \]

\[ \Delta_0(\eta, \xi, e\rho, \nu, w, \theta, \varepsilon) = \]

\[ -K\Delta_a(\eta, \xi, e\rho, w, \theta)[g(\ell(\cdot)/K) - g(\ell_n(\cdot)/K)] + [a(\eta, \xi, e\rho, w, \theta) - a_n(\xi, e\rho, \nu, \varepsilon)]K[\sat(\ell_n(x_s, s, \nu, \varepsilon)/K) - g(\ell_n(x_s, s, \nu, \varepsilon)/K)] \]

\[ -g(\ell_n(x_s, s, \nu, \varepsilon)/K)] + K[a_n(\xi, e\rho) - a_n(\xi, e\rho, \nu, \varepsilon)]g(\ell_n(x_s, s, \nu, \varepsilon)/K) \]

\[ \Delta_1(\eta, \xi, e\rho, \nu, w, \theta, \varepsilon) = \]

\[ \dot{b}(\eta, \xi, e\rho, w, \theta) - \dot{a}(\eta, \xi, e\rho, w, \theta)K\dot{\psi} + \Delta_a(\eta, \xi, e\rho, w, \theta)g(\ell(x_s, s)/K)\dot{l}(\xi, e\rho, \phi) \]

\[ -a(\eta, \xi, e\rho, w, \theta)K\dot{\psi} + \Delta_a(\eta, \xi, e\rho, w, \theta)Kg(\ell(x_s, s)/K). \]
We note that the functions \( \Delta_0 \) and \( \Delta_1 \) are locally Lipschitz in their arguments. This is because \( a_n(\cdot) \) and \( g_\varepsilon(\cdot) \) are continuously differentiable with locally Lipschitz derivatives and globally bounded. Moreover, we can use (3.28) and the definition of \( g_\varepsilon \) to show that \( \Delta_0/\varepsilon \) is locally Lipschitz [16]. We can now write the slow dynamics in a compact form as

\[
\dot{x}_o = f_r(x_o, \nu, \varepsilon, w, \theta),
\]

(3.31)

where \( x_o = \left( x_s \top \ s \right) \top \),

\[
f_r(x_o, \nu, \varepsilon, w, \theta) = \begin{pmatrix}
F(x_s, s, w, \theta) + G(x_s, s, \nu, \varepsilon, w, \theta) \\
\varepsilon K_1 J \nu \rho
\end{pmatrix},
\]

\[
f_2(x_s, s, \nu, \varepsilon, w, \theta) = \begin{pmatrix}
b(x_s, s, w, \theta) + a(\cdot)[K \sat(l_n(\cdot)/K) + \varepsilon K_1 J \nu \rho] \\
\frac{\partial N}{\partial x_s} \begin{pmatrix}
0 \\
Ma(x_s)
\end{pmatrix} \Delta a \sat(l_n(\cdot)/K) - \frac{\partial N}{\partial x_s} F(x_s, s, w, \theta),
\end{pmatrix}
\]

where \( b(x_o, w, \theta) = b(\eta, \xi, e_\rho, w, \theta) - K_1 \psi a(\eta, \xi, e_\rho, w, \theta) \). Then, the reduced system is given by

\[
\dot{x}_o = f_r(x_o, 0, 0, w, \theta),
\]

(3.32)

and it is easily verified that in the compact set \( \Omega \), given by (3.24), this reduced system is identical to the state feedback system (3.19), (3.20), which, due to Proposition 3.1, is exponentially stable with respect to its zero error manifold \( \{ \eta = 0, \ e = 0 \} \). The boundary layer system is given by

\[
\frac{\partial \nu}{\partial \tilde{\tau}} = \bar{A} \nu, \quad \tilde{\tau} = \frac{t}{\varepsilon}.
\]

(3.33)
Since $\bar{A}$ is Hurwitz, the boundary layer system is exponentially stable with respect to $\{\nu = 0\}$.

At this stage in our analysis, we are ready to state some results pertaining to the recovery of the performance of the system (3.19), (3.20) in the case of output feedback as $\varepsilon \to 0$. These results follow directly from the work of [3] because the singularly perturbed system (3.29)–(3.30) is of the same form as the system described by equations (10)–(13) in [3]. Consequently, the analysis of (3.29)–(3.30) can be carried out in the same vein, and so only the results of that analysis by [3] are reproduced below. Moreover, while the structure of system (3.29)–(3.30) is also very similar to that of the closed-loop system analyzed by [16], a key difference between our work and the latter is that both the perturbation terms in the fast dynamics (3.30) are $O(\varepsilon)$ in this work, while there is one term in [16] that is not. Such a term appeared in [16] because the estimate $\hat{\sigma}$ was used in the control $u$, whereas in our work, this quantity is not utilized directly in the static control.

3.2.3.2 Exponential Stability of the Output Feedback System and Trajectory Convergence

Let $(x_O(t, \varepsilon), \nu(t, \varepsilon))$ denote the trajectory of the system (3.30), (3.31) starting from $(x_O(0), \hat{\xi}(0))$. Also, let $x_R(t)$ be the solution of (3.19), (3.20) starting at $x_O(0)$. We know that the trajectory $(x_S(t), s(t))$ of the system (3.19), (3.20) is exponentially stable with respect to $\{(x_S, s) = (0, 0)\}$, so let us suppose that $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be a compact set in the interior of $\mathcal{R}$, and let $\mathcal{Q}$ be a compact subset of $\mathbb{R}^{d+1}$. The recovery of exponential stability of the zero error manifold, and the fact that $x_O(t)$ converges to $x_R(t)$ as $\varepsilon \to 0$, uniformly in $t$, for all $t \geq 0$, is established by the following theorem, [3, Theorems 1, 2, 3 and 5], provided that $x_O(0) \in \mathcal{S}$ and $\hat{\xi}(0) \in \mathcal{Q}$.
Theorem 3.1. Let Assumptions 3.1 through 3.5 hold. Moreover, assume \( x_0(0) \in S \) and \( \nu(0) \in Q \). Then,

1. there exists \( \varepsilon_1^* > 0 \) such that, for every \( 0 < \varepsilon \leq \varepsilon_1^* \), the system (3.30), (3.31) is exponentially stable with respect to the manifold \( \{(x_o,\nu) = (0,0)\} \), and \( S \times Q \) is included in the region of attraction.

2. given any \( \delta > 0 \), there exists \( \varepsilon_2^* > 0 \) such that, for every \( 0 < \varepsilon \leq \varepsilon_2^* \), we have

\[
\|x_o(t, \varepsilon) - x_r(t)\| \leq \delta, \quad \forall t \geq 0.
\] (3.34)

Remark 3.4. Theorem 1 of [3] establishes boundedness of \((x_o,\nu)\) for all trajectories starting in \( S \times Q \). Theorem 2 of [3] shows that the trajectories are ultimately bounded, and that the ultimate bound can be made arbitrarily small by choosing \( \varepsilon \) small enough. Finally, Theorem 5 of [3] establishes that \( \{(x_o,\nu) = (0,0)\} \) is exponentially stable with a region of attraction that is independent of \( \varepsilon \). The combination of the three results yields the first item of the foregoing theorem. The second item follows from Theorem 3 of [3].

### 3.3 Example

We illustrate the design procedure for an extended high-gain observer based sliding mode output feedback regulator using a linearized inverted pendulum on a cart model as an example. The system is given by [10]

\[
\begin{align*}
\dot{z}_1 &= z_2 - \frac{x_2}{h}, \quad \text{(3.35a)} \\
\dot{z}_2 &= \frac{g}{h} z_1, \quad \text{(3.35b)} \\
\dot{x}_1 &= x_2, \quad \text{(3.35c)}
\end{align*}
\]
where $z_1$ is the angle of the pendulum with respect to the vertical, $x_1$ is the position of the cart, $u$ is the control input, $w_1$ is a sinusoidal disturbance of known frequency $\omega$, $m_c$ is the mass of the cart, $m_p$ is the mass of the pendulum (assumed to be concentrated in the bob), $h$ is its length, and $g$ is the acceleration due to gravity. $m_p$ and $h$ are assumed to be uncertain parameters in the ranges $0.19 \text{ kg} \leq m_p \leq 0.23 \text{ kg}$ and $0.55 \text{ m} \leq h \leq 0.67 \text{ m}$. An exosystem generates the sinusoidal disturbance signal $w_1$, and this system is modeled by $\dot{w} = S_0 w$, where

\[
S_0 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.
\]

The goal is to utilize the output measurement $y$ to drive a feedback controller designed to make this system output converge to zero despite the presence of the input disturbance $w_1$ and uncertain parameters $m_p$ and $h$.

We conclude from the steady-state analysis of (3.35) that the steady-state control is $c(w) = -w_1$, and this allows us to determine the $S$ matrix of the servocompensator that satisfies Assumption 3.2 and (3.5), and an appropriate gain matrix $K_1$ to make $S - JK_1$ Hurwitz.

We now proceed with the first step in our design by augmenting (3.35) with the servocompensator

\[
\dot{\zeta} = S\zeta + Jy,
\]

where

\[
S = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Next, we choose a matrix $K_1$ such that the eigenvalues of $(S - JK_1)$ are $-1$ and $-2$. The matrix is thus given by $K_1 = \begin{pmatrix} 3 - \omega^2, & 2 \end{pmatrix}$.

### 3.3.1 The Auxiliary Problem

After applying the change of variable $\zeta \mapsto \psi = \zeta - X\tau$, the auxiliary system associated with the augmented system (3.35), (3.36) is given by

$$\begin{pmatrix} \dot{\psi} \\
\dot{z}_1 \\
\dot{z}_2 \\
\dot{x}_1 \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & J \\
0 & 0 & 1 & 0 \\
g/h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi \\
z_1 \\
z_2 \\
x_1 \end{pmatrix} + \begin{pmatrix} 0 \\
-1/h \\
n \\
1 \end{pmatrix} u_a, \quad (3.37a)$$

$$ya = -\frac{1}{m_c} \begin{pmatrix} -K_1, & m_{pg}, & 0, & 0 \end{pmatrix} \begin{pmatrix} \psi^\top \\
z_1 \\
z_2 \\
x_1 \end{pmatrix}^\top. \quad (3.37b)$$

We now design an observer based controller for (3.37), and it is given by

$$\dot{\phi} = (A - HC_n - BK)\phi + Hy_a), \quad (3.38a)$$

$$u_a = -K\phi, \quad (3.38b)$$

where $C_n = \frac{-1}{\hat{m}_c} \begin{pmatrix} -K_1, & m_{pg}, & 0, & 0 \end{pmatrix}$ is a nominal version of the matrix $C$, $\hat{m}_c$ is a nominal parameter used in lieu of $m_c$, and the gain matrices $K$ and $H$ are designed using the LQR technique to make $A - BK$ and $A - HC_n$ Hurwitz. These gain matrices were designed to be

$$K = \begin{pmatrix} -0.022, & 0.100, & -17.099, & -4.265, & -10.000 \end{pmatrix},$$

$$H = \begin{pmatrix} 0.082, & 0.568, & -4.763, & -19.097, & -0.100 \end{pmatrix}^\top.$$
3.3.2 The Output Feedback Regulator

The EHGO-based output feedback regulator for (3.35) is designed in accordance with Section 3.2.3, and is thus given by

\[
\begin{align*}
\dot{x}_1 &= x_2 + (\hat{\alpha}_1/\varepsilon)(y - \hat{x}_1), \\
\dot{x}_2 &= \hat{\sigma} + v/\hat{m}_c + (\hat{\alpha}_2/\varepsilon^2)(y - \hat{x}_1), \\
\dot{\hat{\sigma}} &= (\hat{\alpha}_3/\varepsilon^3)(y - \hat{x}_1), \\
\dot{\zeta} &= S\zeta + Jy, \\
\dot{\phi} &= (A - HC_n - BK)\phi + H\hat{\sigma}, \\
v &= -\beta\hat{m}_c \text{sat}\left(\frac{\hat{x}_2 + K\phi}{\mu}\right), \\
u &= v - K_1\zeta.
\end{align*}
\]

3.3.3 Numerical Simulations

Simulations were performed for the output feedback design given in Section 3.3.2, and the results were compared with that of the design provided in [10]. The results are depicted in the figures that follow.

3.3.3.1 Discussion

A comparison was made with the design presented in this chapter and the one based on [10] by means of simulations. The results showed that both the high-gain feedback design method of [10] and the EHGO-based design can achieve regulation for the targeted range of uncertainties in the plant parameters \(m_p\) and \(h\). Figure 3.1 to Figure 3.4 show the state and output trajectories for the case when \(\hat{m}_c = m_c = 0.82\) kg. It can be seen from these figures that the EHGO design provides better transient performance and shorter settling times. Another key difference is the performance
in the presence of uncertainty in the nominal plant parameter $m_c$, which also affects the control coefficient, which is basically $1/m_c$, and the range of uncertainty allowed for by each design. The high-gain feedback design of [10] achieves regulation when $m_c \in [0.819, 0.8206]$ kg, which is a range of uncertainty from $-0.12\%$ to $+0.0732\%$, while the EHGO design allows for $m_c \in [0.777, 0.84]$ kg, which is a range of $-5.24\%$ to $+2.44\%$. Moreover, the control effort corresponding to the EHGO method is consistently lower than the method of [10] by a factor of about 5 during the transient phase—this can be seen in Figure 3.5. Figure 3.6 shows the output trajectory of the EHGO design when $m_c = 0.777$ kg—the design of [10] is unstable for this particular parameter value.
3.4 Concluding Remarks

In this chapter, an output feedback regulator design was presented for non-minimum phase systems with uncertainties in the control coefficient, and in the constant system parameters. This design utilizes an extended high-gain observer and sliding mode control. The fundamental idea is to use the knowledge of a stabilizing controller for an associated auxiliary problem in designing the controller for the full system—this approach was first introduced by [21] and later used by [32] to solve the regulation problem. The contribution of this chapter is that it provides a design procedure that reduces the problem of controlling a non-minimum phase nonlinear system to one of designing a controller for a relative degree one minimum phase augmented system under certain assumptions, and uncertainty in the control coefficient is considered, as well. It is shown that this design is able to solve the regulation problem in the
presence of the aforementioned uncertainty, provided it is small enough. Some numerical simulations of a linearized inverted pendulum on a cart model are included to illustrate the procedure and the resulting performance.
Figure 3.4: Trajectories of the derivative of the output for the nominal value of $m_c$. 
Figure 3.5: The control effort of the algorithm from [10] (top) and the EHGO based design (bottom).
Figure 3.6: The response of the EHGO design when $m_c = 0.777$ kg.
Chapter 4

Stabilization and Regulation of the TORA System

4.1 Introduction

The translational oscillator with a rotating actuator (TORA) system, or rotational-translational actuator (RTAC), as it is also known, has been widely used in the literature for the past several years as a benchmark problem for testing various nonlinear control schemes. The problem was introduced in [43, 6]. In [22], two control laws are presented, the first a cascade controller and the second one a feedback passivating controller. A control law based on an $\mathcal{L}_2$ disturbance attenuation approach was proposed in [41]. Output feedback controllers dependent on measurements of the rotational and translational positions were proposed in [23], while controllers requiring only the measurement of the rotational position were presented in [25] and [13]. In [7], four different controllers were experimentally evaluated on an RTAC testbed.

The TORA problem was originally conceived as a simplified version of the dynamics of dual-spin spacecraft [43]. The interaction between rotation and translation in the oscillating eccentric rotor is analogous to the interaction between spin and
Figure 4.1: Translating Oscillator with Rotating Actuator (TORA).

nutation in a dual-spin spacecraft.

4.2 TORA Dynamics: A Simplified Model of a Dual Spin Spacecraft

Figure 4.1 depicts a schematic diagram that is based on experimental setups of the TORA benchmark system.\(^1\) This system consists of a platform or cart of mass \(m_c\) connected to a fixed reference frame via a linear spring with spring constant \(k\). The cart is constrained to only move horizontally. Attached to the cart is an unbalanced mass \(m_r\) with moment of inertia \(I\) about its center of mass, located at a distance \(l_r\) from its rotational axis. Let \(\theta\) be the rotational angle of the unbalanced mass, and \(x_c\) denote the translational position of the center of mass of this proof mass. The control torque applied to the proof mass at its axis of rotation is denoted by \(u\), and we also assume that there is a disturbance torque \(d\) acting at the same point—this disturbance signal could be a model of the effect of friction, or a periodic and

\(^1\)Figure taken from [29].
persistent disturbance torque generated by a collision between the proof mass and an obstacle, for example.

The rotating proof mass can be used to transfer a linear force to the cart, which in turn can be used to dampen the translational oscillations of the platform by a suitable design of the input torque \( u \) to the proof mass.

We now derive the dynamic model for the TORA system. Figure 4.1 shows a free body diagram of the proof mass next to the schematic, and as indicated by this diagram, the pivot where the proof mass is attached to the platform is subject to the external input torque \( u \) and the forces \( F_x \) and \( F_y \) applied by the cart in reaction to \( u \) and \( d \). Hence, Newton’s Second Law yields the following equations at the center of mass of the proof mass [29]

\[
m_r \frac{d^2}{dt^2}(x_c + l_r \sin \theta) = F_x,
\]

\[
m_r \frac{d^2}{dt^2}(-l_r \cos \theta) = F_y,
\]

\[u + F_y l_r \sin \theta - F_x \cos \theta = I \ddot{\theta}.
\]

By Newton’s Third Law, the platform is subject to the forces \(-F_x\) and \(-F_y\), but the force equation along the vertical is trivial and yields no information, since the frame of the testbed is constructed to absorb forces in the vertical direction and constrain the motion of the platform to the horizontal direction only. Newton’s Second Law gives the following dynamic equation governing the motion of the platform.

\[m_c \ddot{x}_C = -F_x - k x_C.
\]

Upon eliminating the internal forces \( F_x \) and \( F_y \) and carrying out the differentiation to the fullest extent possible, the dynamics of the TORA system can be expressed as
follows.

\[
D(\theta) \begin{pmatrix} \dot{\theta} \\ \dot{x}_C \end{pmatrix} = \begin{pmatrix} u \\ m_r l_r \dot{\theta}^2 \sin \theta - k x_C \end{pmatrix},
\]

where

\[
D(\theta) = \begin{pmatrix} I + m_r l_r^2 & m_r l_r \cos \theta \\ m_r l_r \cos \theta & m_r + m_c \end{pmatrix}.
\]

The determinant of \(D(\theta)\) is

\[
\delta(\theta) = (I + m_r l_r^2)(m_r + m_c) - m_r^2 l_r^2 \cos^2 \theta \geq (I + m_r l_r^2)m_c + m_r I > 0,
\]

and so we can solve the above equation for \(\ddot{\theta}\) and \(\ddot{x}_C\) to obtain

\[
\begin{pmatrix} \ddot{\theta} \\ \ddot{x}_C \end{pmatrix} = \frac{1}{\delta(\theta)} \begin{pmatrix} m_r + m_c & -m_r l_r \cos \theta \\ -m_r l_r \cos \theta & I + m_r l_r^2 \end{pmatrix} \begin{pmatrix} u \\ m_r l_r \dot{\theta}^2 \sin \theta - k x_C \end{pmatrix}. \tag{4.1}
\]

We note that since both the \(\ddot{\theta}\) and \(\ddot{x}_C\) equations in (4.1) above depend on \(u\), the obvious choice of state variable, to wit \(q = \begin{pmatrix} x_c & \dot{x}_c & \theta & \dot{\theta} \end{pmatrix}^\top\), will not yield a set of equations for the plant dynamics in the normal form. Consequently, we apply the following change of variables \cite{29}

\[
\begin{align*}
\dot{z}_1 &= x_c + \frac{m_r l_r \sin \theta}{m_r + m_c}, & \dot{z}_2 &= \dot{x}_c + \frac{m_r l_r \dot{\theta} \cos \theta}{m_r + m_c}, & \dot{x}_1 &= \theta, & \dot{x}_2 &= \dot{\theta},
\end{align*}
\tag{4.2}
\]

and it can be verified that (4.2) transforms (4.1) into the following normal form.

\[
\begin{align*}
\dot{z}_1 &= z_2, \quad & \text{\tag{4.3a}}
\end{align*}
\]

\[
\begin{align*}
\dot{z}_2 &= \frac{k}{m_r + m_c} \left( \frac{m_r l_r \sin x_1}{m_r + m_c} - z_1 \right), \quad & \text{\tag{4.3b}}
\end{align*}
\]

\[
\dot{x}_1 = x_2, \quad & \text{\tag{4.3c}}
\]
\[ \dot{x}_2 = \frac{1}{\delta(x_1)} \left\{ (m_r + m_c)u - m_r l_r \cos x_1 \right\} \]
\[ \times \left[ m_r l_r x_2^2 \sin x_1 - k \left( z_1 - \frac{m_r l_r \sin x_1}{m_r + m_c} \right) \right], \]
\[ y = x_1. \]

4.3 Stabilization of the TORA System

In this section, we consider the problem of stabilizing the TORA system about its equilibrium point, which is the origin. The model equations in the normal form are given by (4.3), and we make the substitutions \( \eta_1 = z_1, \eta_2 = z_2, \xi_1 = x_1 \) and \( \xi_2 = x_2 \) for convenience, so that the equations in this section match the notation used in Chapter 2. Moreover, in this section, we do not consider disturbances or reference signals, and so (4.3) now becomes.

\[ \dot{\eta}_1 = \eta_2, \]
\[ \dot{\eta}_2 = \frac{k}{m_r + m_c} \left( \frac{m_r l_r \sin \xi_1}{m_r + m_c} - \eta_1 \right), \]
\[ \dot{\xi}_1 = \xi_2, \]
\[ \dot{\xi}_2 = \frac{1}{\delta(\xi_1)} \left\{ (m_r + m_c)u - m_r l_r \cos \xi_1 \right\} \]
\[ \times \left[ m_r l_r \xi_2^2 \sin \xi_1 - k \left( \eta_1 - \frac{m_r l_r \sin \xi_1}{m_r + m_c} \right) \right], \]
\[ y = \xi_1. \]

Uncertainty in the cart mass \( m_c \) and rotor mass \( m_r \) results in a system with uncertain functions \( a(\cdot) \) and \( b(\cdot) \) corresponding to (2.1)–(2.4). The design is carried out for nominal values of \( m_r \) and \( m_c \), and then the effect of perturbing these parameters is studied via simulation. We assume that the output \( y \) is the only measured variable, and the goal is to stabilize this system about its origin by feeding back the output.
In accordance with the technique presented in Chapter 2, we assume that the signals $\xi$ and $\sigma = b(\eta, \xi, \theta)$ are available, since they can be estimated by the extended high-gain observer to be introduced later, and so we augment (4.4)–(4.8) with a compensator

$$\dot{\phi} = L(\phi, \xi_1) + M(\phi, \xi_1)\sigma,$$

(4.9)

and we define a new variable $s = \xi_2 - N(\phi, \xi_1)$, where $L(\cdot), M(\cdot)$ and $N(\cdot)$ are to be designed with the goal of rendering the resulting system minimum phase when $s$ is viewed as its output. We now perform the change of variable $\xi_\rho \mapsto s$. We thus obtain the augmented system

$$\dot{\eta}_1 = \eta_2,$$

(4.10a)

$$\dot{\eta}_2 = \frac{k}{m_r + m_c} \left( \frac{m_r l_r \sin \xi_1}{m_r + m_c} - \eta_1 \right),$$

(4.10b)

$$\dot{\xi}_1 = \xi_2,$$

(4.10c)

$$\dot{s} = \frac{1}{\delta(\xi_1)} \left\{ (m_r + m_c)u - m_r l_r \cos \xi_1 \right.\right.$$ \begin{equation}
\times \left. \left[ m_r l_r \xi_2^2 \sin \xi_1 - k \left( \eta_1 - \frac{m_r l_r \sin \xi_1}{m_r + m_c} \right) \right]\right.\right. 
\left. \left. - \frac{\partial N}{\partial \xi_1} [s + N(\phi, \xi_1)] \right\},

(4.10d)

$$\dot{\phi} = L(\phi, \xi_1) + M(\phi, \xi_1) \frac{m_r l_r \cos \xi_1}{\delta(\xi_1)} \left[ k \left( \eta_1 - m_r l_r (N(\phi, \xi_1))^2 \sin \xi_1 \right) \right],$$

(4.10e)

which is clearly relative degree one with respect to $s$, and with this signal viewed as the output, we see that the zero dynamics $\{(\eta, \xi_1) \mid s = 0\}$ obey

$$\dot{\eta}_1 = \eta_2,$$

(4.11)

$$\dot{\eta}_2 = \frac{k}{m_r + m_c} \left( \frac{m_r l_r \sin \xi_1}{m_r + m_c} - \eta_1 \right).$$

(4.12)
\[ \dot{\xi}_1 = u_a, \tag{4.13} \]
\[ u_a = N(\phi, \xi_1), \tag{4.14} \]
\[ \dot{\phi} = L(\phi, \xi_1) + M(\phi, \xi_1)\sigma, \tag{4.15} \]
\[ \sigma = \left[k \left(\eta_1 - \frac{m_r l_r \sin \xi_1}{m_r + m_c}\right) - m_r l_r u_a^2 \sin \xi_1\right] \frac{m_r l_r \cos \xi_1}{\delta(\xi_1)}. \tag{4.16} \]

We note that this system is identical to the closed-loop auxiliary system as introduced by Isidori [21]. The problem is now reduced to one of stabilizing (4.11)–(4.16), and we do this in the subsections to follow.

### 4.3.1 Design of the Stabilizing Controller

In order to design the stabilizing compensator (4.14)–(4.16), we shall adopt the strategy of first linearizing the input-output map of (4.11)–(4.13), then designing a linear compensator for the transformed system, and then implementing the inverse transformations to obtain the nonlinear control \{L, M, N\} in the original coordinates. Hence, we define a new output for the auxiliary system as

\[ \tilde{y}_a = \frac{\delta(\xi_1)\sigma}{m_r l_r k \cos \xi_1} + \frac{m_r l_r u_a^2 \sin \xi_1}{k} + \frac{m_r l_r \sin \xi_1}{m_r + m_c} \]

\[ \triangleq \tilde{h}_a(y_a, \xi_1, u_a). \tag{4.17} \]

With the newly defined output \( \tilde{y}_a \), the output equation is given by

\[ \tilde{y}_a = \eta_1. \tag{4.18} \]

The design of a stabilizing feedback controller for the system (4.11)–(4.13) and (4.18) can be simplified by viewing \( \sin \xi_1 \) as a virtual control input to the system (4.11)–(4.12), and then using either backstepping or high-gain feedback to determine the
control $u_a$. We shall use the high-gain feedback approach. To this end, let $u_a = (v_a - \xi_1)/\varepsilon_a$, where $v_a$ is a control input to be designed and $\varepsilon_a$ is a positive number. The open loop auxiliary system can now be expressed in terms of (4.11), (4.12), (4.18) and

$$\varepsilon_a \dot{\xi}_1 = v_a - \xi_1. \quad (4.19)$$

For $\varepsilon_a$ small enough, (4.11), (4.12) and (4.19) will exhibit a two time-scale behavior and can thus be regarded as a singularly perturbed system. The reduced system, obtained by setting $\varepsilon_a = 0 \Rightarrow v_a = \xi_1$, is given by (4.11), (4.12) and (4.18), and for this reduced system, the quantity

$$\sin \xi_1 = \sin v_a \triangleq w_a \quad (4.20)$$

is regarded as the control input, and can now be designed as a linear control. To summarize, the control problem for the auxiliary system reduces to a problem of finding a compensator with a strictly proper transfer function for the linear system

$$\dot{\eta}_1 = \eta_2, \quad (4.21)$$

$$\dot{\eta}_2 = \frac{k}{m_r + m_c} \left( \frac{m_r l_r}{m_r + m_c} w_a - \eta_1 \right), \quad (4.22)$$

$$\tilde{y}_a = \eta_1. \quad (4.23)$$

It can be shown that the above system is observable and controllable, and a state feedback controller $w_a = k_1 \eta_1 + k_2 \eta_2$ was found using LQR theory. The values of $k_1$ and $k_2$ providing near-optimal trade-off between the transient performance of the states and the control effort were computed in terms of a parameter $\rho_1$ used to
minimize the cost function

\[ J = \int_0^\infty \left[ \left( \eta_1 - \frac{m_r l_r}{m_r + m_c} w_a \right)^2 + \rho_1 w_a^2 \right] dt, \]

subject to the constraint \(|w_a| \leq 1\). This parameter was tuned to ensure that \(w_a\) just meets the constraint. This yielded the parameter values of \(k_1 = -1\) and \(k_2 = -8\).

Next, a full-order observer was designed for (4.21)–(4.23) as

\begin{align*}
\dot{\phi}_1 &= \phi_2 + \tilde{\alpha}_1(\ddot{y}_a - \phi_1), \\
\dot{\phi}_2 &= \tilde{\alpha}_2(\ddot{y}_a - \phi_1) - \frac{k\phi_1}{m_r + m_c} + \frac{km_r l_r}{(m_r + m_c)^2} w_a,
\end{align*}

where \(\tilde{\alpha}_1 = 2\) and \(\tilde{\alpha}_2 = 1\). The observer-based control law can now be taken as

\[ w_a = k_1 \phi_1 + k_2 \phi_2, \]

and (4.24)–(4.26) provide the strictly proper dynamic compensator alluded to earlier.

We now substitute the transformations into the compensator equations above and find the stabilizing controller for the auxiliary system (4.11)–(4.16) to be

\begin{align*}
\dot{\phi}_1 &= \phi_2 + \tilde{\alpha}_1(\ddot{h}_a(\sigma, \xi_1, N(\phi, \xi_1)) - \phi_1), \\
\dot{\phi}_2 &= \tilde{\alpha}_2(\ddot{h}_a(\sigma, \xi_1, N(\phi, \xi_1)) - \phi_1) - \frac{k\phi_1}{m_r + m_c} \\
&\quad + \frac{km_r l_r}{(m_r + m_c)^2} \text{sat}(k_1 \phi_1 + k_2 \phi_2), \\
u_a &= N(\phi, \xi_1),
\end{align*}

where \(N(\phi, \xi_1) = \left\{ \sin^{-1}\left[ \text{sat}(k_1 \phi_1 + k_2 \phi_2) \right] - \xi_1 \right\} / \varepsilon_a\). We note that we used the saturated control \(\text{sat}(w_a) = \text{sat}(k_1 \phi_1 + k_2 \phi_2)\) in order to ensure that its contribution to the control effort does not exceed unity magnitude in the presence of peaking.
Thus, the partial state feedback controller for the system (4.4)–(4.7) is given by (4.27)–(4.28) and
\[ u = -\frac{\beta \delta(\xi_1)}{m_r + m_c} \text{sat} \left( \frac{\xi_2 - N(\phi, \xi_1)}{\mu} \right). \] (4.30)

### 4.3.2 Output Feedback Controller for the TORA

The output feedback controller for the system (4.4)–(4.8) can now be designed using an extended high-gain observer.

\[
\begin{align*}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + (\alpha_1 / \varepsilon)(y - \hat{\xi}_1), \quad (4.31) \\
\dot{\hat{\xi}}_2 &= \hat{\sigma} + a_n(y)u + (\alpha_2 / \varepsilon^2)(y - \hat{\xi}_1), \quad (4.32) \\
\dot{\hat{\sigma}} &= (\alpha_3 / \varepsilon^3)(y - \hat{\xi}_1), \quad (4.33) \\
\dot{\phi}_1 &= \phi_2 + \hat{\alpha}_1 \left[ \hat{h}_a(\hat{\sigma}, y, N(\phi, y)) - \phi_1 \right], \quad (4.34) \\
\dot{\phi}_2 &= \hat{\alpha}_2 \left[ \hat{h}_a(\hat{\sigma}, y, N(\phi, y)) - \phi_1 \right] - \frac{k \phi_1}{\hat{m}_r + \hat{m}_c} \\
&\quad + \frac{k \hat{m}_r l_r}{(\hat{m}_r + \hat{m}_c)^2} \text{sat}(k_1 \phi_1 + k_2 \phi_2), \quad (4.35) \\
u &= -\frac{\beta}{a_n(y)} \text{sat} \left( \frac{\hat{\xi}_2 - N(\phi, y)}{\mu} \right), \quad (4.36)
\end{align*}
\]

where \( \delta(\xi_1) = (I + \hat{m}_r l_r^2)(\hat{m}_r + \hat{m}_c) - \hat{m}_r^2 l_r^2 \cos^2 \xi_1 \), the nominal function \( a_n(y) = (\hat{m}_r + \hat{m}_c) / \hat{\delta}(y) \), and
\[
\hat{h}_a(\hat{\sigma}, y, N(\phi, y)) = \frac{\hat{\delta}(y) \hat{\sigma}}{\hat{m}_r l_r k \cos y} + \frac{\hat{m}_r l_r}{k} (N(\phi, y))^2 \sin y + \frac{\hat{m}_r l_r \sin y}{\hat{m}_r + \hat{m}_c},
\]

and \( \hat{m}_r \) and \( \hat{m}_c \) are nominal values for \( m_r \) and \( m_c \), respectively.
4.3.3 Output Feedback Design Based on Isidori [21]

We now follow the procedure of [21] to design an output feedback stabilizer for (4.4)–(4.8) using a high gain approach, and we shall compare the design from the previous subsection with this one by means of numerical simulations. As per [21], the design is again based on the stabilizer for auxiliary system, and the high gain observer based output feedback stabilizer was found to be

\[
\begin{align*}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + (\hat{\alpha}_1/\varepsilon_1)(y - \hat{\xi}_1), \\
\dot{\hat{\xi}}_2 &= (\hat{\alpha}_2/\varepsilon_2)(y - \hat{\xi}_1), \\
\dot{\hat{\phi}}_1 &= \hat{\alpha}_1 \left[ \hat{h}_a(k_I(\hat{\xi}_2 - N(\phi, y)), y, N(\phi, y)) - \hat{\phi}_1 \right] + \phi_2, \\
\dot{\hat{\phi}}_2 &= \hat{\alpha}_2 \left[ \hat{h}_a(k_I(\hat{\xi}_2 - N(\phi, y)), y, N(\phi, y)) - \hat{\phi}_1 \right] \\
&\quad - \frac{k\hat{\phi}_1}{\hat{m}_r + \hat{m}_c} + \frac{k\hat{m}_rl_r}{(\hat{m}_r + \hat{m}_c)^2} \text{sat}(k_1\phi_1 + k_2\phi_2), \\
u &= \frac{\hat{\delta}(y)}{\hat{m}_r + \hat{m}_c} \left[ \hat{N} - (\hat{\xi}_2 - N(\phi, y))k_I \right],
\end{align*}
\]

where \(\hat{N} = k_1p(\phi)\dot{\phi}_1 + k_2p(\phi)\dot{\phi}_2 - \hat{\xi}_2/\varepsilon a\), the constant \(k_I\) is a high gain parameter,

\[
p(\phi) = \frac{u_s(k_1\phi_1 + k_2\phi_2 + 1) - u_s(-k_1\phi_1 - k_2\phi_2 - 1)}{\varepsilon a \sqrt{1 - [\text{sat}(k_1\phi_1 + k_2\phi_2)]^2}},
\]

and \(u_s(\cdot)\) is the unit step function.

4.3.4 Numerical Simulations

Numerical simulations were performed using the system parameters shown in Table 4.1\(^2\) and controller parameters given in Table 4.2. Figures 4.2 through 4.13 show the effects of tuning certain design parameters and of perturbations in the masses of the cart and the proof mass.

\(^2\)These values are taken from [43]
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cart mass</td>
<td>$m_c$</td>
<td>1.3608 kg</td>
</tr>
<tr>
<td>Rotor mass</td>
<td>$m_r$</td>
<td>0.096 kg</td>
</tr>
<tr>
<td>Dist. from rotor pivot to its CM</td>
<td>$l_r$</td>
<td>0.0592 m</td>
</tr>
<tr>
<td>Rotor moment of inertia</td>
<td>$I$</td>
<td>0.0002175 kg m$^2$</td>
</tr>
<tr>
<td>Spring constant</td>
<td>$k$</td>
<td>186.3 N/m</td>
</tr>
</tbody>
</table>

Table 4.1: TORA system parameters.

<table>
<thead>
<tr>
<th>Module</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Auxiliary controller</td>
<td>$\alpha_1$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$\alpha_2$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>$\mu$</td>
<td>${1, 0.1}$</td>
</tr>
<tr>
<td>State feedback controller</td>
<td>$\hat{\alpha}_1$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_2$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$\hat{\alpha}_3$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\epsilon$</td>
<td>${0.01, 0.001}$</td>
</tr>
<tr>
<td>Extended high-gain observer</td>
<td>$\theta(0)$</td>
<td>0 rad</td>
</tr>
<tr>
<td></td>
<td>$x_c(0)$</td>
<td>0.025 m</td>
</tr>
</tbody>
</table>

Table 4.2: TORA stabilization: simulation parameters and initial conditions.
4.3.5 Discussion of the Results

Figures 4.2 through 4.5 show the results of a step-by-step tuning of the design parameters, starting with matching the performance of the fourth-order reduced system in the first step and ending with the tuning of the extended high-gain observer parameter to match the partial state feedback case. In the process, the full output feedback system is made to match the performance of the reduced system, whose state equation is linear and fourth order—however, since the compensator’s eigenvalues are located quite far into the left half-plane relative to those of the reduced system’s plant, the state equation for the latter is essentially second order and linear. Therefore, this
design methodology enabled us to enforce linear second order-like response characteristics on the controlled TORA system. We explain the tuning procedure to recover the reduced system’s performance below. The errors in each step are computed using the system from the previous step as the new target.

1) The top panel in Figure 4.2 shows the target response trajectory we seek to recover. The solid line is the response of the reduced system (4.11)–(4.12), (4.18), (4.24)–(4.26). The dashed lines are the responses of the auxiliary system (4.11)–(4.16), (4.27)–(4.29), and in these cases, high-gain feedback parameter values of $\varepsilon_a = 0.1$ and 0.01 were utilized. The bottom panel shows how reducing $\varepsilon_a$ from 0.1 to 0.01 allows for the auxiliary system to better approximate the response of the reduced system—indeed, the settling time was reduced drastically. The $\varepsilon_a$ parameter is

Figure 4.3: Recovery of the Auxiliary System performance by decreasing $\mu$ (bottom: solid is $\mu = 1$, dash-dot is $\mu = 0.1$).
now fixed at 0.01 and we move to the next step.

2) Now, we try and match the performance of the auxiliary system with the partial state feedback system (4.4)–(4.7), (4.27)–(4.28), (4.30). Figure 4.3 shows the results. The reference trajectory used to compute the error in this step is the auxiliary system state $\xi_1$. The switch was made from $\eta_1$ to $\xi_1$ because the former is not an output of the full system. A $\mu$ value of 0.1 was deemed to provide an acceptably small error, so it was fixed at this value.

3) We now tune the extended high-gain observer parameter $\varepsilon$ to recover the perfor-
Figure 4.5: Recovery of the Partial State Feedback System’s performance by decreasing $\varepsilon$ ($\mu = 0.1$).

mance of the partial state feedback system from the previous step. Figures 4.4 and 4.5 show the recovery of the performance of the state feedback system by reducing $\varepsilon$ from 0.01 to 0.001, in the output feedback system (4.4)–(4.8), (4.31)–(4.36). We note from the bottom panels of these two figures that reducing $\varepsilon$ down to $10^{-3}$ results in a significant reduction in the error. We therefore choose the final design parameter as $\varepsilon = 10^{-3}$.

The above three-step tuning procedure suggests that we pick the design parameters $\varepsilon_a = 0.01$, $\mu = 0.1$ and $\varepsilon = 10^{-3}$ so that the performance of the reduced system is recovered by the output feedback system when the latter is initialized at $y(0) = 0$ (i.e. the trajectory starts out on the sliding manifold, which is also the zero dynamics manifold of the closed loop system when $s$ is viewed as the output).
Next, the EHGO-and-SMC design presented in this paper was compared with the design based on [21]. Figure 4.6 shows the best responses that were achieved after tuning the parameters of both designs. Figure 4.7 shows the control effort required under each method, and it can be seen that they are virtually identical when $y(0) = 0$. Starting with Figure 4.6, the responses of the EHGO + SMC design are shown as solid curves, while those of the high gain feedback design [21] are shown as dash-dotted lines. It should be noted that in the design based on [21], the high gain feedback parameter was tuned to $k_I = 500$, which is comparable to $\beta = 500$ in the EHGO + SMC design. Another point to note is that the high gain observer parameter for the design from [21] needed to be set to $\varepsilon = 10^{-6}$ to ensure stability of the output feedback system, whereas in the EHGO-and-SMC case, a value of $\varepsilon = 10^{-3}$ was
sufficient to guarantee stability and matching of the state feedback performance—this was the case for all subsequent simulations discussed below, as well.

Next, $y(0)$ was chosen to be non-zero, and this immediately revealed differences between the transient performance of the high gain feedback approach [21] and that of the SMC design from our paper. Figure 4.8 shows a comparison of the transient performance when $y(0) = 0.086$. Figure 4.9 shows that the control effort expended by the high gain feedback design is quite large during the transient phase, while the sliding mode control saturates at a magnitude of under 0.3 during this period. Figures 4.10 and 4.11 show the response of the EHGO + SMC design when $y(0) = 0.541$—this initial condition brings the system to the verge of instability. The high gain feedback
design is unstable when initiated at this rotor angle, and so its response is not shown in these two figures. An important point to note is that due to (4.17), the closed loop systems under both designs will have singularities at $2j\pi \pm \pi/2$, $j \in \mathbb{Z}$, and Figures 4.8, 4.10 show the transient response of $y$ approaching $\pi/2$—a saturated control input is preferable in such instances in order to obtain a larger region of attraction.

Finally, in order to investigate the robustness of the two designs, the masses of the cart and the rotor were perturbed and some simulations were run. Figure 4.12 shows the result when the cart and eccentric rotor masses were both perturbed by $+100\%$. Figure 4.13 shows the result when the cart and eccentric rotor masses were perturbed $-50\%$. The $-50\%$ perturbation brought both designs to the verge of instability. The conclusion of this particular investigation is that both designs are able to tolerate
similar amounts of perturbations in the cart and rotor mass parameters.

4.4 Concluding Remarks

In this chapter, the stabilization algorithm from Chapter 2 was employed to stabilize the translational oscillator with rotating actuator (TORA) benchmark system. We were able to simplify the original stabilization problem to one of finding a stabilizing controller for an associated auxiliary system—in order to find this controller, a high-gain feedback approach was employed, which in turn reduced the problem further to a constrained linear control problem for a simple second order linear system. A linear dynamic controller satisfying the control constraint was obtained by optimizing
Figure 4.10: Response of the EHGO + SMC based design when $y(0) = 0.541$.

a quadratic cost function. This immediately led to a set of equations for the nonlinear controller that stabilizes the auxiliary system, and the remainder of the output feedback stabilization problem was solved on the basis of the knowledge of the controller for the auxiliary system, in accordance with the design procedure presented in Chapter 2.

The simulations also show that the performance of the closed-loop reduced system, which is only fourth order, can be asymptotically recovered by appropriate tuning of a single high-gain feedback parameter $\varepsilon_a$—this is possible so long as the initial conditions place the trajectories of the system on the sliding manifold, thus eliminating the reaching phase. In the full output feedback system, the performance can be fine-tuned further by reducing the width of the boundary layer $\mu$ in the sliding mode control, and
also by reducing the extended high-gain observer parameter $\varepsilon$. Through simulations, it was observed that the stabilizing controller design allows for uncertainties between $-50\%$ and $+100\%$ for both the cart mass and the rotor mass.

An important point to note is that decreasing the $\varepsilon$ parameter in the extended high-gain observer too much could potentially pose practical difficulties during implementation due to the effects of measurement noise as discussed in Chapter 2. That being said, the two designs presented in this chapter fully comply with the performance requirements and constraints set forth in [6], to wit: $|x_c| \leq 0.025$ m, and $|u| \leq 0.1$ Nm.

In conclusion, the stabilizing controller designed in this chapter for the TORA
system provides good transient performance and exhibits robustness to perturbations of the cart and rotor masses. The design procedure is fairly simple and systematic in that it provides a framework for converting the problem to one of stabilizing a relative degree one system, followed by order reduction, provided a stabilizing controller can be found for the auxiliary system of order one less than that of the original problem—this proved to be true for the TORA system, thus demonstrating that this technique may be profitably employed to solve other application problems in the future. Furthermore, this design approach was ultimately able to recover the performance characteristics of a full-order observer-based second order linear system, so long as the initial conditions are located on the sliding manifold of the full system.

Figure 4.12: Response under $+100\%$ perturbation in $m_c$ and $m_r$. 
Figure 4.13: Response under −50% perturbation in $m_c$ and $m_r$. 
Chapter 5

Conclusions and Future Work

5.1 Concluding Remarks

This dissertation is concerned with the problems of robust stabilization and regulation, by means of output measurements alone, of non-minimum phase systems that are transformable into the normal form. This work is an extension of the design tool of Isidori [21], and the chapter on regulation is similarly an extension of the technique presented in the paper by Marconi et al.[32]. Nevertheless, this work represents a significant contribution to the control literature because it presents a design methodology that is more versatile than [21, 32, 10], in that any robust control technique can be applied in the design of the control law, unlike the papers by Isidori and coworkers, in which high-gain feedback control was an intrinsic component of the design. This extension to other robust control schemes is possible using our approach due to the fact that an extended high-gain observer is used to estimate an unknown function that renders the zero dynamics of the plant observable, and thus allows us to stabilize the system despite its being non-minimum phase. The contributions of this work are discussed in further detail in Section 5.1.1.

In order to derive the stability results presented in this work, continuously-
implemented sliding mode control was chosen—this was also a natural choice because sliding mode control is known to have good robustness properties, and since the control effort saturates outside the boundary layer, this design precludes the possibility of large spikes during the transient phase of the system response, thus providing protection against the peaking phenomenon, which is a common feature of output feedback systems that employ high-gain observers.

5.1.1 Main Contributions

The primary contribution of this work is the following three-step design procedure for stabilizing the origin of a non-minimum phase system in the normal form.

1) Augment a dynamic compensator \{L, M, N\} to the system and moreover, design this triple such that the resultant system is minimum phase and relative degree one with respect to the virtual output \(s = e_\rho - N(\cdot)\). This task is equivalent to the stabilization of the auxiliary system of Isidori [21], since this closed-loop auxiliary system coincides with the zero dynamics of the aforementioned augmented system with \(s\) viewed as the output.

2) Now use any of several standard robust control techniques such as sliding mode control, saturated high-gain feedback, Lyapunov redesign, etc. to design a static state feedback control law that stabilizes the relative degree one minimum phase system obtained from the previous step.

3) Finally, design an extended high-gain observer to estimate the system output and its first \(\rho - 1\) derivatives, and additionally, the signal \(b\) that is required to solve the auxiliary design problem in step 1). Replace all the states and the signal \(b\) in the state feedback control design with their estimates, and if necessary, saturate the resulting control law outside a compact set of interest, or saturate the estimates obtained from the EHGO.
It is important to note that we were able to separate steps 1) and 2) in the design process in anticipation of the fact that the extended high-gain observer would be used in the final step to estimate $b$ along with $y$ and its first $\rho - 1$ derivatives. The approach pursued by [21, 32, 10] is different; the recovery of the signal $b$ is intrinsic to the design of the control scheme employed in these papers, meaning that steps 1) and 2) go hand-in-hand if this technique is utilized, and hence it becomes necessary to implement their particular high-gain feedback based control law.

Another contribution of this work is the fact that our design allows for uncertainty in the control coefficient $a(\cdot)$, which was not addressed by Isidori and coworkers. This constitutes the main technical difference between the results presented in this work and those of [21]. Handling this uncertainty dominates our analysis in Sections 2.3 and 3.2.2. In both Chapters 2 and 3, we prove exponential stability of the closed-loop system, under the assumption of exponential stability of the closed-loop auxiliary system. The paper [21] proves only practical stabilization, but it requires the less stringent condition of asymptotic stability of the closed-loop auxiliary system.

Simulation examples afforded opportunities to compare our design with that of [21]. The main difference was that in all the examples, the high-gain feedback design of Isidori exhibits a sharp transient response and much larger control effort during this phase, when compared to our designs. Moreover, in the example involving a tangent nonlinearity in Chapter 2, the size of the estimate of the region of attraction was much larger in our case than in the case of the high-gain feedback design of [21].

The stabilization design of Chapter 2 was applied to the nonlinear TORA benchmark system in Chapter 4, and this served to demonstrate that our design procedure can be successfully applied to practical problems and non-trivial physical systems. Once again, comparisons were made between our design method and that of Isidori [21], and the results were consistent with all the other observations made by simulating the aforementioned academic examples. In particular, our design was able to stay
within the performance limitations of $|u| \leq 0.1$ and $|x_c| \leq 0.025$ as specified in [6] over a much wider range of initial conditions than the high-gain feedback design of Isidori [21].

### 5.2 Future Directions

The stabilization and regulation designs that we presented for non-minimum phase systems could be extended in several directions, and could potentially also be unified with other control schemes that employ extended high-gain observers. The following are some ideas for immediate extensions of this work.

- Apply the conditional integrator in conjunction with the extended high-gain observer to solve the problem of causing a non-minimum phase system to track a constant reference and reject constant disturbances.

- Extend the above problem to output regulation of non-minimum phase systems using a conditional servocompensator. This could potentially allow for the design of the stabilizing compensator for the plant alone, without considering the dynamics of the servocompensator, thus paving the way for reusing stabilization designs in regulation problems. Moreover, the conditional servocompensator may be able to asymptotically recover the performance of “ideal” sliding mode control.

- Extend our techniques to incorporate control schemes such as saturated high-gain feedback, Lyapunov redesign, etc.

- Combine the extensions of the results presented in this work with that of Freidovich and Khalil [16] to obtain a general framework that provides for the application of extended high-gain observers in solving tracking and disturbance rejection problems for systems that are possibly non-minimum phase.
APPENDIX
Proofs and Technical Details

A.1 Locally Lipschitz property of $\Delta_0/\varepsilon$

In this section, we illustrate the locally Lipschitz property of the quantity $\Delta_0/\varepsilon$ of Chapter 2 by first looking at the sub-expression

\[
\frac{1}{\varepsilon}[g_\varepsilon(l(\hat{\xi}, \phi)/K) - g_\varepsilon(l(\xi, \phi)/K)].
\]  

(A.1)

To simplify the analysis, we define

\[ G(\xi, \phi) \triangleq g_\varepsilon(l(\xi, \phi)/K). \]

Hence, by the Fundamental Theorem of Calculus, we have

\[
\frac{1}{\varepsilon}[G(\hat{\xi}, \phi) - G(\xi, \phi)] = \frac{1}{\varepsilon} \left[ \left. G(\xi + \lambda(\hat{\xi} - \xi), \phi) \right|_{\lambda=1} - \left. G(\xi + \lambda(\hat{\xi} - \xi), \phi) \right|_{\lambda=0} \right] \\
= \frac{1}{\varepsilon} \int_0^1 \frac{\partial}{\partial \xi} G(\xi + \lambda(\hat{\xi} - \xi), \phi) \, d\lambda(\hat{\xi} - \xi) \\
= -\int_0^1 \frac{\partial}{\partial \xi} G(\xi + \lambda(\hat{\xi} - \xi), \phi) \, d\lambda \\
\times \left( \varepsilon^{\rho-1} \nu_1, \ldots, \varepsilon^{\nu_{\rho-1}}, \nu_{\rho} \right)^T. \]  

(A.2)

We note that both $g_\varepsilon(\cdot)$ and $l(\cdot)$ are differentiable and have locally Lipschitz derivatives with respect to their arguments, and hence (A.2) is locally Lipschitz uniformly
in $\varepsilon$. We can use similar arguments for the remaining terms of $\Delta_0/\varepsilon$ to show that they are also locally Lipschitz uniformly in $\varepsilon$.

A.2 Solution of the PDE (2.14)

A.2.1 Properties of the solution

In this subsection, we establish the existence of the transformation $q = T(x_a, \phi, s, \theta)$ as a solution of (2.14), and determine the conditions under which it is a diffeomorphism. Since $a(\cdot) \neq 0$, we can write (2.14) as

$$
\begin{align*}
\left(\frac{\partial T}{\partial \phi} - \frac{\partial T}{\partial s} \frac{\partial N}{\partial \phi}\right) M \frac{\Delta a}{a} + \frac{\partial T}{\partial s} = 0.
\end{align*}
$$

(A.3)

Let $\varphi = \begin{pmatrix} \phi^\top & s \end{pmatrix}^\top$ and

$$
\bar{g}(\varphi, \theta) = \begin{pmatrix} M \frac{\Delta a}{a} \\ 1 - \frac{\partial N}{\partial \phi} M \frac{\Delta a}{a} \end{pmatrix}.
$$

We note that for sufficiently small $|\Delta a/a|$, $\bar{g}$ has full rank, and so we shall make this assumption henceforth. The PDE (A.3) can now be written as

$$
\frac{\partial T}{\partial \varphi} \bar{g}(\varphi, \theta) = 0.
$$

Since $\bar{g}$ is of full rank, the Frobenius theorem [19] guarantees the existence of a solution to the above PDE. Now, let

$$
\tilde{\varphi} = \tilde{T}(x_a, \varphi, s, \theta) = \begin{pmatrix} q^\top & s \end{pmatrix}^\top = \left(\begin{pmatrix} (T(x_a, \phi, s, \theta))^\top & s \end{pmatrix}^\top \right).
$$
We seek conditions under which $\tilde{T}$ is a diffeomorphism. We note that its Jacobian matrix is

$$\frac{\partial \tilde{T}}{\partial \phi} = \begin{pmatrix} \frac{\partial T}{\partial \phi} & \frac{\partial T}{\partial s} \\ 0 & 1 \end{pmatrix},$$

so that $\frac{\partial \tilde{T}}{\partial \phi}$ is nonsingular if and only if $\frac{\partial T}{\partial \phi}$ is nonsingular. The latter condition will be utilized when finding the solution to (A.3) in the following subsection.

Finally, we conclude by the inverse function theorem [19] that if $\frac{\partial \tilde{T}}{\partial \phi}$ is nonsingular for some $\phi$ in the domain of interest, then there exists an open neighborhood of $\phi$ in this domain such that the restriction of $\tilde{T}$ to this neighborhood is a diffeomorphism.

### A.2.2 Details of the solution

We begin by noting that when $\Delta a = 0$, a particular solution of (A.3) can be taken to be $T = \phi$.

Now, for $\Delta a \neq 0$, we need to employ the method of characteristics for (quasi-)linear differential equations, which reduces (A.3) to a system of ODE along so-called characteristic curves of the PDE. To this end, we rewrite (A.3) as a system of scalar differential equations as follows.

$$\left( \frac{\partial T_i}{\partial \phi} - \frac{\partial T_i}{\partial s} \frac{\partial N}{\partial \phi} \right) M \frac{\Delta a}{a} + \frac{\partial T_i}{\partial s} = 0, \quad 1 \leq i \leq r, \quad (A.4)$$

where $T_i$ is the $i$th component of $T$. A solution $T_i = q_i(x_a, \phi, s, \theta)$ defines an integral surface in the $\phi, s, T_i$-space. The direction of the normal to the surface is $\left( \frac{\partial T_i}{\partial \phi}, \frac{\partial T_i}{\partial s}, -1 \right)^\top$, so that (A.4) can be interpreted as the condition that each point on the integral surface has the property that the vector

$$\left( \frac{\Delta a}{a} M^\top, 1 - \frac{\Delta a}{a} \frac{\partial N}{\partial \phi} M, 0 \right)^\top \quad (A.5)$$

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at that point is tangent to the surface. Thus (A.4) defines a direction field given by (A.5), called the characteristic directions, having the property that a surface

\[ T_i = q_i(x_a, \phi, s, \theta) \]

is an integral surface if and only if, at each of its points, the tangent hyperplane contains the characteristic direction. The integral curves of this field, i.e. the family of curves whose tangent lines coincide with the characteristic directions, are called characteristic curves. Now, for a PDE of the form

\[ \sum_{i=1}^{m} a_i \frac{\partial u}{\partial x_i} = a, \]

the characteristic directions tangent to the integral surface are given by

\[ du : dx_m : \cdots : dx_2 : dx_1 = a : a_m : \cdots : a_2 : a_1, \]

and hence, we can now reduce the above PDE to a system of \( m \) ODE

\[ \frac{du}{dp} = a, \quad \frac{dx_i}{dp} = a_i, \quad 1 \leq i \leq m, \]

that describes a family of \( m \) characteristic curves. In our problem, the characteristic ODE are given by

\[ \frac{dT_i}{dp} = 0, \quad \frac{ds}{dp} = 1 - \Delta \frac{\partial N}{a} \frac{\partial M}{\partial \phi}, \quad \frac{d\phi}{dp} = \frac{M \Delta a}{a}, \quad 1 \leq i \leq r. \quad (A.6) \]

The integral surface can now be constructed by solving the following initial value problem: let an \( r \)-dimensional manifold \( C \) be given by

\[ \phi = \vartheta, \quad s = 0, \quad T_i = T_0^i(\vartheta), \quad 1 \leq i \leq r, \]
where \( \vartheta = \left( \vartheta_1 \ \vartheta_2 \ \ldots \ \vartheta_r \right)^\top \) is a vector of parameters, and suppose that the rank of the matrix
\[
\left( \begin{array}{cc}
\left( \frac{\partial \phi}{\partial \vartheta} \right)^\top & \left( \frac{\partial s}{\partial \vartheta} \right)^\top
\end{array} \right)^\top
\]
is \( r \). We assume that the projection \( C_0 \) of this manifold on the \((\phi, s)\)-space is free of double points, that is, different points on \( C_0 \) correspond to different sets of \( \vartheta \). We also note that the boundary conditions \( T_i = T_i^0 \) on the manifold \( C \) must be chosen such that the condition \( \frac{\partial T_i}{\partial \phi} \neq 0 \) obtained in the previous subsection is satisfied.

We obtain the solutions \( \phi(\vartheta, p) \), \( s(\vartheta, p) \) and \( T_i(x_a, \vartheta, p, \theta) \) of the system of characteristic ODE (A.6), which, at \( p = 0 \), coincide with the prescribed functions of \( \vartheta \) describing the initial manifold \( C \). We now consider the quantities \( \vartheta, p \) to be expressed in terms of \( \phi \) and \( s \), respectively, and substitute them into \( T_i(x_a, \vartheta, p, \theta) \), so that the \( T_i \) appear as functions of \( \phi \) and \( s \). This introduction of \( \phi \) and \( s \) as new independent variables is possible if the following inequality holds.

\[
\Delta \triangleq \begin{vmatrix}
\frac{\partial \phi}{\partial p} & \frac{\partial s}{\partial p} \\
\frac{\partial \phi}{\partial \vartheta} & \frac{\partial s}{\partial \vartheta}
\end{vmatrix}_{p=0} \neq 0.
\] (A.7)

We note that when \( \Delta_a = 0 \), we have \( \frac{\partial s}{\partial p} = 1 \), \( \frac{\partial \phi}{\partial \vartheta} = I \) and \( \frac{\partial s}{\partial \vartheta} = 0 \), so the above inequality (A.7) holds, because \( \Delta = -1 \) in this case. Consequently, (A.7) also holds for small enough \( |\Delta_a/a| \). Let us now define

\[
\Psi \triangleq \int \frac{\Delta_a}{a - \frac{\partial N}{\partial \phi} M \Delta_a} \, ds,
\] (A.8)
and

\[
\tilde{\Delta} \triangleq \frac{\partial N}{\partial \phi} M \Psi.
\] (A.9)
The solutions of (A.6) are therefore given by

\[ \phi = \vartheta + M\Psi, \quad s = p - \tilde{\Delta}, \quad T_i = T_i^0(\vartheta). \]

\[ \Rightarrow p = s + \tilde{\Delta}, \quad \vartheta = \phi - M\Psi, \quad T_i(x_a, \phi - M\Psi, s + \tilde{\Delta}, \theta) = T_i^0(\phi - M\Psi). \]

We now consider the case when \( \Delta_a = 0 \). We then have a solution \( T_i = T_i^0(\phi) \); moreover, we can choose \( T^0 = \phi \), which yields the solution \( T = \phi \) (where \( T \) and \( T^0 \) are vectors comprising the elements \( T_i \) and \( T_i^0 \), respectively). We note that with the same choice of initial condition, the solution for the \( \Delta_a \neq 0 \) case is \( T = \phi - M\Psi \). Due to the well known result that an ODE exhibits continuous dependence on parameters [29], we conclude that the solutions of the second and third ODEs in (A.6) will be close to \( s \) and \( \phi \), respectively, provided the perturbation \( |\Delta_a/a| \) is small. Therefore, the resulting solution \( T = \phi - M\Psi \) of the original PDE (A.3) will also be close to \( \phi \).

### A.3 Proof of Proposition 2.1

We pick up the analysis of the closed-loop state feedback system of Chapter 2 where we left off at equation (2.26). We first show that the reduced system and the boundary layer system have exponentially stable equilibrium points at \( x_s = 0 \) and \( s = 0 \), respectively, and then we use a composite Lyapunov function to show that the origin of the whole system is also an exponentially stable equilibrium point.

Item 3) of Assumption 2.4 states that when \( s = 0 \), the origin of (2.20) is locally exponentially stable. Hence, the origin of the reduced system in (2.26) is locally exponentially stable by hypothesis, and by a converse Lyapunov theorem [29, Theorem 4.14], there exists a Lyapunov function \( V_1(x_s, \theta) \) that satisfies the following
inequalities
\[ c_1 \|x_s\|^2 \leq V_1(x_s, \theta) \leq c_2 \|x_s\|^2, \]  
(A.10a)
\[ \frac{\partial V_1}{\partial x_s} F(x_s, 0, \theta) \leq -c_3 \|x_s\|^2, \]  
(A.10b)
\[ \left\| \frac{\partial V_1}{\partial x_s} \right\| \leq c_4 \|x_s\|, \]  
(A.10c)
in some neighborhood \( N_{x_s} \) of \( x_s = 0 \). The boundary layer system is given by
\[ \frac{ds}{d\tau} = -\frac{g_b}{a_n} \beta s = -\frac{a}{a_n} \left( 1 - \frac{\partial N}{\partial \phi} M \frac{\Delta a}{a} \right) s. \]  
(A.11)
Let \( V_2(s) = s^2/2 \) be a Lyapunov function candidate for the boundary layer system (A.11). We then obtain
\[ \frac{\partial V_2}{\partial s} \cdot \frac{ds}{d\tau} = -\frac{a}{a_n} \left( 1 - \frac{\partial N}{\partial \phi} M \frac{\Delta a}{a} \right) s^2 \leq -\beta_0 k_0 k_\beta s^2. \]
Hence, the origin of the boundary layer system is exponentially stable uniformly in \( x_s \). By the smoothness of \( F \) and \( f_b \) and the fact that each function vanishes at the origin, we have
\[ \|F(x_s, s, \theta) - F(x_s, 0, \theta)\| \leq k_1 |s|, \]  
(A.12a)
\[ \|f_b(x_s, s, \theta)\| \leq k_2 \|x_s\| + k_3 |s| \]  
(A.12b)
in some neighborhood \( N \) of \((x_s, s) = (0, 0)\). Suppose \( \mu \) is chosen small enough such that \( \Omega_{\mu} \subset N_{x_s} \) and \( \Omega_{\mu} \subset N \). We now consider the composite Lyapunov function candidate
\[ W(x_s, s, \theta) = V_1(x_s, \theta) + s^2/2. \]
From (A.10) and (A.12), it can be shown that \[29\]

\[
\dot{W} \leq -c_3 \|xs\|^2 + c_4 k_1 \|xs\| |s| + k_2 \|xs\| |s| + k_3 s^2 - \beta_0 k_0 k_1^\dagger / \mu
\]

\[
= - \left( \begin{array}{c} \|xs\| \\ |s| \end{array} \right)^\top \left( \begin{array}{cc} c_3 & -c_4 k_1 - k_2 \\ -c_4 k_1 - k_2 & \beta_0 k_0 k_1^\dagger / \mu - k_3 \end{array} \right) \left( \begin{array}{c} \|xs\| \\ |s| \end{array} \right). \tag{A.13}
\]

There exists a $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, the right-hand side of (A.13) is negative definite in $\Omega_\mu$, which then yields

\[
\dot{W} \leq -2\gamma W
\]

for some $\gamma > 0$. Therefore,

\[
W(x_s(t), s(t), \theta) \leq W(x_s(t_0), s(t_0), \theta)e^{-2\gamma(t-t_0)}
\]

and, from the properties of $V_1$ and $V_2$, we obtain

\[
\begin{bmatrix} x_s(t) \\ s(t) \end{bmatrix} \leq \begin{bmatrix} x_s(t_0) \\ s(t_0) \end{bmatrix} K^* e^{-\gamma(t-t_0)},
\]

for some $K^* > 0$.

This completes the proof of Proposition 2.1.

### A.4 Proof of Proposition 3.1

It was already shown prior to stating the proposition that the trajectories of the closed-loop state feedback system (3.19), (3.20) reach the set $\{s| \leq \mu\}$ in finite time and remain inside thereafter, and so we continue the analysis by considering the set $\Omega$ defined in (3.24) below the statement of the proposition.
Assumption 3.4 calls for the existence of a Lyapunov function \( V(x_s, w, \theta) \) for the internal dynamics. The derivative of \( V \) is given by

\[
\dot{V} = \frac{\partial V}{\partial x_s} F(x_s, s, w, \theta) + \frac{\partial V}{\partial w} S_0 w.
\]

Thus, on the boundary \( V = c_0 \), we have

\[
\dot{V} = \frac{\partial V}{\partial x_s} F(x_s, s, w, \theta) + \frac{\partial V}{\partial w} S_0 w \leq -\alpha_3(\|x_s\|) \leq -\alpha_3(\alpha_2^{-1}(c_0)).
\]

Hence, \( \Omega \) is a positively invariant set for the system (3.19).

As a consequence of the preceding analysis, all trajectories starting in \( \Omega \), stay in \( \Omega \) and reach the set \( \Omega \cap \{|s| \leq \mu\} \) in finite time. Inside this set, we have

\[
\dot{V} \leq -\alpha_3(\|x_s\|), \; \forall \|x_s\| \geq \gamma(\mu).
\]

The above analysis shows that the trajectories enter the positively invariant set

\[
\Omega_\mu = \{V(x_s, w, \theta) \leq \alpha_2(\gamma(\mu))\} \times \{|s| \leq \mu\}
\]

in finite time. Now, inside the set \( \Omega_\mu \), the closed-loop system is given by the following equations in the singularly perturbed form.

\[
\dot{x}_s = F(x_s, s, w, \theta), \tag{A.14a}
\]

\[
\mu \dot{s} = \mu f_s(x_s, s + N, w, \theta) - \frac{\beta(x_s, s)}{\alpha_n(x_s, s)} g_s(x_s, s + N, w, \theta)s. \tag{A.14b}
\]

Item 2) of Assumption 3.4 states that when \( s = 0 \), the origin of the \( \dot{x}_s \)-equation in (A.14) is locally exponentially stable. Hence, the origin of the reduced system in (A.14) is locally exponentially stable by hypothesis, and by a converse Lyapunov the-
orem [29, Theorem 4.14], there exists a Lyapunov function $V_1(x_s, w, \theta)$ that satisfies the following inequalities

\begin{align}
  c_1 \left\| x_s \right\|^2 &\leq V_1(x_s, w, \theta) \leq c_2 \left\| x_s \right\|^2, \\
  \frac{\partial V_1}{\partial x_s} F(x_s, 0, w, \theta) + \frac{\partial V_1}{\partial w} S_0 w &\leq -c_3 \left\| x_s \right\|^2, \\
  \left\| \frac{\partial V_1}{\partial x_s} \right\| &\leq c_4 \left\| x_s \right\|,
\end{align}

in some neighborhood $N_{x_s}$ of $x_s = 0$. The boundary layer system is given by

\begin{equation}
  \frac{ds}{d\tau} = -\frac{g_s}{a_n} \beta s = -\frac{a}{a_n} \left( 1 - \frac{\partial N}{\partial \phi} M \frac{\Delta a}{a} \right) s. \tag{A.16}
\end{equation}

Let $V_2(s) = s^2/2$ be a Lyapunov function candidate for the boundary layer system (A.16). We then obtain

\begin{equation}
  \frac{\partial V_2}{\partial s} \cdot \frac{ds}{d\tau} = -\frac{a}{a_n} \left( 1 - \frac{\partial N}{\partial \phi} M \frac{\Delta a}{a} \right) s^2 \leq -\beta_0 k_k \dot{s}^2,
\end{equation}

due to (3.22), (3.12) and Assumption 3.3. Hence, the origin of the boundary layer system is exponentially stable uniformly in $x_s$. By the smoothness of $F$ and $f_s$ and the fact that each function vanishes at the origin, we have

\begin{align}
  \| F(x_s, s, w, \theta) - F(x_s, 0, w, \theta) \| &\leq k_1 |s|, \tag{A.17a} \\
  \| f_s(x_s, s + N, w, \theta) \| &\leq k_2 \| x_s \| + k_3 |s| \tag{A.17b}
\end{align}

in some neighborhood $N$ of $(x_s, s) = (0, 0)$. Suppose $\mu$ is chosen small enough such that $\Omega_\mu \subset N_{x_s}$ and $\Omega_\mu \subset N$. We now consider the composite Lyapunov function candidate

\begin{equation}
  W(x_s, s, w, \theta) = V_1(x_s, w, \theta) + s^2/2.
\end{equation}
From (A.15) and (A.17), it can be shown that [29]

\[
\dot{W} \leq -c_3 \|x_s\|^2 + c_4 k_1 \|x_s\| |s| + k_2 \|x_s\| |s| + k_3 s^2 - \beta_0 k_0 k_1 s^2 / \mu
\]

\[
= - \left( \frac{c_3}{|s|} \right)^\top \left( \begin{array}{cc}
- c_4 & -c_4 k_1 - k_2 \\
-\frac{c_4 k_1 - k_2}{2} & -\frac{\beta_0 k_0 k_1}{\mu} - k_3
\end{array} \right) \left( \begin{array}{c}
\|x_s\| \\
|s|
\end{array} \right).
\]  

(A.18)

There exists a \( \mu^* > 0 \) such that for all \( 0 < \mu < \mu^* \), the right-hand side of (A.18) is negative definite in \( \Omega_\mu \), which then yields

\[
\dot{W} \leq -2\gamma_r W
\]

for some \( \gamma_r > 0 \). Therefore,

\[
W(x_s(t), s(t), w(t), \theta) \leq W(x_s(t_0), s(t_0), w(t_0), \theta)e^{-2\gamma_r(t-t_0)}
\]

and, from the properties of \( V_1 \) and \( V_2 \), we obtain

\[
\left\| \frac{x_s(t)}{s(t)} \right\| \leq K_r e^{-\gamma_r(t-t_0)},
\]

for some \( K_r > 0 \).

This completes the proof of Proposition 3.1.
BIBLIOGRAPHY


