Resource-Aware Model Predictive Control of Spatially Distributed Processes Using Event-Triggered Communication

Zhiyuan Yao and Nael H. El-Farra†
Department of Chemical Engineering & Materials Science
University of California, Davis, CA 95616 USA

Abstract—This work focuses on the design of a resource-aware model predictive control (MPC) system for spatially distributed processes with low-order dynamics and a limited number of output measurements. A reduced-order model that captures the dominant process dynamics is initially obtained and used to design a Lyapunov-based output feedback MPC. A finite-dimensional state observer is included in the sensors to generate estimates of the slow states of the infinite-dimensional systems which are broadcast over the network to update the states of the reduced-order model used in the MPC controller at each sampling time. Precise conditions that guarantee closed-loop stability under plant-model mismatch are derived and used to devise an event-triggered sensor-controller communication strategy that minimizes network utilization without jeopardizing closed-loop stability. The key idea is to monitor the model estimation error at each sampling time and suspend communication when the prescribed stability bounds are satisfied based on a forecast of the future evolution of the Lyapunov function are satisfied. At times when the model estimation error fails to satisfy the projected bound on the evolution of the Lyapunov function, the sensors are prompted to proactively transmit the observer-generated state estimates to update the model states and avert instability. Finally, the design and implementation of the proposed event-triggered MPC are illustrated using a diffusion-reaction process example.

I. INTRODUCTION

The integration of wireless sensor networks in process control systems is an appealing goal that promises to expand the capabilities of existing control technology beyond what is possible with wired devices alone, and is a key mechanism for enabling the nascent vision of smart plant operations [1]. An important challenge with this evolution is the development of resource-aware control methods that can systematically balance the desired closed-loop stability and performance requirements against the intrinsic constraints on the resources of the measurement system and the communication medium. Despite the significant and growing body of research work on networked control systems, the majority of existing methods have targeted lumped parameter systems described by ordinary differential or difference equations (e.g., see [2]–[6] and the references therein). By contrast, results for distributed parameter systems modeled by partial differential equations (PDEs) are limited at present.

Over the past few years, research work has begun to address the development of networked control and scheduling methods for spatially distributed processes that arise commonly in the modeling of transport-reaction processes and fluid flows. Major bottlenecks in the design of networked control systems for such processes include their infinite-dimensional nature, as well as the complex dynamics and uncertainties that characterize their models. An effort to address this problem was initiated in [7] where a model-based networked control architecture for linear distributed processes was developed. The methodology was subsequently generalized in [8] to account for process nonlinearities, and in [9] to incorporate sensor transmission scheduling. Additionally, the problem of networked control using dynamic communication was studied in [10], and the problem of handling communication delays was addressed in [11]. The central aim of these studies has been to achieve closed-loop stability with minimal network resource utilization.

Beyond closed-loop stability, performance optimization considerations and control constraint handling capabilities were explicitly incorporated in [12] using a networked model predictive control (MPC) framework. The framework, which leverages event-triggered communication, was developed based on full-state feedback which requires the accessibility of the full state for measurement at all positions within the spatial domain. This requirement restricts the applicability of the networked MPC scheme, since for many practical applications, the full state cannot be measured and only a finite number of measurement sensors can be placed across the spatial domain. The lack of full state measurements places important restrictions on the design and implementation of both the control system and communication logic.

Motivated by these considerations, we present in this work a networked MPC framework for spatially distributed processes under output feedback control and event-triggered sensor-controller communication. A Lyapunov-based output feedback MPC is initially designed based on a suitable reduced-order model that captures the dominant dynamics of the infinite-dimensional system. Within the controller, a finite-horizon optimal control problem subject to control and suitable Lyapunov stability constraints is repeatedly solved using the reduced-order model at each sampling time. A finite-dimensional state observer is included to predict the slow states using the available output measurements and transmit its estimate to update the reduced-order model in the MPC controller at each sampling time. To minimize the information transfer between observer and controller,
an event-triggered communication strategy is developed to determine when to switch on or off the sensor-controller communication. The key idea is to monitor the error between the model and observer states at each sampling time and suspend communication for periods of time when the prescribed stability conditions based on the forecast of future process state are satisfied. At times when the model estimation error fails to satisfy the projected bound on the evolution of the Lyapunov function, the sensors are prompted to proactively send the observer estimates to update the model states and ensure closed-loop stability. Finally, the theoretical results are illustrated using a diffusion-reaction process example.

II. PRELIMINARIES

We focus on spatially-distributed processes modeled by nonlinear parabolic PDEs of the form:

\[
\frac{\partial \bar{x}}{\partial t} = \alpha \frac{\partial^2 \bar{x}}{\partial z^2} + \beta \bar{x} + f(\bar{x}) + \omega \sum_{i=1}^{m} b_i(z)u_i(t)
\]

(1)

\[
y_{s}(t) = \int_{0}^{\pi} q_{s}(z)\bar{x}(z,t)dz, \quad \kappa = 1, \ldots, n
\]

subject to the boundary and initial conditions:

\[
c_{1}\bar{x}(0,t) + d_{1}\frac{\partial \bar{x}}{\partial z}(0,t) = 0
\]

\[
c_{2}\bar{x}(\pi,t) + d_{2}\frac{\partial \bar{x}}{\partial z}(\pi,t) = 0
\]

(2)

where \(\bar{x}(z,t) \in \mathbb{R}\) denotes the process state variable, \(z \in [0,\pi], t \in [0,\infty), u_i \in U_s \subset \mathbb{R}\) denotes the i-th constrained manipulated input which satisfies the constraint \(u := [u_1, u_2, \cdots, u_m]^T \in \{u \in \mathbb{R} : \|u\| \leq u_{\text{max}}\}\), \(y_{s}(t)\) represents the \(\kappa\)-th measured output, \(m\) and \(n\) are the numbers of manipulated inputs and measured outputs; \(b_i(\cdot)\) and \(q_{s}(\cdot)\) are the actuator and sensor distribution functions; \(\alpha > 0\) and \(\beta, c_1, c_2, d_1, d_2\) are all real constant parameters, and \(\bar{x}_{0}(z)\) is a smooth function of \(z\).

Based on the aid of operator theory [13], the PDE of (1)-(2) can be formulated as an infinite-dimensional system:

\[
\dot{\bar{x}}(t) = \mathcal{F}(x(t)) + Bu(t), \quad y(t) = Qx(t), \quad x(0) = x_{0}
\]

(3)

where \(x(t) = \bar{x}(z,t), t > 0, z \in [0,\pi]\) is the state function defined on the Hilbert space \(\mathcal{H} = L_{2}(0,\pi)\) endowed with inner product and norm:

\[
\langle \omega_1, \omega_2 \rangle = \int_{0}^{\pi} \omega_1(z)\omega_2(z)dz, \quad \|\omega_1\|_2 = \langle \omega_1, \omega_1 \rangle^{\frac{1}{2}}
\]

(4)

where \(\omega_1, \omega_2\) are two elements of \(L_{2}(0,\pi), \mathcal{F}(x) = Ax + f(x)\), \(A\) is the differential operator defined by \(A\phi = \alpha \frac{\partial^2 \phi}{\partial z^2} + \beta \phi, z \in [0,\pi]\), where \(\phi(\cdot)\) is an element of \(L_{2}(0,\pi)\) and satisfies the boundary conditions. The nonlinear term \(f(\cdot)\) is assumed to locally Lipschitz and satisfies \(f(0) = 0\). \(B\) is the input operator defined by \(Bu = \omega \sum_{i=1}^{m} b_i(\cdot)u_i\) with \(\|u\| \leq u_{\text{max}}, x_{0} = \bar{x}_{0}(z), y = [y_1, \cdots, y_m]^T\) and \(Q\) denotes the output operator defined by \(Qx = [Q_1 x, \cdots, Q_m x]^T\).

Based on the solution of the eigenvalue problem associated with the linear differential operator, \(A\phi_k = \lambda_k \phi_k, k \in \{1, \cdots, \infty\}\), it can be verified that the spectrum of \(A\) can be partitioned into a finite set containing the first \(m\) slow eigenvalues, and an infinite complement that contains the remaining fast eigenvalues, which implies that the dominant dynamics of the PDE can be approximated by a finite-dimensional system, and thus justifies the application of modal decomposition techniques (e.g., see [14]) to decompose the infinite-dimensional system of (3) as:

\[
\dot{x}_s = \mathcal{F}_s(x_s, x_f) + B_s u, \quad x_s(0) = \mathcal{P}_s x_0
\]

(5)

\[
\dot{x}_f = \mathcal{F}_f(x_s, x_f) + B_f u, \quad x_f(0) = \mathcal{P}_f x_0
\]

(6)

\[
y = Q_s x_s + Q_f x_f
\]

(7)

where \(x_s = \mathcal{P}_s x \in \mathcal{H}_s := \text{span}\{\phi_1, \cdots, \phi_m\}\) is the state of a finite dimensional system that describes the evolution of the slow eigenmodes, \(x_f = \mathcal{P}_f x \in \mathcal{H}_f := \text{span}\{\phi_{m+1}, \phi_{m+2}, \cdots\}\) is the state of an infinite-dimensional system that captures the evolution of the fast eigenmodes, \(\mathcal{H}_s, \mathcal{H}_f\) are modal subspaces of \(\mathcal{A}\), and \(\mathcal{P}_s\) and \(\mathcal{P}_f\) are the orthogonal projection operators, where \(\mathcal{F}_s(x_s, x_f) = A_s x_s + f_s(x_s, x_f), A_s = \mathcal{P}_s A\) is an \(m \times m\) diagonal matrix of the form \(A = \text{diag}\{\lambda_1, \cdots, \lambda_m\}, f_s(\cdot) = \mathcal{P}_s f(\cdot), B_s = \mathcal{P}_s B\) and \(Q_s = \mathcal{P}_s Q, \mathcal{F}_f(x_s, x_f) = A_f x_f + f_f(x_s, x_f), A_f = \mathcal{P}_f A\) is an unbounded differential operator which is exponentially stable, \(f_f(\cdot) = \mathcal{P}_f f(\cdot), B_f = \mathcal{P}_f B\) and \(Q_f = \mathcal{P}_f Q\). Neglecting the fast and stable \(x_f\)-subsystem of (6), the following approximate, \(m\)-dimensional slow system can be obtained:

\[
\dot{x}_s = \mathcal{F}_s(x_s, 0) + B_s u, \quad \hat{y} = Q_s \hat{x}_s
\]

(8)

III. AN AUXILIARY BOUNDED LYAPUNOV-BASED OUTPUT FEEDBACK CONTROLLER DESIGN

A. Model-based Bounded Lyapunov-based Control

To realize the desired networked MPC, we consider a finite-dimensional approximate model of (8) of the form:

\[
\dot{\hat{x}}_s = \mathcal{F}_s(\hat{x}_s) + \hat{B}_s u
\]

(9)

where \(\hat{x}_s\) represents an estimates of \(\hat{x}_s, \mathcal{F}_s(\cdot)\) and \(\hat{B}_s\) are models of \(\mathcal{F}_s\) and \(B_s\). Referring to the approximate model of (9), we assume that an appropriate stabilizing controller which satisfies the control constraints for all \(\hat{x}_s\) within a well-defined stability region, \(\Omega_s\), has already been designed. This requirement is formulated in the following assumption.

Assumption 1: There exists a nonlinear feedback control law of the general form:

\[
u = \mathcal{K}(\hat{x}_s)
\]

(10)

such that for all \(\hat{x}_s \in \Omega_s, \|\mathcal{K}(\hat{x}_s)\| \leq u_{\text{max}}\) and the origin of the closed-loop model of (9)-(10) is asymptotically stable.

From converse Lyapunov theorems, Assumption 1 implies the existence of a continuously differentiable function \(V(\hat{x}_s)\) such that for all \(\hat{x}_s \in \Omega_s\), the following inequalities hold:

\[
\alpha_1(\|\hat{x}_s\|) \leq V(\hat{x}_s) \leq \alpha_2(\|\hat{x}_s\|)
\]

(11)

\[
\frac{dV(\hat{x}_s)}{d\hat{x}_s} |\mathcal{F}_s(\hat{x}_s) + \hat{B}_s \mathcal{K}(\hat{x}_s)| \leq -\alpha_3(\|\hat{x}_s\|)
\]

(12)

\[
\frac{dV(\hat{x}_s)}{d\hat{x}_s} \leq \alpha_4(\|\hat{x}_s\|)
\]

(13)

where \(\alpha_i(\cdot), i \in \{1, 2, 3, 4\}\) are class \(\mathcal{K}\) functions of their arguments. To simplify our subsequent analysis, and without loss of generality, we choose as an estimate of the closed-loop stability region, \(\hat{\Omega} \triangleq \hat{x}_s \in \mathcal{H}_s : V(\hat{x}_s) \leq c\), where \(c\) is the largest positive number such that \(\Omega \subseteq \hat{\Omega}_s\).
B. Closed-loop Model Stability Analysis under Discrete Controller Implementation

In this subsection, we analyze the stability properties of the closed-loop model when the controller of (10) is implemented on the model in a discrete (sample-and-hold) fashion. This analysis is important because it provides the critical link between the auxiliary Lyapunov-based controller design presented in this section and the Lyapunov-based MPC to be presented in Section IV. Under discrete implementation, the control action is calculated as follows:

\[ u(t) = K(\hat{x}_s(t_j)), \quad t \in [t_j, t_{j+1}), \quad j \in \{0, 1, 2, \cdots \} \]  

where \( t_{j+1} - t_j = \Delta \) is the sampling period. The following proposition establishes that, under this discrete controller implementation, practical closed-loop stability of the finite-dimensional model of (9) remains preserved in the sense that any \( \bar{x}_s \in \Omega \) can be steered to a small terminal neighborhood of the origin in finite time, if the sampling period is chosen sufficiently small. The proof is omitted due to space limitations.

**Proposition 1:** Consider the approximate model of (9) subject to the controller of (14) and let \( \Omega' \subset \Omega \) be a compact subset of \( \Omega \) defined by

\[ \bar{x}_s \in \mathcal{H}_s : V(\bar{x}_s) \leq c' \]  

where \( c' < c \). Then, given any \( \sigma \in (0, \alpha_3(\alpha_2^{-1}(c')))) \) and \( \epsilon \in (c', c) \), there exists \( \Delta' > 0 \), such that, for all \( \bar{x}_s(t_0) \in \Omega \) and \( \Delta \in [0, \Delta') \), \( \bar{x}_s(t) \in \Omega \) for all \( t \in [t_0, \infty) \), and

\[ \limsup_{t \to \infty} V(\bar{x}_s(t)) \leq \epsilon. \]  

Furthermore, when \( \bar{x}_s(t_j) \in \Omega \setminus \Omega' \), the following bound \( V(\bar{x}_s(t)) \leq -\sigma \) holds for all \( t \in [t_j, t_{j+1}) \).

**Remark 1:** Proposition 1 places constraints on the choice of sampling period which must be satisfied in order to overcome the destabilizing tendency of the model state due to the sample-and-hold implementation of the controller. However, in this situation, only practical stability can be achieved and \( \bar{x}_s \) will be ultimately bounded within a subset of \( \Omega \) defined by \( Y \equiv \{ \bar{x}_s \in \mathcal{H}_s : V(\bar{x}_s) \leq \epsilon \} \). It can be proven that the sampling period \( \Delta \) determines the size of terminal region \( Y \). Specially, a smaller \( \Delta \) leads to a smaller terminal region. From the definition of \( \epsilon \) and \( \sigma \), we conclude that \( \Delta \) has great influence on the stability properties of the closed-loop model; therefore, \( \Delta \) must be sufficiently small to ensure the satisfaction of these two inequalities.

C. Output Feedback Implementation of Discrete Bounded Controller on the Process

Due to the unavailability of full-state measurements, a finite-dimensional state observer of the following form is designed and embedded in the measurement sensors:

\[ \hat{\eta} = \bar{F}_s(\hat{\theta}) + \bar{B}_s u + L(y - Q \hat{\eta}) \]  

where \( \hat{\eta} \) is the observer state that generates estimates of \( \bar{x}_s \), using \( \bar{y} \), and \( \mathcal{L} \) is the observer gain. An important property of this observer is that for an appropriate choice of \( \mathcal{L} \), if \( \bar{x}_s(t) \in \Omega \), for \( t \in [t_0, \infty) \), the observer estimation error, defined as \( e = \bar{x}_s - \hat{\eta} \), satisfies the following inequality:

\[ || e(t) || \leq \beta_e(t_0), t - t_0 + \varphi \]  

where \( \beta(\cdot, \cdot) \) is a class \( \mathcal{KL} \) function, \( \varphi \) is a positive constant which is dependent on the size of the uncertainty. This bound on the observer estimation error can be obtained by applying the result of Proposition 1 in \([8]\). By virtue of the limited access of the communication medium and the consideration of efficient energy utilization, the observer-generates estimate can only be received by the controller and used to immediately update the state of model when it becomes available from the network. In this situation, the networked controller is implemented as follows:

\[ \begin{align*}  
\hat{x}_s(t) &= \bar{F}_s(\hat{\theta}(t)) + \bar{B}_s u(t) + L(\bar{y}(t) - Q \hat{\eta}(t)) \\
\hat{\eta}(t) &= \bar{F}_s(\hat{\eta}(t)) + \bar{B}_s u(t) + L(y(t) - Q \hat{\eta}(t))  
\end{align*} \]  

(17)

As a consequence of the model uncertainty, a choice of \( \Delta \) which guarantees the practical stability of the closed-loop model does not necessarily enforce convergence and ultimate boundedness of the closed-loop process state and observer estimate. To deal with this problem, additional constraints on the sampling period \( \Delta \) and the initial observer estimate error \( e(t_0) \) need to be imposed in order to guarantee closed-loop stability, and are given in the following theorem.

**Theorem 1:** Consider the slow system (8) subject to the model-based controller of (17). Given any \( \epsilon \in (0, \sigma) \) and \( \omega \in (\epsilon, c) \), there exist \( e(t_0) \geq 0 \) and \( \Delta^* \geq 0 \) such that if \( \bar{x}_s(t_0) \in \Omega \), \( \bar{x}_s(t_0) \in \Omega \), \( \eta(t_0) \in \Omega \), \( e(t_0) \in [0, e(t_0)) \), and \( \Delta \in [0, \Delta^*) \), then \( \bar{x}_s(t) \in \Omega \), for \( t \in [t_0, \infty) \), and

\[ \limsup_{t \to \infty} V(\bar{x}_s(t)) \leq \epsilon. \]

Furthermore, when \( \bar{x}_s(t_j) = \eta(t_j) \in \Omega \setminus \Omega' \), the following inequality holds: \( V(\bar{x}_s(t)) \leq -\sigma \) for all \( t \in [t_j, t_{j+1}) \).

**Proof:** Part 1: From the definition of \( \hat{\eta}(t) \) and using (9) and (15), the time-derivative of \( \hat{\eta}(t) \) is given by

\[ \dot{\hat{\eta}}(t) = \mathcal{L} Q \bar{x}_s e(t), \forall t \in [t_j, t_{j+1}) \]

Integrating and taking norm of both sides, we obtain:

\[ || \dot{\hat{\eta}}(t) || \leq \int_{t_j}^{t} || \mathcal{L} Q \bar{x}_s || || e(t') || dt' \leq || \mathcal{L} Q \bar{x}_s || \sup_{[t_j, t)} || e(t') || \Delta \]

where we used the fact that \( || \dot{\hat{\eta}}(t) || = 0 \), since the model state is updated by observer estimate at time \( t_j \). Then, substituting (16) into the above inequality gives:

\[ || \dot{\hat{\eta}}(t) || \leq || \mathcal{L} Q \bar{x}_s || || e(t_j) + \mathcal{F}(e(t_0)) || \Delta \leq \Delta^* \leq \mathcal{E}^* || e(t_0) || \Delta \]  

(18)

Part 2: We first assume that \( \hat{x}_s(t) \) and \( \hat{\eta}(t) \) stay within \( \Omega \) for all \( t \in [t_0, t_1) \). Then for \( \hat{x}_s(t_0) = \eta(t_0) \in \Omega \setminus \Omega' \), the time-derivative of the Lyapunov function along the trajectories of the closed-loop system of (8) and (17) satisfies:

\[ V(\hat{x}_s) = \frac{dV(\hat{x}_s)}{dt} \left[ \begin{array}{c} \mathcal{F}_s(\hat{x}_s) + \bar{B}_s u \\ \mathcal{D}_s \end{array} \right] + \frac{dV(\hat{x}_s)}{dt} \left[ \begin{array}{c} \mathcal{F}_s \hat{x}_s \\ \mathcal{D}_s \end{array} \right] - \frac{dV(\hat{x}_s)}{dt} \hat{\theta}_s \]  

(19)

where \( \mathcal{F}_s(\cdot) = \mathcal{F}_s(\cdot) - \mathcal{F}_s(\cdot), \mathcal{B}_s = \mathcal{B}_s - \mathcal{B}_s, \Theta_\mathcal{F}_s \leq \max x_s \in \Omega \left[ \mathcal{F}_s(\hat{x}_s) \right], \) and we used the fact that \( || \hat{x}_s - \bar{x}_s || \leq || e(t) || + || \hat{\eta}(t) || \leq || \Theta_\mathcal{F}_s || (e(t_0), \Delta) \leq \beta_e(e(t_0), 0) + c_{max} || e(t_0) || \Delta + \varphi. \)

Given any \( \epsilon \in (0, \sigma) \), there must exist \( e(t_0) \geq 0 \) and \( \Delta^* \geq 0 \) such that, for \( e(t_0) \in [0, e(t_0)) \) and \( \Delta \in [0, \Delta^*) \),
\[ \alpha_4(\alpha_1^{-1}(c))(\Theta_{\bar{x}} + \| \bar{B}_s \| u_{\text{max}}) + (\Phi_1 + \Phi_2 \| \bar{B}_s \| u_{\text{max}})\zeta(e(t_0), \Delta) \leq \sigma - \varsigma \] (20)

which leads to \( \dot{V}(\bar{x}_s(t)) \leq -\varsigma < 0 \). For \( \bar{x}_s(t_0) = \eta(t_0) \in \Omega \), there exists a positive constant \( \rho \) such that a Taylor series expansion of \( V(\bar{x}_s) \) around \( \bar{x}_s \) holds for all \( \bar{x}_s, \bar{x}_s \in \Omega \):

\[
V(\bar{x}_s) \leq V(\bar{x}_s) + \frac{dV}{d\bar{x}_s} \| \bar{x}_s - \bar{x}_s \| + \rho \| \bar{x}_s - \bar{x}_s \|^2
\]

where the term \( \rho \| \bar{x}_s - \bar{x}_s \|^2 \) bounds the second and higher-order terms of the Taylor expansion. Substituting the bound on \( \| \bar{x}_s - \bar{x}_s \| \), we obtain

\[
V(\bar{x}_s) \leq V(\bar{x}_s) + \varepsilon \| \bar{x}_s - \bar{x}_s \| + \rho \| \bar{x}_s - \bar{x}_s \|^2
\]

which also validates the assumption made initially on the boundedness of \( \bar{x}_s \). In order to verify the boundedness of \( \eta \in \Omega \), we can repeat the above analysis on the evolution of \( V(\eta) \). Specifically, it can be shown that for \( \bar{x}_s(t_j) \in \Omega \setminus \Omega' \), there exist \( e^s(t_0) \geq 0 \) and \( \Delta^s \geq 0 \) such that, for \( e(t_0) \in [0, e^s(t_0)] \) and \( \Delta \in [0, \Delta^s] \),

\[
\alpha_4(\alpha_1^{-1}(c))\zeta(e(t_0), \Delta) + \rho |\zeta(e(t_0), \Delta)|^2 \leq \varepsilon - \varepsilon
\] (22)

and thus \( V(\bar{x}_s) \leq \varepsilon < c \). Therefore, given \( \bar{x}_s(t_0), \bar{x}_s(t_0), \eta(t_0) \in \Omega \), \( \bar{x}_s(t) \) will be bounded within \( \Omega \) for all \( t \in [t_0, t_1) \). The above analysis can be performed recursively for all future intervals to show that the process state, \( \bar{x}_s(t) \), stays within \( \Omega \) for all \( t \in [t_0, \infty) \) which also validates the assumption made initially on the boundedness of \( \bar{x}_s \).

Remark 2: As can be observed from (20) and (22), besides the size of the sampling period \( \Delta \), the stability properties of the closed-loop system are also influenced by the accuracy of the embedded approximate model as well as the initial observer estimation error \( e(t_0) \). Therefore, initializing the model and observer states close to the actual process state allows the possibility of using a less accurate model and/or a larger sampling period without loss of stability. Alternatively, for a fixed model and observer, one needs to reduce the initial observer estimation error and/or choose a smaller \( \Delta \) to improve the control system performance.

IV. RESOURCE-AWARE NETWORKED MODEL PREDICTIVE CONTROL

A. Lyapunov-based Model Predictive Control

In this section, the networked MPC scheme is developed based on the Lyapunov-based networked output feedback controller where the finite-horizon optimization problem is formulated as follows:

\[
\min_{u(t)} \int_{t_j}^{t_j+N} \left[ \| \bar{x}_s(s) \|_Q^2 + \| u(s) \|_{\mathcal{R}}^2 \right] ds
\] (23a)

s.t. \( \bar{x}_s(t) = \bar{x}_s(t) + \bar{B}_s u(t), \forall t \in [t_j, t_j+1) \)

\( \bar{x}_s(t_j) = \eta(t_j), j \in \{0, 1, 2, \ldots \} \) \hspace{1cm} (23b)

\( \dot{V}(\bar{x}_s(t)) \leq -\varsigma, \text{ if } V(\bar{x}_s(t_j)) > c' \) \hspace{1cm} (23d)

\( \dot{V}(\bar{x}_s(t)) \leq \varsigma, \text{ if } V(\bar{x}_s(t_j)) \leq c' \) \hspace{1cm} (23e)

where \( Q \) and \( R \) are appropriate penalty weights associated with \( \bar{x}_s \) and \( u \); the control and prediction horizons are both chosen as \( N \), and the initial states of the process, the model and the observer satisfy \( \bar{x}_s(t_0), \bar{x}_s(t_0), \eta(t_0) \in \Omega \).

Remark 3: The Lyapunov-based MPC scheme of (23) is designed based on the discrete (sample-and-hold) implementation of the Lyapunov-based bounded controller discussed in the previous section. According to the analysis in Section III-C, satisfying the constraints on the initial states, \( \bar{x}_s(t_0), \bar{x}_s(t_0), \eta(t_0) \), the sampling period guarantees boundedness of the model states and the observer estimates within \( \Omega \) for all \( t \in [t_0, \infty) \), and therefore the optimization problem of (23) is always feasible. Moreover, from Theorem 1, the desired networked MPC scheme with appropriate choices of \( e(t_0) \) and \( \Delta \) not only preserves the stability properties of the closed-loop model, but also guarantees the convergence and ultimate boundedness of the closed-loop process state.

B. Event-triggered Communication Policy

Our aim in this subsection is to develop an event-triggered communication policy that further reduces the information transfer from the observer to the controller. Ideally, if the approximate model of (9) perfectly matches the actual process of (8), it is reasonable to suspend the feedback, i.e., remove (23c) from the MPC controller. In this case, the observer estimation error \( e(t_j) \) will not be reset to zero and the optimization problem is solved by completely relying on the model. However, in practice an accurate model is unobtainable; and, therefore, if model updates are suspended at some sampling instances, the error bound in (18) needs to be modified, since its validity requires \( \|ar{e}(t_j)\| = 0 \) which no longer holds if the model state is not updated at \( t_j \). By re-examining the calculation leading to that error bound, a new bound on \( \|ar{e}(t)\| \) can be obtained as follows:

\[ \|ar{e}(t)\| \leq \theta(\|ar{e}(t_j)\|, e(t_0), \Delta) \]

where \( \theta(\|ar{e}(t_j)\|, e(t_0), \Delta) \) is a continuous function which satisfies \( \theta(0, e(t_0), \Delta) = e_{\text{max}}(e(t_0), \Delta) \) and increases as \( \|ar{e}(t_j)\| \) grows. Applying an similar approach in the proof of Theorem 1, the evolution of the process state at time \( t \in [t_j, t_j+1) \) can be captured by the following bounds:

For \( \eta(t_j) \in \Omega \setminus \Omega' \):

\[
\dot{V}(\bar{x}(s)) \leq -\sigma + \alpha_4(\alpha_1^{-1}(c))(\Theta_{\bar{x}} + \| \bar{B}_s \| u_{\text{max}}) + (\Phi_1 + \Phi_2 \| \bar{B}_s \| u_{\text{max}})\zeta(e(t_0), \Delta)
\]

\[
\leq \theta_1(\|ar{e}(t_j)\|, e(t_0), \Delta, t \in [t_j, t_j+1])
\] (24)

For \( \eta(t_j) \in \Omega' \):

\[
\dot{V}(\bar{x}(s)) \leq \epsilon + \alpha_4(\alpha_1^{-1}(c))\vartheta(\|ar{e}(t_j)\|, e(t_0), \Delta)
\]

\[
+ \rho |\vartheta(\|ar{e}(t_j)\|, e(t_0), \Delta)|^2
\]

\[
\leq \theta_2(\|ar{e}(t_j)\|, e(t_0), \Delta, t \in [t_j, t_j+1])
\] (25)

where

\[ \vartheta(\|ar{e}(t_j)\|, e(t_0), \Delta) = \theta(\|ar{e}(t_j)\|, e(t_0), \Delta) + \beta(e(t_0), 0) + \varphi \]

and \( \vartheta(0, e(t_0), \Delta) = \zeta(e(t_0), \Delta) \).

Comparing (24) and (25) with (19) and (21), we can see how the growth of \( \bar{e}(t_j) \) influences closed-loop stability due to the perturbation effect of suspending communication between the observer and the controller. Due to the presence
of process-model mismatch, \( \| \hat{c}(t_j) \| \) may grow significantly and, therefore, the right-hand side of (24) and (25) may become large enough such that over the next sampling period the process state is not able to maintain the decreasing tendency and/or may even escape the stability region. Once this trend is detected, the observer state estimate must be promptly broadcast to the controller to reset the model-observer estimation error \( \hat{c}(t_j) \) to zero. This corrective action will convert the right-hand side of (24) and (25) into those of (19) and (21), which preserves the prescribed stability bounds on the closed-loop state. The following theorem summarizes the proposed event-triggered communication policy.

**Theorem 2:** Consider the closed-loop system of (8) under the networked MPC of (23) (excluding the constraints of (23c)) with the initial conditions satisfying \( \bar{x}(t_0) \in \Omega \), \( \bar{x}_s(t_0) = \eta(t_0) \in \Omega \). Then, for certain \( \Delta \in [0, \Delta^* \) , \( e(t_0) \in [0, \epsilon(t_0) \), at any \( t_j \), \( \| \hat{c}(t_j) \| \) satisfies:

\[
q_1(\| \hat{c}(t_j) \|, e(t_0), \Delta) > -\varsigma, \text{ for } \eta(t_j) \in \Omega \setminus \Omega' \quad (26)
\]

\[
q_2(\| \hat{c}(t_j) \|, e(t_0), \Delta) > \varsigma, \text{ for } \eta(t_j) \in \Omega' \quad (27)
\]

then the update law given by \( \bar{x}_s(t_j) = \eta(t_j) \) ensures that for \( t \in [t_j, t_{j+1}) \), \( \bar{V}(\bar{x}_s(t)) \leq -\varsigma \), for \( \eta(t_j) \in \Omega \setminus \Omega' \), and \( \bar{V}(\bar{x}_s(t)) \leq \varsigma \) for \( \eta(t_j) \in \Omega' \), for \( t \in [t_j, t_{j+1}) \).

**Proof:** The proof can be obtained by noting that at any sampling time when model state is updated, we have \( \| \hat{c}(t_j) \| = 0 \). Substituting this result into (24) and (25) yields (19) and (21). From Theorem 1, the choices of \( \Delta \) and \( e(t_0) \) guarantee that \( \bar{V}(\bar{x}_s(t)) \leq -\varsigma \), for \( \eta(t_j) \in \Omega \setminus \Omega' \), and \( \bar{V}(\bar{x}_s(t)) \leq \varsigma \) for \( \eta(t_j) \in \Omega' \), for \( t \in [t_j, t_{j+1}) \).

**Remark 4:** The event-triggered communication policy described in Theorem 2 is developed based on an analysis of the future evolution of the closed-loop system. This is necessary because even if the current observer state estimate raises no alarm regarding stability of the closed-loop system at some sampling instant, the control action generated by the networked MPC based on the approximate model may lead to an increase in \( \bar{V}(\bar{x}_s(t)) \) during the next sampling interval and drive the process state outside the stability region at the next sampling instant. Once this happens, there is no guarantee that the process state will converge back to the terminal region even if the MPC scheme is switched back to the sample-and-hold fashion since Theorem 1 is valid only when the process state stays within the stability region.

**Remark 5:** When the proposed networked MPC with event-triggered communication policy is implemented on the infinite-dimensional system of (5)-(7), the evolution of \( \bar{V}(x_s(t)) \) will be influenced by the fast states, \( x_f \), due to the inherent coupling between the slow and fast subsystems through the nonlinear terms, \( \bar{F}_s(x_s, x_f) \) and \( \bar{F}_f(x_s, x_f) \), and the fact that \( y = Q_s x_s + Q_f x_f \). To ensure the infinite-dimensional networked closed-loop stability, the update laws of (26) and (27) need to be modified. Specifically, the left hand sides of (26) and (27) will be enlarged by additional class \( K \) functions, \( \sigma_1(\epsilon) \) and \( \sigma_2(\epsilon) \), respectively, to account for the discrepancy between \( \bar{x}_s \) and \( x_s \). This in turn imposes additional restrictions on the choice of the sampling period \( \Delta \), the tolerable initial observer estimation error \( e(t_0) \), as well as the maximum allowable process-model mismatch. Furthermore, the separation between the slow and fast eigenmodes determines the value of \( \epsilon \), and thus plays an important role in deciding the closed-loop stability and performance properties. If the number of slow eigenmodes is chosen to be large enough such that \( \epsilon \) is sufficiently small, the fast eigenmodes will decay quickly, and thus the errors by using \( x_s \) instead of \( \bar{x}_s \) can be negligible. This argument can be justified using singular perturbation techniques, and the technical details are omitted due to space limitations.

**V. SIMULATION EXAMPLE**

In this section, we apply the proposed networked output feedback MPC scheme under event-triggered communication policy to a typical diffusion-reaction process. We consider the following example which describes a long, thin catalytic rod in a reactor with a zeroth-order exothermic reaction taking place on the rod. The evolution of the dimensionless rod temperature is governed by:

\[
\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} - \beta_U \bar{x} + \beta_T [e^{-\gamma/(1+\bar{x})} - e^{-\gamma}] + \beta_U b(z) u
\]

subject to the boundary conditions \( \bar{x}(0, t) = \bar{x}(\pi, t) = 0 \), where \( \bar{x} \) denotes the dimensionless temperature, \( \beta_T = 50.0 \), \( \gamma = 2.0 \), \( \beta_U = 4.0 \) are some process dynamic coefficients; \( u(t) \) denotes the dimensionless temperature of the cooling medium, \( y(t) \) is the measured output. The functions \( b(z) \) and \( q(z) \) are the actuator and sensor distribution functions, respectively. It can be verified that the operating steady state \( x(z, t) = 0 \) (with \( u = 0 \)) is unstable. The control objective is to thus stabilize the temperature profile at this unstable steady state by manipulating the temperature of the cooling medium, \( u(t) \), subject to the constraint \( \| u(t) \| \leq u_{max} = 2.5 \).

The solution of the eigenvalue problem for the linear differential operator yields \( \lambda_j = -j^2 - \beta_U \), \( \phi_j(z) = \sqrt{\frac{j}{\pi}} \sin(jz) \), \( j \in \{1, 2, \cdots, \infty \} \). We consider the first eigenvalue as the dominant one and apply Galerkins method to derive an ODE that describes the approximate temporal evolution of the amplitude of the first eigenmode:

\[
\hat{a}_1 = \lambda_1 \hat{a}_1 + f(\hat{a}_1) + g(z_a) a_1, \quad \tilde{y} = q(z_a) a_1
\]

where \( \bar{x}(z, t) = \sum_{j=1}^{\infty} a_j(t) \phi_j(z) \), \( f(\hat{a}_1) = \beta_U \langle \phi_1(z), h(\hat{a}_1) \rangle \), \( g(z_a) = \beta_U \langle \phi_1(z), b(z) \rangle \), and a single point actuator (with finite support) centered \( z_a = 1.5 \) is used for stabilization. In addition, a point sensor (with finite support) is implemented to measure the output variable, which gives \( q(z_a) = \phi_1(z) \). The ODE of (28) is used to design the Lyapunov-based MPC which is then implemented on a 30th order Galerkin discretization of the PDE (similar results were obtained using higher-order discretizations). Based on this setup, a networked Lyapunov-based MPC formulation of (23) is designed to achieve the aforementioned control objective with \( Q = 8.79, R = 0.01, N = 5 \) and \( \Delta = 0.002 \).

As can be seen from Fig.1 which depicts the closed-loop state profiles under the event-triggered networked MPC scheme (a) and under a conventional Lyapunov-based output feedback MPC (b) where the state of embedded model is
updated at each sampling time, both MPC approaches can stabilize the closed-loop state around the desired zero steady state with comparable closed-loop performances. This conclusion can also be reached by comparing the first eigenmode amplitude and manipulated input profiles shown in (c)-(d).

Fig. 2(a) shows the state of the model embedded in the networked MPC needs to be updated, where the binary variable “Update” denotes the model update signal. Under the proposed event-triggered communication policy, the model update rate (ratio between the number of updates and the number of sampling instants) can be reduced to 18.3%, which indicates a large saving in network resource utilization. One can see from Fig.2(a) that initially the MPC controller is executed in a sample-and-hold fashion until \( t = 0.5 \). After that time, \( a_1 \) converges inside the desired terminal set and communication is suspended until \( \| \hat{e}(t_j) \| \) grows sufficiently large to satisfy (26) or (27). To illustrate how the event-triggered communication policy works, we investigate the dynamics of the closed-loop process state and the update status for \( t \in [1.15, 1.17] \). The evolution of \( V(a_1) \), \( \| \hat{e}(t_j) \| \) and the update signal during that time interval are depicted in Figs.2(b)-(d). It can be observed that the value of \( \| \hat{e}(t_j) \| \) keeps increasing for \( t \in [1.15, 1.162] \); however, the condition of (27) is still not met at any sampling instant during that period, and therefore the control action is generated completely based on the model with no updates taking place. This can be verified from the profile of \( V(a_1) \) which implies that the closed-loop state, \( a_1 \), remains within the terminal set \( \Psi \triangleq \{ a_1 \in \mathbb{R} : V(a_1) \leq 0.0018 \} \) for \( t \in [1.15, 1.162] \). At \( t = 1.162 \), the increase in \( \| \hat{e}(t_j) \| \) is significant enough to breach the prescribed stability threshold \( (\| \hat{e}(t_j) \| \leq 0.0005) \), which indicates that (27) now holds such that \( a_1 \) may escape from \( \Psi \) in the next sampling period. To avoid this, the state of embedded model needs to be updated using the observer estimate in order to suppress the escaping tendency of \( a_1 \) (see Fig.2(a)). This corrective action resets \( \hat{e}(t_j) \) to zero and ensures the satisfaction of (27).

![Figure 1: Evolution of the closed-loop state profiles under networked MPC with event-triggered communication policy (a) and under a conventional Lyapunov-based output feedback MPC (b). Plot (c) depicts a comparison of the manipulated inputs, while plot (d) shows a comparison of the amplitudes of the first eigenmode profiles.](image1)

![Figure 2: Plot (a): Update times of the model embedded in the MPC controller under the event-triggered communication policy. Plots (b)-(d): Profiles of the update signal (b), \( V(a_1) \) (c) and \( \| \hat{e} \| \) (d), where blue circle markers represent \( e(t_j) \) right before potential updates and the blue square makers represent \( e(t_j) \) at the tie of potential updates, on the time interval [1.15, 1.17](image2)

**REFERENCES**


