Note

Semi-hyper-connected edge transitive graphs

Zhao Zhang, Jixiang Meng*

Colleges of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China

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Abstract

A graph $G$ is said to be hyper-connected if the removal of every minimum cut creates exactly two connected components, one of which is an isolated vertex. In this paper, we first generalize the concept of hyper-connected graphs to that of semi-hyper-connected graphs: a graph $G$ is called semi-hyper-connected if the removal of every minimum cut of $G$ creates exactly two components. Then we characterize semi-hyper-connected edge transitive graphs.

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1. Introduction and Notation

Let $G = (V, E)$ be a connected undirected graph. A cut of $G$ is a vertex set $C \subseteq V(G)$ such that the removal of $C$ results in either a disconnected graph or a trivial graph. The minimum size of a cut of $G$ is called the connectivity of $G$, denoted by $\kappa(G)$. It is well known that $\kappa(G)$ is one of the vulnerability measures of a network [2]. Recently, many variations on connectivity have been proposed. A graph $G$ is said to be superconnected [2], if for any minimum cut $C$ of $G$, $G - C$ has isolated vertices. A graph $G$ is called hyper-connected [3], if the removal of each minimum cut of $G$ creates exactly two components, one of which is an isolated vertex. In this paper, we generalize the concept of hyper-connected graphs to that of semi-hyper-connected graphs.

Call a graph $G$ semi-hyper-connected, if the removal of each minimum cut of $G$ results in exactly two components. The removal of some vertices can be viewed as the failure of some nodes in a network. Clearly, when the failure is such that some information from one part of the network is unable to reach another part, then the fewer the components, the less the damages to the network. Thus, semi-hyper-connectivity can be used as one measure of network vulnerability. Obviously, a graph is hyper-connected if and only if it is both superconnected and semi-hyper-connected.

In [6], Meng characterized vertex and edge transitive hyper-connected graphs. In this paper, we characterize edge transitive semi-hyper-connected graphs.

Let us introduce some notation used in this paper.

The neighbor set of $N \subseteq V(G)$ is $\delta_G(N) = \{v \in V(G) \setminus N | uv \in E(G) \text{ for some } u \in N\}$. Denote by $\delta_G(N) = V(G) \setminus (N \cup \delta_G(N))$. When the graph under consideration is clear, we use $\delta(N)$ and $\delta(N)$ instead of $\delta_G(N)$ and $\delta_G(N)$. The subgraph induced by $N$ is denoted by $G[N]$.

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* Corresponding author.

E-mail addresses: zhzhao@xju.edu.cn (Z. Zhang), mjx@xju.edu.cn (J. Meng).
If $G$ is not semi-hyper-connected, then there exists a minimum cut $C$ such that $G \setminus C$ has at least three components. Call such a minimum cut a semi-hyper-cut. A vertex set $N \subseteq V(G)$ with $\delta(N)$ being a semi-hyper-cut is called a semi-hyper-fragment. If $N$ is a semi-hyper-fragment, so is $\delta(N)$. A semi-hyper-fragment with least cardinality is called a semi-hyper-atom. Clearly, the subgraph induced by a semi-hyper-atom is connected. The ideas of fragments and atoms were originated by Mader [11] Watkins [10], and are important tools in investigating various connectivities of graphs (see [5,7–9]).

Denote the automorphism group of $G$ by $\text{Aut}(G)$. A graph $G$ is called vertex transitive, if for any two vertices $u, v \in V(G)$ there exists some automorphism $\phi \in \text{Aut}(G)$ such that $\phi(u) = v$. A vertex transitive graph is always regular. A graph is called edge transitive, if for any two edges $e_1, e_2 \in E(G)$, there exists some automorphism $\phi \in \text{Aut}(G)$ such that $\phi(e_1) = e_2$.

Notation and definitions not given here can be found in [1,4].

The paper is organized as follows. In Section 2, a Key Lemma is proved, and some preliminary results are given. In Section 3 we characterize graphs which are both edge transitive and vertex transitive. In Section 4 we characterize graphs which are edge transitive but not vertex transitive.

2. Preliminary results

The following lemma is crucial for carrying out certain proof in later sections.

**Lemma 2.1.** Let $G$ be a connected but not semi-hyper-connected graph, let $M$ be a semi-hyper-atom of $G$, and let $N$ be a semi-hyper-fragment of $G$. Then either $M \cap N = \emptyset$ or $M \subseteq N$.

**Proof.** Suppose $M \cap N \neq \emptyset$, we are going to prove that $M \subseteq N$.

Without loss of generality we may assume that $G[N]$ is connected. In fact, when $M \cap N \neq \emptyset, M$ must have nonempty intersection with the vertex set of some component of $G[N]$, say $G[K]$. Replacing $N$ by $K$ in the following deduction, it can be seen that $M \subseteq K$, and thus $M \subseteq N$.

Since $\delta(N)$ is a semi-hyper-cut, the connectedness of $G[N]$ implies that $G[\delta(N)]$ has at least two components. Denote by $R_1 = M \cap \delta(N)$, $R_2 = \delta(M) \cap N$, $R_3 = \delta(M) \cap \delta(N)$, $R_4 = \delta(M) \cap \delta(N)$, and $R_5 = \delta(M) \cap \delta(N)$.

\[
\begin{array}{c|ccc}
    N & \delta(N) & \emptyset(N) & \emptyset(M) \\
    \hline
    M & R_1 & & \\
    \delta(M) & R_2 & R_3 & R_4 \\
    \emptyset(M) & & R_5 & \\
\end{array}
\]

**Claim 1.** $|R_1| = |R_4|$ and $|R_2| = |R_5|$.

Suppose $|R_1| < |R_4|$. Since $\delta(M \cap N) \subseteq R_1 \cup R_2 \cup R_3$, we have

$$|\delta(M \cap N)| \leq |R_1| + |R_2| + |R_3| < |R_2| + |R_3| + |R_4| = |\delta(M)| = \kappa(G).$$

So $\delta(M \cap N)$ is a cut of $G$ with cardinality less than $\kappa(G)$, which is impossible.

Suppose $|R_1| > |R_4|$. Then it follows from

$$|R_1| + |R_3| + |R_5| = |\delta(N)| = \kappa(G) = |\delta(M)| = |R_2| + |R_3| + |R_4|$$

that $|R_3| < |R_2|$. If furthermore $\delta(M) \cap \delta(N) \neq \emptyset$, then similar to the above, it can be deduced that $\delta(\delta(M) \cap \delta(N))$ is a cut of $G$ with cardinality less than $\kappa(G)$, a contradiction. So $\delta(M) \cap \delta(N) = \emptyset$. On the other hand, since $\delta(N)$
is a semi-hyper-fragment, we have \(|\delta(N)| \geq |M|\). Combining this with

\[ |\delta(N) \setminus \delta(M)| = |\delta(N) \cap M| + |R_4| < |\delta(N) \cap M| + |R_1| = |M \setminus N| < |M|, \]

we have \(|\delta(N) \cap \delta(M)| = |\delta(N)| - |\delta(N) \setminus \delta(M)| > 0\), which implies \(\delta(N) \cap \delta(M) \neq \emptyset\), again a contradiction.

So \(|R_1| = |R_4|\). By equality (1), we have \(|R_2| = |R_5|\).

**Claim 2.** \(M \cap \delta(N) = \emptyset\).

Suppose by contradiction that \(M \cap \delta(N) \neq \emptyset\).

First, we show that \(|R_1| = |R_2|\). In fact, if \(|R_1| < |R_2|\), then similar to the proof of Claim 1, it can be deduced that \(\partial(M \cap \delta(N))\) is a cut of \(G\) with cardinality less than \(\kappa(G)\). Whereas when \(|R_1| > |R_2|\), we have \(|R_5| < |R_4|\) by equality (1), and thus \(\delta(M) \cap N = \emptyset\) since otherwise \(\partial(M \cap N)\) is a cut of \(G\) with fewer than \(\kappa(G)\) vertices. On the other hand,

\[ |N \setminus \delta(N)| = |N \cap M| + |R_2| < |N \cap M| + |R_1| = |M \setminus \delta(N)| < |M|. \]

Combining this with the fact \(|N| \geq |M|\), we have \(|\delta(M) \cap N| > 0\), which implies \(\delta(M) \cap N \neq \emptyset\), a contradiction.

Now by Claim 1, we have \(|R_1| = |R_2| = |R_4| = |R_5|\). By

\[ \kappa(G) \leq |\partial(M \cap \delta(N))| \leq |R_1| + |R_3| + |R_4| = |R_2| + |R_3| + |R_4| = \kappa(G), \]

we see that \(\partial(M \cap \delta(N))\) is a minimum cut of \(G\) with

\[ \partial(M \cap \delta(N)) = R_1 \cup R_3 \cup R_4. \] (2)

Note that \(M \cap \delta(N)\) is a proper subset of \(M\). By the minimality of semi-hyper-atom, \(M \cap \delta(N)\) cannot be a semi-hyper-fragment. So \(G - \partial(M \cap \delta(N))\) has exactly two components. In particular, \(G[M \cap \delta(N)]\) is connected. From (2), \(R_4 \subseteq \partial(M \cap \delta(N))\). Combining this with the connectedness of \(G[M \cap \delta(N)]\), we see that \(G[\delta(N) \setminus \delta(M)] = G[M \cap \delta(N)] \cup R_4\) is connected. Since \(G[\delta(N)]\) has at least two components, there is a component \(H\) of \(G[\delta(N)]\) which is completely contained in \(G[\delta(N) \cap \delta(M)]\). Hence, \(\partial(V(H)) \subseteq R_3 \cup R_5\). If \(R_4 \neq \emptyset\), then

\[ |\partial(V(H))| \leq |R_3| + |R_5| < |R_3| + |R_5| + |R_4| = |R_3| + |R_5| + |R_1| = |\delta(N)| = \kappa(G). \]

This contradiction shows that \(R_4 = \emptyset\), and thus \(R_1 = \emptyset\) by Claim 1. But then \(G[M]\) is not connected, again a contradiction.

**Claim 3.** \(R_4 = \emptyset\).

Suppose by contradiction that \(R_4 \neq \emptyset\).

We first show that \(\delta(M) \cap \delta(N) = \emptyset\). Suppose this is not true. Denote by \(H_1, H_2, \ldots, H_t\) the components of the subgraph of \(G\) induced by \(\delta(M) \cap \delta(N)\). If every vertex \(v \in R_4\) has a neighbor in every \(H_i\) (1 \(\leq i \leq t\), then \(G[R_4 \cup (\delta(M) \cap \delta(N))]\), which is \(G[\delta(N)]\) by Claim 2, is connected. This contradicts our assumption that \(G[\delta(N)]\) has at least two components. So there exists a vertex \(v \in R_4\) and a component \(H_{i_0}\) of \(G[\delta(M) \cap \delta(N)]\) such that \(v\) has no neighbor in \(H_{i_0}\). Then \(\partial(V(H_{i_0}))\) is a proper subset of \(R_3 \cup R_4 \cup R_5\), and thus it follows from Claim 1 that

\[ |\partial(V(H_1))| < |R_3| + |R_4| + |R_5| = |R_3| + |R_1| + |R_5| = |\delta(N)| = \kappa(G), \]

a contradiction.

Combining \(\delta(M) \cap \delta(N) = \emptyset\) with Claim 2, we see that \(\delta(N) = R_4\). Then a contradiction arises since

\[ |M| \leq |\delta(N)| = |R_4| = |R_1| < |M|. \]

Combining Claim 3 with Claim 1, we have \(R_1 = \emptyset\). This, together with Claim 2, implies \(M \cap N = M\). Hence \(M \subseteq N\). □

In the following, we need the concept of imprimitive blocks \([9]\). An *imprimitive block* of a graph \(G\) is a proper nonempty subset \(A\) of \(V(G)\) such that for any \(\phi \in Aut(G)\), either \(\phi(A) = A\) or \(\phi(A) \cap A = \emptyset\). The following proposition is known:
Theorem A (Tindell [9]). Let $G$ be a connected edge transitive graph and let $Y$ be the subgraph of $G$ induced by an imprimitive block $A$. Then $A$ is an independent subset of $G$.

3. Edge and vertex transitive graphs

Graphs considered in this section are both edge transitive and vertex transitive.

**Lemma 3.1.** Let $G$ be a connected edge transitive graph which is not semi-hyper-connected. Then every semi-hyper-atom of $G$ is a singleton.

**Proof.** Let $M$ be a semi-hyper-atom of $G$. For any automorphism $\phi \in \text{Aut}(G)$, $\phi(M)$ is also a semi-hyper-atom of $G$. If $\phi(M) \cap M \neq \emptyset$, then it follows from Lemma 2.1 that $\phi(M) = M$. So $M$ is an imprimitive block, and thus an independent set by Theorem A. Combining this with the fact that $G[M]$ is connected, we see that $M$ is a singleton. □

**Lemma 3.2.** Let $G$ be a connected edge and vertex transitive graph which is not semi-hyper-connected, and $u$ a vertex of $G$. Denote the components of $G - \bar{\partial}(u)$ by $G_1, \ldots, G_t$, where $G_1 = G[u]$ and $G_t$ is the largest one. Then $t \geq 3$, and all the components $G_2, \ldots, G_{t-1}$ are singletons.

**Proof.** Let $M$ be a semi-hyper-atom of $G$. By Lemma 3.1, $M$ is a singleton. By vertex transitivity of $G$, every vertex is a semi-hyper-atom; in particular, $u$ is a semi-hyper-atom. So $G - \bar{\partial}(u)$ has at least three components, and thus $t \geq 3$.

If not all the components $G_2, \ldots, G_{t-1}$ are singletons, suppose, for example, that $G_2$ has order at least 2. Choose $v \in V(G_2)$ such that $\bar{\partial}(v) \cap \bar{\partial}(u) \neq \emptyset$. Since $G_2$ is connected, we see that $\bar{\partial}(v) \subsetneq \bar{\partial}(u)$, and thus it follows from $|\bar{\partial}(u)| = |\bar{\partial}(v)|$ that $\bar{\partial}(u) \setminus \bar{\partial}(v) \neq \emptyset$. Let $w \in \bar{\partial}(u) \setminus \bar{\partial}(v)$. Since $\bar{\partial}(u)$ is a minimum cut, $w$ is adjacent to every component of $G_i$ ($1 \leq i \leq t$). So, the subgraph of $G$ induced by $\left( \bigcup_{i \neq 2} V(G_i) \right) \cup \{w\}$ is connected. But then, there is a component of $G - \bar{\partial}(v)$ larger than $G_t$, contradicting that $G - \bar{\partial}(u) \cong G - \bar{\partial}(v)$. □

A graph $G$ is said to be reducible if there exist vertices $u, v \in V(G)$ with $\bar{\partial}(u) = \bar{\partial}(v)$. Otherwise $G$ is said to be irreducible.

**Theorem 3.1.** Let $G$ be a connected edge and vertex transitive $k$-regular graph which is neither the complete graph $K_{k+1}$ nor the complete multipartite graph $K_{2,2,\ldots,2}$. Then $G$ is semi-hyper-connected if and only if $G$ is irreducible.

**Proof.** Suppose $G$ is not semi-hyper-connected. Let $u$ be a vertex of $G$. By Lemma 3.2, there is a component of $G - \bar{\partial}(u)$ other than $G[u]$, which is a singleton. Denote by $v$ the vertex in this component. Then $\bar{\partial}(v) = \bar{\partial}(u)$, and thus $G$ is reducible.

Conversely, suppose $G$ is reducible. Then there exist $u, v \in V(G)$ with $\bar{\partial}(u) = \bar{\partial}(v)$. By the edge transitivity of $G$, we have $\kappa(G) = k$ (see [10] for references). So $\bar{\partial}(u)$ is a minimum cut. Clearly, $G[u]$ and $G[v]$ are two distinct components of $G - \bar{\partial}(u)$. Since $G \not\cong K_{k+1}$ and $G \not\cong K_{2,2,\ldots,2}$, we have $|V(G)| > k + 2 = |\{u, v\} \cup \bar{\partial}(u)|$. So there are at least three components in $G - \bar{\partial}(u)$, and thus $G$ is not semi-hyper-connected. □

As to the two exceptions in the above theorem, $K_{k+1}$ is irreducible but not semi-hyper-connected, $K_{2,2,\ldots,2}$ is semi-hyper-connected but reducible.

Define an equivalence relation $R$ on $V(G)$ by letting $uRv$ if and only if $\bar{\partial}(u) = \bar{\partial}(v)$.

According to this equivalence, $V(G)$ is partitioned into some nonempty sets, say $A_1, \ldots, A_p$. For any vertex $u \in A_i$, $A_i$ is the set of vertices having the same neighbor with $u$ in $G$. Clearly, each $A_i$ is an imprimitive block of $G$.

Define a quotient graph $\bar{G} \equiv G/R$ of $G$: The vertices of $\bar{G}$ are $A_i$ ($i = 1, 2, \ldots, p$), and $A_iA_j$ is an edge in $\bar{G}$ if and only if some vertex in $A_i$ is adjacent to some vertex in $A_j$ in $G$. By the imprimitivity of $A_i$'s, we have

**Lemma 3.3.** Let $G$ be a connected edge and vertex transitive graph. Then $\bar{G}$ is also a connected edge and vertex transitive graph.
Proof. By Lemma 3.4, Theorem 4.2.

Theorem 4.2. Let \( G \) be a connected edge and vertex transitive graph. Then the quotient graph \( \bar{G} \) of \( G \) is either complete or semi-hyper-connected.

Remark 3.1. From Lemma 3.2, Theorems 3.1, and 3.2, it can be seen that any connected edge and vertex transitive graph \( G \) which is not semi-hyper-connected can be constructed as follows: let \( \bar{G} \) be a connected edge and vertex transitive graph which is irreducible, duplicate every vertex the same number of times, and let the duplicated vertices have the same neighborhood as the original one.

4. Edge transitive but not vertex transitive graphs

In this section, graphs are assumed to be edge transitive but not vertex transitive. The following theorem is well known

Theorem B (Biggs [1]). If a connected graph \( G \) is edge transitive but not vertex transitive, then \( G \) is bipartite. Furthermore, vertices in a same part are transitive under \( \text{Aut}(G) \).

So, we may suppose that \( G \) is a bipartite graph with bipartition \((X, Y)\), and vertices in a same part have the same degree.

A bipartite graph \( G \) is said to be bi-reducible, if there exist vertices \( u, v \in V(G) \) with minimum degree \( \delta(G) \) such that \( \hat{\delta}(u) = \hat{\delta}(v) \). Otherwise \( G \) is said to be bi-irreducible.

Theorem 4.1. Let \( G = (X, Y) \) be a connected edge transitive but not vertex transitive graph. Then \( G \) is semi-hyper-connected if and only if \( G \) is bi-irreducible.

Proof. Suppose \( G \) is not semi-hyper-connected. Let \( M \) be a semi-hyper-atom of \( G \). By Lemma 3.1, \( M \) contains only one vertex \( u \). Since \( \hat{\delta}(u) \) is a minimum cut of \( G \), we have \( d(u) = \delta(G) \). Suppose, without loss of generality that \( u \in X \). Then every vertex in \( X \) has degree \( \delta(G) \) and is a semi-hyper-atom. Write \( G_1, \ldots, G_t \) (\( t \geq 3 \)) the components of \( G - \hat{\delta}(u) \), where \( G_1 = G[u] \) and \( G_t \) is the largest one. Similar to the proof of Lemma 3.2, we can show that all the components \( G_2, \ldots, G_{t-1} \) are singletons. In fact, if this is not true, suppose, without loss of generality that \( |V(G_2)| \geq 2 \). Choose a vertex \( v \in V(G_2) \) with \( \hat{\delta}(v) \cap \hat{\delta}(u) \neq \emptyset \). Then \( v \in X \), since \( G \) is bipartite. So, \( v \) is also a semi-hyper-atom of \( G \), and \( G - \hat{\delta}(u) \cong G - \hat{\delta}(v) \). By a similar argument as in the proof of Lemma 3.2, there is a larger component than \( G_t \) in \( G - \hat{\delta}(v) \), a contradiction. Write \( v \) the only vertex in \( G_2 \). Then \( \hat{\delta}(v) = \hat{\delta}(u) \), and thus \( G \) is bi-reducible.

Note that if \( |V(G)| = \delta(G) + 1 \), then \( G \) is a complete graph; if \( |V(G)| = \delta(G) + 2 \), then \( G \) is a cycle of length 4 since \( G \) is bipartite. In either case, \( G \) is also vertex transitive. So, under the assumption of this theorem, we have \( |V(G)| > \delta(G) + 2 \). Then the necessity follows by a similar argument as in the proof of Theorem 3.1. \( \square \)

Lemma 4.1. Let \( G \) be a connected edge transitive graph (which is not necessarily vertex transitive), then the quotient graph \( \bar{G} \) is also connected and edge transitive. If \( G \) is a bipartite graph, then \( \bar{G} \) is also a bipartite graph.

Theorem 4.2. Let \( G \) be a connected edge transitive but not vertex transitive graph, then \( \bar{G} \) is semi-hyper-connected.

Proof. By Lemma 3.4, \( \bar{G} \) is irreducible, and thus is also bi-irreducible. By Lemma 4.1, \( \bar{G} \) is connected and edge transitive. If it is also vertex transitive, then by Theorem 3.2, \( \bar{G} \) is either complete or semi-hyper-connected. But the first case cannot occur since \( \bar{G} \) is bipartite. If \( \bar{G} \) is not vertex transitive, then the result follows from Theorem 4.1. \( \square \)

Remark 4.1. From Lemma 4.1, Theorems 4.1, and 4.2, it can be seen that every connected edge transitive but not vertex transitive graph which is not semi-hyper-connected can be constructed in the following way: let \( \bar{G} = (\bar{X}, \bar{Y}) \) be a connected graph which is not vertex transitive, and let \( \bar{G} \) be a connected edge and vertex transitive graph. Then the quotient graph \( \bar{G} \) of \( G \) is either complete or semi-hyper-connected.

connected edge transitive bipartite graph which is semi-hyper-connected. Duplicate each of the vertices in a same part the same number of times, and let the duplicated vertices have the same neighborhood as the original one. If, in this process, only one part, say \( \bar{X} \), has its vertices duplicated, then the times duplicated must be such that in the resulting graph the vertices in \( \bar{X} \) have degree not more than those in \( \bar{Y} \).

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References