Multi-Keyhole Effect in MIMO AF Relay Downlink Transmission with Space-Time Block Codes

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Abstract—Multi-keyhole bridges the gap between single keyhole and full-scattering multiple-input multiple-output (MIMO) channels. In this paper, we therefore investigate the multi-keyhole effect on the MIMO amplify-and-forward (AF) relay downlink transmission with orthogonal space-time block codes. In particular, we derive the analytical symbol error rate (SER) expression for the considered system with arbitrary number of keyholes. Moreover, SER approximations in the high SNR regime for several important special scenarios of multi-keyhole channels are further derived. These asymptotic results provide important insights into the impact of system parameters on the SER performance. Our analysis is confirmed by comparing with Monte-Carlo simulations.

I. INTRODUCTION

In recent years, a large body of research has studied the transmission of orthogonal space-time block code (OSTBC) in relay networks to explore the spatial diversity gain in a distributed manner (see, e.g., [1]–[4] and the references therein). In particular, by incorporating the line-of-sight effect into the second hop, the symbol error rate (SER) over dual-hop Rayleigh-Rician fading channels has been investigated in [1]. The error probability of multiple-input multiple-output (MIMO) systems using OSTBCs and semi-blind amplify-and-forward (AF) relays over dual-hop Rayleigh fading channels has been presented in [2]. For CSI-assisted AF relay, the bit error rate (BER) of MIMO system with OSTBC transmission has been presented in [3].

Although rich-scattering conditions have been widely employed in the existing literature on MIMO, the assumption fails to model some practical scenarios with lack of scattering objects. The later propagation phenomenon can be characterized as a keyhole or a pinhole MIMO channel. The keyhole channel has been validated by conducting many measurement campaigns [5]. The keyhole modeling describes the most extreme case of channel fading, i.e., the channel matrix is unit-rank regardless of the number of transmit/receive antennas. As a consequence, keyhole effects substantially decrease the performance of MIMO systems.

Despite of this important practical phenomenon, however, to the best of the authors’ knowledge, little is known on the effects of keyhole propagation in dual-hop relay systems. Very recently, the effect of keyhole on the performance of relay systems has been investigated in [6]–[8]. In [6], [7], the authors have shown that using the decode-and-forward relay can mitigate the deleterious effect of keyhole. In [8], we adopted the same practical system model assumed in [6], [7] and investigated the effect of antenna correlation on the performance of single keyhole MIMO AF relay systems. It has been shown that the antenna correlation has no impact on the diversity order. In this paper, we take a step further and investigate the effects of multi-keyhole propagation for OSTBC transmission in AF relay systems. In many practical scenarios, the degeneration effect of poor-scattering environment can be more generally modeled as multi-keyhole channels. This generalization embraces both rich-scattering and keyhole conditions in which the rank of channel matrix now can vary from unit to full-rank [9], [10].

Our new contributions are as follows. We first characterize the end-to-end instantaneous received signal-to-noise ratio (SNR), which allows us to obtain the exact moment generating function (MGF) of end-to-end SNR. By utilizing this result, we derive an analytical expression for the exact SER and an approximate expression for a downlink system where an $n_S$-antenna base station (source) communicates with an $n_D$-antenna mobile station (destination) through the assistance of an $n_R$-antenna relay, i.e., $n_S > \min(n_R,n_D)$. The derived MGF expression is also useful to study additional important performance criteria such as the outage probability and the ergodic capacity. We have also presented a high SNR analysis where both diversity and array gain are quantified in several special cases of interest. It is demonstrated that under the considered downlink scenario, the diversity order is $\min(n_K,n_R)$ for a multi-keyhole MIMO/multiple-input single-output (MISO) channel, i.e., $n_D = 1$, and $n_K$ is the number of keyholes.

Notation: Vectors and matrices are denoted in lower case/upper case boldface, respectively. $I_n$ represents the $n \times n$ identity matrix and $\|A\|_F$ defines the Frobenius norm of the matrix $A$. $E\{\cdot\}$ is the expectation operator. $\det(A)$ means the determinant of the matrix $A$. $K_\nu(\cdot)$ is the modified Bessel function of the second kind [11, Eq. (8.432.3)] and $B(\cdot,\cdot)$ is the Beta function [11, Eq. (8.380.1)].

II. SYSTEM AND CHANNEL MODEL

A. Protocol Description

We consider a dual-hop MIMO relay system with source, relay and destination terminals having $n_S$, $n_R$ and $n_D$ antennas respectively. Let $H_1 \in \mathbb{C}^{n_R \times n_S}$ be the channel matrix between the source and the relay, and $H_2 \in \mathbb{C}^{n_D \times n_R}$ be the channel matrix between the relay and the destination. The channel gains are assumed to be fixed during a block of $T_c$ symbols.
and slowly changed over independent blocks. The system under consideration is downlink in which the source acts as a base station and the destination is a mobile station, i.e., $n_S > \min(n_R, n_D)$. Moreover, we assume that the direct link does not exist in this system possibly due to high shadowing and large path loss effects.

An OSTBC is generated at the source before transmission over the dual-hop AF relay network. A sequence of information symbols $x_1, x_2, ..., x_N$ is selected from a constellation $S$ with average transmit power per symbol $E_s$, i.e., $E \{ |x_i|^2 \} = E_s$. These symbols are then encoded into an OSTBC matrix denoted by an $T_c \times n_S$ transmission matrix $G$ with the property that columns of $G$ are orthogonal. The input-output relationship for the communication of the source-relay link is expressed as

$$Y_1 = H_1 X + W_1,$$  

where the OSTBC transmission matrix at the source is defined as $X = G^T$ and $W_1$ denotes the additive white Gaussian noise (AWGN) matrix at the relay. The relay then multiplies the received signal $Y_1$ with a fixed amplifying gain $G$, and retransmits the resulting signal to the destination. At the destination, the input-output relationship is given by

$$Y_2 = H_2 Y_1 + W_2,$$  

where $W_2$ is the AWGN matrix at the destination. In this paper, without loss of generality, we assume that the relay terminal operates in semi-blind mode and consumes the same number of power as the source, leading to the amplifying gain $G = \sqrt{\gamma / [nr(1 + \bar{\gamma})]}$, where $\gamma = E_s / N_0$ is the average SNR. The end-to-end instantaneous SNR is written as [1], [12]

$$\gamma_D = a^2 \| \left( I_{n_D} + G^2 H_2 H_2^H \right)^{-1/2} H \|_F^2,$$  

where $a = T_c / (N n_S)$.

**B. Channel Model**

We consider a downlink cellular network where the source and the relay terminals are fixed base stations and can be installed at strategic locations by network operators. In contrast, the destination is a mobile station. In the considered system model, it is reasonable to assume that the source-relay link enjoys a rich-scattering environment yielding its corresponding channel matrix $H_1$ to have Rayleigh fading with entries $CN(0, 1)$. On the other hand, we consider a practical scenario in which the mobile station is located in a poor scattering environment, e.g., due to mobility. In order to characterize this scenario, the channel matrix $H_2$ is assumed to undergo multi-keyhole fading with $n_K$ number of keyholes. The multi-keyhole channel model extends the widely used single keyhole channel model (see e.g., [6]–[8]). In fact, the multi-keyhole channel model can be viewed as a generalized model which bridges the gap between single-keyhole and rich scattering MIMO channels. The multi-keyhole fading channel can be mathematically represented as

$$H_2 = \sum_{k=1}^{n_K} \sqrt{\sigma_k} h_{k,k} h_{k,k}^H = H_{\kappa} A H_{\kappa}^H,$$  

where $\sigma_k$ is the power gain of the $k$-th keyhole and $A$ is the diagonal matrix whose $k$-th diagonal element is $\sqrt{\sigma_k}$. Moreover, $H_{\kappa} = [h_{\kappa,1}, \ldots, h_{\kappa,n_K}]$ and $H_{\kappa} = [h_{\kappa,1}, \ldots, h_{\kappa,n_K}]$ are mutually independent matrices. Note that, depending on the keyhole number $n_K$, the rank of matrix $H_2$ can vary from one to full-rank.

**III. PERFORMANCE ANALYSIS**

**A. The MGF of the end-to-end instantaneous SNR**

In this subsection, we will derive the MGF of $\gamma_D$, which will be utilized to obtain the system performance. By the definition, the MGF is the inverse Laplace transform of the probability density function (PDF), i.e., $\Phi_{\gamma_D} (s) = E_{\gamma_D} \{ \exp(-s \gamma_D) \}$. As can be observed from (3), the MGF of $\gamma_D$ can be expressed as

$$\Phi_{\gamma_D} (s) = E \left[ \frac{\det \left( I_{n_D} + G^2 H_2 H_2^H \right)^{-1/2}}{\det \left( I_{n_D} + G^2 (1 + s \omega) H_2 H_2^H \right)^{-1/2}} \right]^{n_K}.$$  

From (6), and using the property that $H_{\kappa} H_{\kappa}^H$ is a positive-definite matrix, (6) can be rewritten as

$$\Phi_{\gamma_D} (s) = E_{\Lambda} \left\{ \prod_{i=1}^{q} \left( \frac{1 + G^2 \lambda_i}{1 + G^2 (1 + \gamma_D \lambda_i)} \right)^{n_S} \right\},$$  

where $\lambda_i, i = 1, \ldots, q$, are the $q$ non-zero ordered eigenvalues of $H_{\kappa} H_{\kappa}^H$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_q)$. Averaging in (7) requires the joint PDF of $\lambda_1, \ldots, \lambda_q$ for which a combined general expression is presented in the appendix.

**B. Exact Symbol Error Rate**

From (7), (36), and using Lemma 1 in [13], the MGF of $\gamma_D$ can be readily derived as

$$\Phi_{\gamma_D} (s) = \frac{\det(S)}{K \det(s/\lambda_D)},$$  

where $K = \prod_{i=1}^{n_K} \Gamma(i) \Gamma(n_D - i) \Gamma(n_S - 1) / \Gamma(n_D - i + 1)$ and $S$ is an $n_K \times n_K$ matrix whose entries are given by

$$S_{i,j} = \left\{ \begin{array}{ll} \sigma_i^{j-i-1} & 1 \leq i < j, \\ 2 \sigma_j^{n_K} & 2 \sigma_j^{n_K} - 2 \sigma_j^{n_K-1} - \sigma_j^{n_K} + 1 - t - q - 1, \\ \times K_{n_K-n_S} \left( \frac{2 \sigma_j^{n_K-1}}{1 + G^2 \sigma_j^{n_K-1}} \right)^{-n_S} & t < i \leq n_K. \end{array} \right.$$  

In this paper, we consider $M$-PSK modulated source symbols. Therefore, the average SER can be expressed as [14]

$$P_e = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_{\gamma_D} \left( \frac{g}{\sin^2 \theta} \right) d\theta,$$  

where $g = \sin \left( \frac{\theta}{2} \right)^2$. Unfortunately, to the best of our knowledge, (9) cannot be evaluated in closed-form. However, note that it is a convenient result and allows us to numerically evaluate the system’s average SER performance.
C. High SNR Symbol Error Rate

To provide further insight into the impact of multi-keyhole phenomenon on the performance of the system, we will further scrutinize two important special scenarios: i) Single keyhole MIMO/MIMO relay channels \( (n_k = 1) \) and ii) Multi-keyhole MIMO/MISO relay channels \( (n_D = 1) \). An asymptotic expression for \( \Phi_{\gamma_{\Omega}} (s) \) for arbitrary combinations of \( n_S, n_R, n_D, \) and \( n_K \) appears prohibitively complicated to obtain, if not impossible.

1) Single keyhole MIMO/MIMO Channels \( (n_K = 1) \): In this special scenario, \( \Phi_{\gamma_{\Omega}} (s) \) given in (7) becomes

\[
\Phi_{\gamma_{\Omega}} (s) = \int_0^{\infty} \frac{2^{R_{\gamma} + 1}}{\Gamma(n_R) \Gamma(n_D)} \left( \frac{x}{\sigma} \right)^{n_R + n_D - 1} \times K_{n_R - n_D} \left( \frac{x}{\sigma} \right) \left( 1 + \frac{G^2 a^2 x_s}{1 + G^2 x} \right)^{-n_S} \, dx. \tag{10}
\]

To approximate \( \Phi_{\gamma_{\Omega}} (s) \) in the high SNR, we employ [15]

\[
K_n (2 \sqrt{z}) \approx \begin{cases} 
- \frac{1}{2} \ln (z) & \text{for } n = 0, \\
\frac{\Gamma(n/2)}{2^{n/2} \Gamma(n/2)} & \text{for } n \neq 0.
\end{cases} \tag{11}
\]

- \( n_R = n_D \): By substituting (11) into (10) and exchanging the variable \( \gamma x = y \), we have

\[
\Phi_{\gamma_{\Omega}} (s) \approx \frac{(\sigma \gamma)^{n_R - n_D}}{\Gamma(n_R) \Gamma(n_D)} \int_0^{\infty} \ln (\sigma \gamma) y^{n_R - 1} (1 + G^2 a^2 y_s)^{-n_S} \, dy - \frac{1}{2} \ln (s) \times \psi(n_S - n_R), \tag{12}
\]

where (12) is obtained by neglecting the small term in (10). By applying the results of [11, Eq. (3.194.3)] and [16, Eq. (2.6.4.7)] with the condition that \( n_D > n_R \), the MGF of \( \gamma \) for the case \( n_R = n_D \) can be expressed as

\[
\Phi_{\gamma_{\Omega}} (s) \approx \frac{(\sigma \gamma)^{n_R - n_D}}{\Gamma(n_R) \Gamma(n_D)} \left[ \Gamma(n_S - n_R) \Gamma(n_D) \times \ln (\sigma \gamma) - \psi(n_S - n_R) \right], \tag{13}
\]

where \( a_1 = \sigma^2 G a \).

- \( n_R \neq n_D \): Similarly, we can find the MGF of \( \gamma_D \) as

\[
\Phi_{\gamma_D} (s) \approx \frac{(\sigma \gamma)^{n_D - n_S}}{\Gamma(n_D) \Gamma(n_D)} \int_0^{\infty} \ln (\sigma \gamma) y^{n_D - 1} (1 + G^2 s)^{-n_D} \, dy, \tag{14}
\]

where \( m_1 = \min(n_R, n_D) \). Note that the integral in (14) converges when \( n_D > \min(n_R, n_D) \) which then yields

\[
\Phi_{\gamma_D} (s) \approx \frac{(\sigma \gamma)^{n_D - n_S}}{\Gamma(n_D) \Gamma(n_D)} \left[ \Gamma(n_S - n_R) \Gamma(n_D) \times \ln (\sigma \gamma) \right], \tag{15}
\]

By combining the two cases, the asymptotic SER can be expressed as

\[
P_e \approx \sum_{1}^{n_R} \frac{\Gamma(n_S - m) \Gamma(m - 1)}{\Gamma(n_S) \Gamma(m - 1) \Gamma(m)} \times \left[ \frac{\Gamma(n_S - m) \Gamma(m - 1)}{\Gamma(n_S) \Gamma(m - 1) \Gamma(m)} \right]. \tag{16}
\]

with the condition that \( n_S > \min(n_R, n_D) \) and \( \Xi_1 \) is defined, for example with M-PSK modulation as

\[
\Xi_1 = \begin{cases} 
\pi^{-\frac{n}{2}} \frac{1}{\sin \frac{\gamma_D}{2}} \left[ \frac{n}{\sin \frac{\gamma_D}{2}} \ln \left( \frac{n}{\sin \frac{\gamma_D}{2}} \right) \right]^{n_D - n_R} - \psi(n_R) + \psi(n_S - n_R), & \text{for } n_R = n_D, \\
\frac{\Gamma(n_S - m) \Gamma(m - 1)}{\pi} \left[ \frac{\sin \frac{\gamma_D}{2}}{\sin \frac{\gamma_D}{2}} \right]^{n_D - n_R}, & \text{for } n_R \neq n_D.
\end{cases} \tag{17}
\]

2) Multi-keyhole MIMO/MISO Channels \( (n_D = 1) \): The case of multi-keyhole MIMO/MISO system setup is reasonable in the context of current cellular networks where for e.g., due to space constraints mobile station is only equipped with a single antenna. In this special scenario, i.e., \( n_D = 1 \), the elements of matrix \( S \) in (8) are represented as \( S_{ij} = \sigma_j \) for \( i = 1, \ldots, n_k - 1 \) and the last row of \( S \) is given by

\[
S_{n_k, j} = \int_0^{\infty} 2^s e^{-s^2 - 1} x^{n_k - 1} \times K_{n_k - 1} \left( \frac{x}{\sigma} \right) \left( 1 + G^2 a^2 x_s \right)^{-n_S} \, dx. \tag{17}
\]

Due to the multi-linear property of the determinant, to calculate \( \det(S) \) for large \( \gamma \), we use the series representation of the Bessel function as

\[
K_n (z) = \sum_{k=0}^{\infty} \frac{(z)^{n+k-1}}{\Gamma(k+1)} \left( \frac{z}{2} \right)^{-n-2k} + \sum_{k=0}^{\infty} \frac{(z)^{n+k-1}}{\Gamma(k+1)} \left( \frac{z}{2} \right)^{-n-2k-2} \tag{18}
\]

From (17), (18), and after several calculations together with the help of [11, Eq. (3.194.3)] and [16, Eq. (2.6.4.7)], we can express the entries of the last row of matrix \( S \) as two terms \( I_1 \) and \( I_2 \), i.e., \( S_{n_k, j} = I_1 + I_2 \), respectively shown as follows:

\[
I_1 \approx \sum_{k=0}^{n_k - 2} (-1)^k \Gamma(n_k - 1 - k) \times \frac{1}{\Gamma(n_k - k)} = \frac{1}{\Gamma(n_k - k)}, \tag{19}
\]

\[
I_2 \approx (-1)^{n_k + 1} \Gamma(n_k - k) \times \psi(n_k - n_k - k) + \psi(k + 1) \tag{20}
\]

To asymptotically approximate the determinant of matrix \( S \), we consider two separate cases: i) \( n_k > n_k \) and ii) \( n_k \leq n_k \)

- \( n_k > n_k \): The minimum exponent of \( \gamma \) to make \( \det(S) \) non-zero in \( I_1 \) is \( k = n_k - 1 \) and in \( I_2 \) is \( k = n_k - 1 \). Hence, \( I_2 \) can be neglected as compared to \( I_1 \), which results in

\[
\Phi_{\gamma_{\Omega}} (s) \approx \frac{\Gamma(n_S - n_k) \Gamma(n_k) \det(S^1)}{kT(n_S) (G^2 a^2)^n \det(S^1)}, \tag{20}
\]

where \( S^1 \) is an \( n_k \times n_k \) matrix whose entries are

\[
S^1_{i, j} = \begin{cases} 
\sigma_{i, j - 1} & 1 \leq i \leq n_k - 1, \\
(-1)^{n_k - 1} \sigma_{i, j - 1} & i = n_k.
\end{cases}
\]

Since \( \det(S^1) = \prod_{k=1}^{n_k} \sigma_{k}^{-1} \prod_{l=1}^{n_k} (\sigma_{k} - 1) \), we can rewrite (20) as

\[
\Phi_{\gamma_{\Omega}} (s) \approx \frac{\Gamma(n_S - n_k) \Gamma(n_k) \prod_{k=1}^{n_k} \sigma_{k}(G^2 a^2)^{n_k}}{kT(n_S) \prod_{k=1}^{n_k} \sigma_{k}(G^2 a^2)^{n_k}}. \tag{21}
\]
For the OSTBC transmission, we apply the general approach
and the minimum exponent of $\gamma$ in $I_2$ can be selected as $k = 0$, which yields
\[
\Phi_{ns}(s) = \frac{\Gamma(n_S - n_R)}{\Gamma(n_S)(G^2a^2)s^n}\det\left(\sigma_i^{-1}\right),
\]
where $\sigma_i^{-1}$ is an $n_K \times n_K$ matrix whose entries are as
\[
\sigma_i^{-1} = \begin{cases} \sigma_i^{-1}, & 1 \leq i \leq n_K - 1, \\ (-1)^{n+1}\sigma_i^{-n-1}\ln\left(G^2a^2\sigma_i\right), & i = n_K. \end{cases}
\]

Additionally, we can compute $\det(S^2)$ as
\[
\det(S^2) = (-1)^{n+1}\ln\left(G^2a^2s\right) + \psi(n_S - n_K) + \psi(1) 
\times \det(S^1) + \det(S^3),
\]
where $S^3$ is an $n_K \times n_K$ matrix whose entries are as
\[
S^3_{ij} = \begin{cases} \sigma_i^{-1}, & 1 \leq i \leq n_K - 1, \\ (-1)^{n+1}\sigma_i^{-n-1}\ln\left(G^2a^2\sigma_i\right), & i = n_K. \end{cases}
\]

From (22) and (23), $\Phi_{ns}(s)$ can be asymptotically approximated as
\[
\Phi_{ns}(s) \approx \frac{\Gamma(n_S - n_R)}{n_K\Gamma(n_S)(G^2a^2)s^n}\det\left(\sigma_i^{-1}\right) + (-1)^{n+1}
\ln\left(G^2a^2s\right) + \psi(n_S - n_K) + \psi(1) 
\times \prod_{k=1}^{n_K}\sigma_i^{-1}. \tag{24}
\]

By combining (21), (24) with (9), the average SER can be asymptotically approximated as
\[
P_e = \frac{\Gamma(n_K)}{\Gamma(n_K)
\ln\left(G^2a^2s\right) + \psi(n_S - n_K) + \psi(1) 
\times \prod_{k=1}^{n_K}\sigma_i^{-1}.}
\]

\[
IV. \text{ Numerical Results}
\]

In this section, we provide numerical results for some representative scenarios to validate our analysis in previous section. For notational brevity, we define a MIMO AF relay system as $(n_S, n_R, n_D)$ and all results are shown for 8-PSK modulation. For the OSTBC transmission, we apply the general approach presented in [17] for arbitrary $n_S$, for e.g., when $n_S=4$ and 5, the code rate $R_c = 3/4$ and 1/2, respectively.

Fig. 1 displays the SER performance versus SNR for (4, 3, 2)-MIMO AF relay system when $n_K = 1, 2, 3$ and $\{\sigma_i\}_{i=1}^{n_K} = \{0.5, 0.3, 0.2\}$. As can be observed from Fig. 1, the analytical results show a good agreement with simulations. We see that the performance improves with the number of keyholes $n_K$.

Fig. 2 compares the asymptotic and analytical SER for keyhole channels, i.e., $n_K = 1$ and $\sigma = 1$. Results are shown for different MIMO AF relay systems, i.e., Case 1: $(5, 3, 2)$, Case 2: $(5, 3, 3)$, Case 3: $(5, 4, 3)$. The asymptotic curves precisely converge to analytical ones in the high SNR regime. We see that when increasing $n_K$ from three to four, the diversity gain does not increase as the SER curves for Case 2 and Case 3 are parallel. This observation is inline with the our derivation in previous section as the diversity gain depends on $\min(n_K, n_R)$.

Fig. 3 shows the SER performance for multi-keyhole MIMO/MISO AF relay systems. Results are shown for various antenna configurations and number of keyholes, i.e., Case 4: $n_K = 4$ and $n_K = 2$, Case 5: $n_K = 2$ and $n_K = 4$, Case 6: $n_K = 3$ and $n_K = 4$, Case 7: $n_K = 6$ and $n_K = 8$. The number of antenna at source $n_S = 5$ and multi-keyhole powers are $\{\sigma_i\}_{i=1}^{n_K} = \{0.4, 0.3, 0.2, 0.1\}$. We can observe that the asymptotic approximations are accurate in the high SNR regime and the SER performance is determined by the
the performance approaches to that of Rayleigh MIMO AF
expected. Moreover, the performance is significantly impro-
vanced when $n$ comparison, the case of MIMO AF relay channel where both
value of $K$ is increased from 1 to 8, while additional keyholes,
$K$, $n$-MIMO AF relay systems with multi-keyhole versus
$\min(\mu_k, n_R)$. The tightly converging asymptotic results provide
multi-keyhole and system’s configuration on the SER performance.

**APPENDIX**

In this appendix, we derive the PDF of $\lambda= [\lambda_1, \lambda_2, \ldots]$. Due to
of $n_R > n_D$.

**A. The $n_D > n_K$ Case**

Let $0 \leq \lambda_1 \leq \ldots \leq \lambda_{n_K} \leq \infty$ be the eigenvalues of $H_1^*H_2$
with $H_2 \in \mathbb{C}^{n_R \times n_K}$. Define $\Sigma = A A^\dagger$ and $B = A^\dagger H_1^*H_2A$.
Hence, we can write

$$H_1^*H_2 = H_1^\dagger B H_1. \quad (26)$$

Let $D_{ord} = (x_1, \ldots, x_{n_K})$ be the ordered eigenvalues of $B$
with $0 \leq x_1 \leq \ldots \leq x_{n_K} \leq \infty$. Conditioned on $D_{ord}$, the
joint PDF of the non-zero eigenvalues of $H_1^*H_2$ is given by
the joint PDF of the eigenvalues of $B^2 H_1^*H_2 B^2$. This has the same
form as $A^\dagger H_1^*H_2 A$ and so the PDF comes from [18, Eq. (42)]
as

$$f(\lambda_1, \ldots, \lambda_{n_K}|x_1, \ldots, x_{n_K}) = \prod_{i=1}^{n_K} \lambda_i^{n_{n_K} - n_K} \det(\lambda^{n_{n_K} - j}) \det\left(\frac{1}{x_j} \frac{1}{x_R} \right), \quad (27)$$

Therefore,

$$f(\lambda) = C_{mk1} \int_{D_{ord}} \prod_{i=1}^{n_K} x_i^{n_{n_K} - n_K} \prod_{j=1}^{n_{n_K}} \left(\frac{1}{x_j} - \frac{1}{x_R} \right)$$

$$\times \det\left(\frac{1}{x_{n_K} - j} \right) \det\left(\frac{1}{x_R - j} \right) \det\left(\frac{1}{x_j - j} \right) dx_1 \ldots dx_{n_K}, \quad (28)$$

where $\sigma_i$ are the diagonal elements of $\Sigma$ and

$$C_{mk1} = \frac{\det(\Sigma)^{-n_K} \prod_{i=1}^{n_K} \lambda_i^{n_{n_K} - n_K} \det\left(\lambda_i^{n_{n_K} - j}\right)}{\prod_{i=1}^{n_K} (n_R - i)! (n_D - i)! \prod_{j=1}^{n_{n_K}} (\frac{1}{\sigma_j} - \frac{1}{\sigma_R})}. \quad (29)$$

V. CONCLUSION

In this paper, we have investigated the multi-keyhole effect
on the SER performance of MIMO AF relay systems by
deriving the analytical SER expression. The analytical results enable us to investigate the multi-keyhole effect which en-
compasses a variety of MIMO fading channels from keyhole
to full-scattering environment. We have also obtained the
asymptotic approximation of SER for several important cases
including single keyhole MIMO/MISO AF relay systems,
multi-keyhole MIMO/MISO AF relay systems. Specifically,
we have shown that for multi-keyhole MIMO/MISO AF relay
channels ($n_D = 1$), the diversity gain is solely determined by
the minimum among the number of keyholes and the number
of antennas at the relay, i.e., $\min(\mu_k, n_R)$. The tightly conver-
ging asymptotic results provide insights into the effect of multi-
keyhole and system’s configuration on the SER performance.

minimum of $n_K$ and $n_R$. Specifically, among the four cases,
Case 7 surpasses Case 4, 5, and 6 since Case 7 has the highest
value of $\min(n_K, n_R)$.

Finally, to further understanding the effect of multi-keyhole
on the SER performance, Fig. 4 displays the performance
of $(5, 3, 2)$-MIMO AF relay systems with various $n_K$. For
comparison, the case of MIMO AF relay channel where both
hops undergo Rayleigh fading is also plotted in Fig. 4. It is
observed that as $n_K \leq 2$ the diversity gain does not increase as
expected. Moreover, the performance is significantly improved
when $n_K$ is increased from 1 to 8, while additional keyholes,
$(n_K > 8)$, has a diminishing impact on reducing the SER.
When the number of keyholes increases in the limit $n_K \to \infty$,
the performance approaches to that of Rayleigh MIMO AF
relay channels [12].

Fig. 3. SER of $(5, n_K, n_D)$-MIMO AF relay systems with keyhole versus
SNR for 8-PSK modulation and $n_D = 1$.

Fig. 4. SER of $(5, 3, 2)$-MIMO AF relay systems with multi-keyhole versus
SNR for 8-PSK modulation and $n_K = 1, 2, 4, 8, 12.$
Using Corollary 2 in [19], we obtain
\[
f(\lambda) = \frac{2^{\kappa \lambda}}{\prod_{i=1}^{K} \lambda_{i}^{\kappa - n_{\lambda}} \det \left( \lambda_{i}^{\kappa - j} \right)} \prod_{i=1}^{n_{\lambda}} \frac{1}{(n_{\lambda} - i)!} \det \left( \mathbf{A}_{\lambda}^{n_{\lambda} - i} \right) \det \left( \mathbf{A}^{n_{\lambda} - i} \right)
\times \det \left( \lambda_{i,j}^{n_{\lambda} - n_{\lambda}} \mathbf{K}^{n_{\lambda} - n_{\lambda}} \left( \frac{\lambda_{i,j}}{\sigma_{j}} \right) \right),
\]
where (30) is obtained from [11, Eq. (3.471.9)].

B. The n_D ≤ n_K Case
When \(n_R \geq n_K \geq n_D\) or \(n_R \leq n_K \geq n_D\) based on [20, Eq. (25)], we have joint ordered eigenvalues \(x_1 \geq x_2 \geq \cdots \geq x_{n_D}\) as
\[
f(x_1, \ldots, x_{n_D}) = \frac{\det(x_{i}^{1-1}) \det(V)}{\prod_{i=0}^{n_{D}} \Gamma(n_{D} - i + 1) \det(\sigma_{i}^{-1})},
\]
where matrix \(V\) is defined as
\[
V_{i,j} = \begin{cases} 
\sigma_{i,j}^{1-1}, & 1 < j \leq n_K - n_D, \\
\sigma_{i,n_{D}}^{n_{D} - n_{D} - 1} e^{-x_{i,j} - n_{D} - 1}, & n_K - n_D < j \leq n_K.
\end{cases}
\]
for \(i, j = 1, \ldots, n_K\). From [18, Eq. (42)], we have
\[
f(\lambda) = C_{mk2} \int_{n_{D}}^{n_{K}} \det(e^{\frac{1}{\lambda_{i,j}^{-1}}}) \times \det(V) \prod_{i=1}^{n_{D}} x_{i}^{n_{D} - n_{D} - 1} dx_1 \ldots dx_{n_{D}},
\]
where \(C_{mk2}\) is given by
\[
C_{mk2} = \frac{\prod_{i=1}^{n_{D}} \lambda_{i}^{n_{D} - n_{D}} \det(\lambda_{i}^{-1})}{\prod_{i=1}^{n_{D}} \Gamma(n_{D} - i + 1) \Gamma(n_{R} - i + 1) \det(\sigma_{i}^{-1})}.
\]
Using [13, Lemma 2], the integral in (33), can be solved as
\[
I = \det(\phi),
\]
where \(\phi\) is the \(n_K \times n_K\) matrix whose entries are given by
\[
\phi_{i,j} = \begin{cases} 
\sigma_{i,j}^{1-1}, & 1 < j \leq n_K - n_D, \\
\sigma_{i,1}^{n_{D} - n_{D} - 1} e^{-x_{i,1} - n_{D} - 1}, & n_K - n_D < j \leq n_K.
\end{cases}
\]
Pulling everything together, we have the following expression for the joint ordered eigenvalue given by
\[
f(\lambda) = \frac{\Gamma(n_{D} + 1) \prod_{i=1}^{n_{D}} \lambda_{i}^{n_{D} - n_{D}} \det(\lambda_{i}^{-1}) \det(\phi)}{\prod_{i=1}^{n_{D}} \Gamma(n_{D} - i + 1) \Gamma(n_{R} - i + 1) \det(\sigma_{i}^{-1})}.
\]
Finally, by combining (30) and (35), we can obtain the PDF of \(\lambda\) for the general case as
\[
f(\lambda) = \frac{\prod_{i=1}^{n_{D}} \lambda_{i}^{n_{D} - q} \det(\lambda_{i}^{-1}) \det(\Omega)}{\prod_{i=1}^{n_{D}} \Gamma(n_{D} - i + 1) \Gamma(n_{R} - i + 1) \det(\sigma_{i}^{-1})}_{n_{K}},
\]
where \(q = \min(n_D, n_K)\) and \(\Omega\) is an \(n_K \times n_K\) matrix with entries
\[
\Omega_{i,j} = \begin{cases} 
\sigma_{i,j}^{1-1}, & 1 \leq j \leq t, \\
\sigma_{i,n_{D}}^{n_{D} - n_{D} - 1} e^{-x_{i,n_{D} - n_{D} - 1}}, & t < j \leq n_K.
\end{cases}
\]
where \(t = \max(0, n_K - n_D)\).

REFERENCES