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Existence of Double Vortex Solutions

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Abstract

We show analytically and numerically the existence of double vortex solutions in two-Higgs systems. These solutions are generalizations of the Nielsen-Olesen vortices and exist for all values of the parameters in the Lagrangians considered. We derive analytically the asymptotic behavior of the solutions and confirm it numerically by solving the field equations. Finally, we show that these solutions can be embedded in realistic theories like the two-doublet extension of the standard model.

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It has recently been shown[1, 2] that the Nielsen-Olesen vortex solution[3, 4] can be embedded in the standard electroweak model. It was also shown[5] that the resulting electroweak vortices are dynamically (but not topologically) stable for a finite range of parameters of the electroweak model Lagrangian. This range of parameters however is not included within the limits allowed by current experiments, assuming the simplest form of the Higgs sector (one Higgs doublet).

There are recent theoretical and experimental indications[6] that the Higgs sector of the standard model may need to be extended to involve at least two Higgs multiplets for the electroweak symmetry breaking. This is required, for example, in order to achieve consistent unification of the gauge couplings on high energy scales[6]. The prospect of this possibility raises the question of what (if any) soliton-like objects exist in these extended, multiple Higgs theories.

Clearly the Nielsen-Olesen vortex which involves a single Higgs can not be directly embedded in multiple Higgs systems unless it is generalized first. Recent studies[7] have shown the existence of an extra global symmetry in realistic two-Higgs doublet models, which breaks during the electroweak symmetry breaking. The breaking of this extra symmetry may lead to *topologically* stable embedded *double vortices* (the term *'double'* here refers to the number of Higgs fields that wind non-trivially and not to the topological charge or the flux of the vortices). It is therefore important to study how can the Nielsen-Olesen vortex be generalized to multiple Higgs systems, what are the properties of the new solutions and how can they be embedded in realistic two doublet models. These issues consist the focus of the present work.

Previous studies [8] attempting to generalize the Nielsen-Olesen solution have claimed that such generalized solutions exist only for certain values of the parameters of the two-Higgs Lagrangian. These studies, which did not make any numerical confirmation of their results, used a very constraining ansatz for the asymptotic behavior of their candidate solutions which led to incorrect conclusions about the existence of solutions.

Here, we avoid the use of *any* ansatz and show that vortex solutions exist for any value of parameters in the Lagrangian. We derive the asymptotic behavior of these solutions and point out new features that are not present in the Nielsen-Olesen vortices. We also obtain the solutions numerically and confirm the derived asymptotic behavior. Finally, we consider the realistic two doublet electroweak model and show that the obtained double vortex solution can be embedded in this model.

Consider the Abelian-two-Higgs Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}|D_\mu\Phi_1|^2 + \frac{1}{2}|D_\mu\Phi_2|^2 - V_0(\Phi_1,\Phi_2) \tag{1}$$

where Φ_1, Φ_2 are complex singlets (the Higgs fields), $D_\mu = \partial_\mu - ieA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ and

$$V_0(\Phi_1,\Phi_2) = \frac{\lambda_1}{4}(|\Phi_1|^2 - v_1^2)^2 + \frac{\lambda_2}{4}(|\Phi_2|^2 - v_2^2)^2 + \frac{\lambda_3}{4}(|\Phi_1|^2 + |\Phi_2|^2 - v_1^2 - v_2^2)^2 \tag{2}$$

Consider now the ansatz:

$$\begin{aligned} \Phi_1 &= v_1 f_1(r) e^{i\theta} && (3)\\ \Phi_2 &= v_2 f_2(r) e^{i\theta} && (4)\\ A_\mu &= \hat{e}_\theta \frac{v(r)}{er} && (5) \end{aligned}$$

Using the ansatz (3)-(5) and a rescaling of the radial coordinate $r \to \frac{r}{\sqrt{v_1 v_2 e}}$ the field equations of motion obtained from (1) become

$$\begin{aligned} f_1''(r) + \frac{1}{r}f_1'(r) - \frac{(1-v(r))^2}{r^2}f_1(r) - q_1(f_1(r)^2-1)f_1(r) \\ -q_3(f_2(r)^2-1)f_1(r) = 0 && (6)\\ f_2''(r) + \frac{1}{r}f_2'(r) - \frac{(1-v(r))^2}{r^2}f_2(r) - q_2(f_2(r)^2-1)f_2(r) \\ -q_3(f_1(r)^2-1)f_2(r) = 0 && (7)\\ v''(r) - \frac{1}{r}v'(r) + q_5(1-v(r))f_1(r)^2 + q_6(1-v(r))f_2(r)^2 = 0 && (8) \end{aligned}$$

where

$$\begin{aligned} q_1 &= \frac{\lambda_1+\lambda_3}{e^2}(\frac{v_1}{v_2}) & q_2 &= \frac{\lambda_2+\lambda_3}{e^2}(\frac{v_2}{v_1})\\ q_3 &= \frac{\lambda_3}{e^2}(\frac{v_2}{v_1}) & q_4 &= \frac{\lambda_3}{e^2}(\frac{v_1}{v_2})\\ q_5 &= (\frac{v_1}{v_2}) & q_4 &= (\frac{v_2}{v_1}) \end{aligned}$$

Single valuedness for the fields Φ_1, Φ_2 imply the following boundary conditions for the system (6)-(8)

$$f_1 \to 0 \quad f_2 \to 0 \quad v \to 0 \quad \text{for} \quad r \to 0 \tag{9}$$

and

$$f_1 \to 1 \quad f_2 \to 1 \quad v \to 1 \quad \text{for} \quad r \to \infty \tag{10}$$

It is easy to see that for $r \ll 1$, $f_1(r) \sim f_2(r) \sim r$ and $v(r) \sim r^2$.

The asymptotic behavior of the fields may also be obtained in a straightforward way. Define δf_1, δf_2 and δv by

$$f_i \to 1 + \delta f_i \quad v \to 1 + \delta v \quad \text{for} \quad r \gg 1 \tag{11}$$

where $i = 1, 2$. By keeping only lowest order terms in δf_1, δf_2, the system (6)-(8) becomes:

$$\delta f_1''(r) + \frac{1}{r}\delta f_1'(r) - \frac{\delta v(r)^2}{r^2} - 2q_1\delta f_1(r) - 2q_3\delta f_2(r) = 0 \tag{12}$$

$$\delta f_2''(r) + \frac{1}{r}\delta f_2'(r) - \frac{\delta v(r)^2}{r^2} - 2q_2\delta f_2(r) - 2q_4\delta f_1(r) = 0 \tag{13}$$

$$\delta v''(r) - \frac{1}{r}\delta v'(r) + (q_5 + q_6)\delta v(r) = 0 \tag{14}$$

Equation (16) shows that δv decays exponentially. This implies that the system of (12) and (13) becomes mathematically similar to a system of coupled harmonic oscillators driven by the same 'force'. This is a simple problem but we will go through it in some detail in order to show that there is always a solution and thus resolve the controversy with Ref. [8]. Notice that the term $\frac{\delta v(r)^2}{r^2}$ has been kept in (12) and (13). Neglecting this term, as was done in the original paper of Nielsen-Olesen , leads to an asymptotic behavior for the fields, which is incorrect for large values of the self coupling when this term dominates. For a proper treatment of this term in the Nielsen-Olesen case see Ref. [9].

Clearly the solution to (14) may be approximated for $r \gg 1$ by

$$\delta v = r^{\tau} e^{-\sigma r}(c_1^v + c_2^v r^{-1}) \tag{15}$$

Using this ansatz in (14) leads to

$$\tau = \frac{1}{2} \quad \text{and} \quad \sigma = \sqrt{q_5 + q_6} \tag{16}$$

This gives the behavior of the 'driving force' in the system (12), (13). The homogeneous solution of this system may now be obtained by using an ansatz of the form

$$\delta f_i = c_i^f r^\alpha e^{-\beta r} \tag{17}$$

which leads to

$$\begin{aligned} c_1^f(\beta^2 - 2q_1) - c_2^f 2q_3 &= 0 \qquad (18)\\ -c_1^f 2q_4 + c_2^f(\beta^2 - 2q_2) &= 0 \qquad (19) \end{aligned}$$

Demanding that this system has a solution implies that the secular determinant vanishes which in turn leads to

$$\beta_\pm = [(q_1 + q_2) \pm \sqrt{(q_1 - q_2)^2 + 4q_3 q_4}\ \ ]^{\frac{1}{2}} \tag{20}$$

The corresponding ratio $\frac{c_2^f}{c_1^f}$ is

$$(\frac{c_2^f}{c_1^f})_\pm = \frac{(q_2 - q_1) \pm \sqrt{(q_2 - q_1)^2 + 4q_3 q_4}}{2q_3} \equiv \gamma_\pm \tag{21}$$

By definition $q_3 q_4 \le q_1 q_2$ (for $\lambda_1, \lambda_2, \lambda_3 \ge 0$) which implies $\beta_\pm \in \Re$ and $\gamma_- < 0 < \gamma_+$. Therefore, the homogeneous solution to the system (12),(13) is

$$\begin{aligned} \delta f_1^h &= (c_{1+}^f e^{-\beta_+ r} + c_{1-}^f e^{-\beta_- r}) r^\alpha \qquad (22)\\ \delta f_2^h &= (c_{1+}^f \gamma_+ e^{-\beta_+ r} + c_{1-}^f \gamma_- e^{-\beta_- r}) r^\alpha \qquad (23) \end{aligned}$$

where α is a constant that can be determined numerically. The particular solution of the system is of the form

$$\delta f_i^p = d_i^f \frac{e^{-2\sigma r}}{r} \tag{24}$$

where $i = 1, 2$. The ansatz (24) leads for $r \gg 1$ to the inhomogeneous system

$$\begin{aligned} d_1^f(4\sigma^2 - 2q_1) - d_2^f 2q_3 &= (c_1^v)^2 \qquad (25)\\ -d_1^f 2q_4 + d_2^f(4\sigma^2 - 2q_2) &= (c_1^v)^2 \qquad (26) \end{aligned}$$

where c_1^v is defined in (15). The constants d_1^f, d_2^f can be obtained by solving the system (25), (26) which always has a solution. Therefore, the form of δf_i, δv is:

$$\delta f_1 = (c_{1+}^f e^{-\beta_+ r} + c_{1-}^f e^{-\beta_- r}) r^\alpha + d_1^f \frac{e^{-2\sigma r}}{r} \tag{27}$$

$$\delta f_2 = (c_{1+}^f \gamma_+ e^{-\beta_+ r} + c_{1-}^f \gamma_- e^{-\beta_- r}) r^\alpha + d_2^f \frac{e^{-2\sigma r}}{r} \tag{28}$$

Clearly, the term with the lowest value of the exponent will dominate. We may therefore distinguish four cases

1. $\lambda_1 \neq \lambda_2$ **(or** $v_1 \neq v_2$**) and** $\beta_- < 2\sigma$**.** In this case the term proportional to $e^{-\beta_- r}$ dominates and we have

$$\delta f_1 = c_{1-}^f e^{-\beta_- r} r^\alpha \tag{29}$$

$$\delta f_2 = c_{1-}^f \gamma_- e^{-\beta_- r} r^\alpha \tag{30}$$

$$\delta v = c_1^v e^{-\sigma r} r^{\frac{1}{2}} \tag{31}$$

Notice that since $\gamma_- < 0$, δf_2 is negative in this case and therefore $f_2(r)$ is approaching its asymptotic value (=1) from *above*. This is a new feature which does not appear in the Nielsen-Olesen solution for *any* value of the parameters. We have checked the validity of this point numerically (see below).

2. $\lambda_1 \neq \lambda_2$ **(or** $v_1 \neq v_2$**) and** $\beta_- > 2\sigma$**.** In this case the term proportional to $e^{-2\sigma r}$ dominates and

$$\delta f_i = d_i^f \frac{e^{-2\sigma r}}{r} \tag{32}$$

$$\delta v = c_1^v e^{-\sigma r} r^{\frac{1}{2}} \tag{33}$$

where $i = 1, 2$.

3. $\lambda_1 = \lambda_2$**,** $v_1 = v_2$ **and** $\beta_+ < 2\sigma$**.** This implies by symmetry that $\delta f_1 = \delta f_2$ and therefore, since $\gamma_- < 0$, we must have $c_{1-}^f = 0$. Therefore

$$\delta f_i = c_{1+}^f e^{-\beta_+ r} \tag{34}$$

$$\delta v = c_1^v e^{-\sqrt{2} r} r^{\frac{1}{2}} \tag{35}$$

since $\gamma_+ = 1$, $\sigma = \sqrt{2}$ for $\lambda_1 = \lambda_2$ and $v_1 = v_2$.

4. $\lambda_1 = \lambda_2$**,** $v_1 = v_2$ **and** $\beta_+ > 2\sigma$**.** The dominant term now is proportional to $e^{-2\sigma r}$ and $\delta f_1 = \delta f_2$ by symmetry. Thus

$$\delta f_i = d_1^f \frac{e^{-2\sqrt{2}r}}{r} \tag{36}$$

$$\delta v = c_1^v e^{-\sqrt{2}r} r^{\frac{1}{2}} \tag{37}$$

We have verified the above cases by plotting $\frac{ln(|\delta f_i|)}{r}$ and $\frac{ln(|\delta v|)}{r}$ vs r for several different values of parameters. In Figures 1 and 2 we show those plots for parameter values $\lambda_1 = \lambda_2 = \lambda_3 = 1$ (case 3) and $\lambda_1 = \lambda_3 = 1$, $\lambda_2 = 2$ (case 1) respectively. The asymptotic values for all fields are consistent with the corresponding predictions of exponents. Indeed, for Figure 1 the relevant exponent for δf_1, δf_2 is $\beta_+ = 2.45$ (case 3) while for Figure 2 the corresponding exponent is $\beta_- = 1.7$ (case 1). In both cases we had $v_1 = v_2$ which implies that the relevant exponent for δv is $\sigma = \sqrt{2}$. Notice also the singularity of $\frac{ln(|\delta f_2|)}{r}$ in Figure 2, shown as a sharp minimum due to the discrete nature of the numerically constructed plot. This is due to the fact that $f_2(r)$ crosses the $f_2(r) = 1$ line (leading to $\delta f_2 = 0$) in order to approach its asymptotic value $f_2 = 1$ from above as predicted by (30) (since $\gamma_- < 0$).

It is instructive to study how does the above analysis get modified in the case of more general two-Higgs potentials appearing in realistic cases. Consider for example the potential:

$$V(\Phi_1, \Phi_2) = V_0(\Phi_1, \Phi_2) + \lambda_4(|\Phi_1|^2|\Phi_2|^2 - |\Phi_1^*\Phi_2|) + \lambda_5|\Phi_1^*\Phi_2 - v_1 v_2|^2 \tag{38}$$

where $V_0(\Phi_1, \Phi_2)$ is defined in (2). Using the same ansatz as in (3)-(5) we obtain the field equations for $f_1(r)$, $f_2(r)$ and $v(r)$ as:

$$f_1''(r) + \frac{1}{r}f_1'(r) - \frac{(1-v(r))^2}{r^2}f_1(r) - q_1(f_1(r)^2 - 1)f_1(r) - q_3(f_2(r)^2 - 1)f_1(r) - p(f_1(r)f_2(r) - 1)f_2(r) = 0 \tag{39}$$

$$f_2''(r) + \frac{1}{r}f_2'(r) - \frac{(1-v(r))^2}{r^2}f_2(r) - q_2(f_2(r)^2 - 1)f_2(r) - q_3(f_1(r)^2 - 1)f_2(r) - p'(f_1(r)f_2(r) - 1)f_1(r) = 0 \tag{40}$$

$$v''(r) - \frac{1}{r}v'(r) + q_5(1 - v(r))f_1(r)^2 + q_6(1 - v(r))f_2(r)^2 = 0 \tag{41}$$

where

$$p = \frac{\lambda_5}{e^2}(\frac{v_2}{v_1}) \quad \text{and} \quad p' = \frac{\lambda_5}{e^2}(\frac{v_1}{v_2}) \tag{42}$$

and q_i as defined previously. It is easy to show that the equations for δf_i, δv may be obtained from (12)-(14) by substituting

$$2q_1 \rightarrow 2q_1 + p \quad 2q_2 \rightarrow 2q_2 + p' \tag{43}$$
$$2q_3 \rightarrow 2q_3 + p \quad 2q_4 \rightarrow 2q_4 + p' \tag{44}$$

and the analysis proceeds in exactly the same way but with different parameters.

We now show that the obtained solutions can be embedded in realistic theories. The main steps are similar to those in Ref. [1] which showed that the Nielsen-Olesen vortex is a solution of the equations of motion obtained from the standard electroweak Lagrangian involving a single Higgs doublet. We therefore follow the notation of Ref. [1] with minor modifications.

Consider the bosonic sector of the two-Higgs doublet electroweak Lagrangian:

$$L = L_W + L_B + L_{\Phi_1,\Phi_2} - V(\Phi_1, \Phi_2) \tag{45}$$

with

$$L_W = -\frac{1}{4} G_{\mu\nu a} G^{\mu\nu a} \tag{46}$$
$$L_B = -\frac{1}{4} F_{B\mu\nu} F^{B\mu\nu} \tag{47}$$
$$L_{\Phi_1,\Phi_2} = \sum_{i=1}^{2} |D_\lambda \Phi_i|^2 = \sum_{i=1}^{2} |(\partial_\lambda - \frac{1}{2} ig\tau^a W_\lambda^a - \frac{1}{2} ig' B_\lambda)\Phi_i|^2 \tag{48}$$

where $G_{\mu\nu a}$, $F_{B\mu\nu}$ are the usual $SU(2)$ and $U(1)_Y$ gauge tensors and W_λ^a, B_λ are the corresponding gauge fields. Also $V(\Phi_1, \Phi_2)$ is given by (38) with Φ_1, Φ_2 being complex doublets.

The field equations obtained from (45) are

$$\partial_\nu G^{\mu\nu a} - g\varepsilon^{abc} G^{\mu\nu b} W_\nu^c = \sum_{i=1}^{2} \frac{i}{2} g(\Phi_i^\dagger \tau^a D^\mu \Phi_i - (D^\mu \Phi_i)^\dagger \tau^a \Phi_i) \tag{49}$$
$$\partial_\nu F^{\mu\nu} = \sum_{i=1}^{2} \frac{i}{2} g'(\Phi_i^\dagger D^\mu \Phi_i - (D^\mu \Phi_i)^\dagger \Phi_i) \tag{50}$$
$$D_\mu D^\mu \Phi_i = -\frac{\delta V(\Phi_1, \Phi_2)}{\delta \Phi_i^\dagger} \tag{51}$$

which are almost identical to the ones shown in Ref. [1] apart from the two-Higgs potential term. Define now e, θ_w, Z^μ and A^μ as usual *i.e.*

$$g \equiv e\cos\theta_w \quad g' \equiv e\sin\theta_w \tag{52}$$
$$Z^\mu \equiv \cos\theta_w W^{\mu 3} - \sin\theta_w B^\mu \tag{53}$$
$$A^\mu \equiv \sin\theta_w W^{\mu 3} + \cos\theta_w B^\mu \tag{54}$$

and consider the ansatz

$$A^\mu = W^{\mu 1} = W^{\mu 2} = 0 \tag{55}$$
$$Z^\mu = \hat{e}_\theta \frac{v(r)}{er} \tag{56}$$
$$\Phi_1 = v_1 e^{i\theta} f_1(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{57}$$
$$\Phi_2 = v_2 e^{i\theta} f_2(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{58}$$

It is straightforward to show that the ansatz (55)-(58) substituted in (49)-(51) leads to the equations (39)-(41) for $f_1(r)$, $f_2(r)$ and $v(r)$. Therefore, the double vortex solution is also a solution of the field equations in the two Higgs doublet electroweak model.

It has been argued[7] that a certain type of the embedded double vortex is topologically stable due to an extra global symmetry that amounts to the freedom of independent $U(1)$ phase transformations of the two doublets. The breaking of this symmetry during the electroweak transition leads to stability of the embedded double vortices. In the Lagrangians we have considered however, the presence of the λ_5 term does not allow such global transformations and it explicitly breaks the extra global symmetry. Even in the absence of this term the investigation of the stability of the embedded vortices needs special attention. In fact, instabilities may be present even in the topological case towards repulsion of the centers of the two vortices. This is similar to the Nielsen-Olesen n-vortex (n being the winding number) which even though topological is unstable towards decay to n 1-vortices for large self coupling. In the case of double vortices it would cost logarithmically infinite energy to place the two centers infinite distance apart. This is due to the fact that the gauge field can not effectively screen the angular gradient energy of the two centers when they are separated. This however leaves open the possibility of repulsion at small distances which would be favoured by the two-Higgs potential term. These issues of stability are currently under investigation.

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Figure Captions

Figure 1: The dependence of $\frac{ln(|x|)}{r}$ on r for $x = \delta f_1$, $x = \delta f_2$ and $x = \delta v$ with $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $v_1 = v_2$ (case 3). The predicted asymptotic values are $\beta_+ = 2.45$ for δf_1, δf_2 and $\sigma = \sqrt{2}$ for δv. Clearly, the numerically obtained asymptotic values are in good agreement with the predicted ones.

Figure 2: Same as Fig. 1 with $\lambda_1 = \lambda_3 = 1$, $\lambda_2 = 2$, $v_1 = v_2$ (case 1). The predicted asymptotic values are $\beta_- = 1.7$ for δf_1, δf_2 and $\sigma = \sqrt{2}$ for δv.

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