Upper bounds for harmonious colourings

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Abstract

A harmonious colouring of a simple graph $G$ is a colouring of the vertices such that adjacent vertices receive distinct colours and each pair of colours appears together on at most one edge. The harmonious chromatic number $h(G)$ is the least number of colours in such a colouring. We improve an upper bound on $h(G)$ due to Lee and Mitchem, and give upper bounds for related quantities.
1 Introduction

A proper colouring of a (simple) graph $G$ is a colouring of the vertices such that adjacent vertices receive distinct colours. A harmonious colouring of $G$ is a proper colouring such that each pair of colours appears together on at most one edge. The harmonious chromatic number $h(G)$ is the least number of colours in such a colouring.

For a recent survey on harmonious colourings see Wilson [8]. Clearly $h(G) \geq \Delta + 1$ where $\Delta$ is the maximum degree of a vertex. Also $h(G) \leq n$ where $n$ is the number of vertices; and $h(G) = n$ if and only if $G$ has diameter at most 2. Lee and Mitchem [5] show that

$$h(G) \leq (\Delta^2 + 1)[n^{\frac{1}{2}}].$$

We shall improve this upper bound on $h(G)$, and give some related results. Let us call a graph non-trivial if it has at least one edge.

**Theorem 1** For any non-trivial graph $G$,

$$h(G) \leq 2 \Delta (n - 1)^{\frac{1}{2}}.$$

The following example is of interest here.

*Example (see for example Bollobás [2], page 176)*

Let $q$ be a prime power and let $V$ be the set of points $(x, y, z)$ of the projective plane $PG(2, q)$. Form a graph $G = (V, E)$ by setting two points $(x, y, z)$ and $(x', y', z')$ adjacent if $xx' + yy' + zz' = 0$. Then $n = |V| = |E| = q^2 + q + 1$, $G$ is regular of degree $q+1$, and $G$ has diameter 2 so that $h(G) = n$. Thus $h(G) \simeq \Delta n^{\frac{1}{2}} \ll \Delta^2 n^{\frac{1}{2}}$ for $q$ large.
For any graph $G$ let $\hat{\delta}(G)$ denote the maximum over all induced subgraphs of the minimum vertex degree. Of course $\hat{\delta} \leq \Delta$. It is well known that the chromatic number $\chi(G)$ is at most $\hat{\delta} + 1$.

**Theorem 2** For any non-trivial graph $G$,

$$h(G) \leq \left\{ \begin{array}{ll} \frac{33}{16} \frac{\hat{\delta} \Delta (n - 1))^{1/2}}{ \hat{\delta} \Delta (n - 1))^{1/2}} & \text{if } \hat{\delta} \leq 8. \end{array} \right.$$

**Corollary 3** If $G$ is a non-trivial tree then

$$h(G) \leq 2(\Delta n)^{1/2}.$$ 

How good is the bound in this last result? For trees $G$ do we have $h(G) = 0(\Delta + n^{1/2})$? For example, let $r, d \geq 2$ with $d$ even and let $G$ be the complete $r$-ary tree of depth $d$, so that $n = (r^{d+1} - 1)/(r - 1)$ and $\Delta = r + 1$. It may be shown that

$$h(G) \leq 2 \left( \frac{r}{r - 1} \right)^{1/2} n^{1/2}.$$ 

(For the case $r = 2$ a tighter bound is given in [6].)

In the next section we shall prove both theorems 1 and 2 from one lemma, by considering how to colour the vertices of $G$ sequentially. After that we consider a generalisation of harmonious colourings when we allow a given pair of colours to appear together on at most $k$ edges, and develop corresponding upper bounds. Now the best proof method is non-constructive and relies on the Erdős-Lovász local lemma.
2 Proof of theorems 1 and 2

Lemma 2.1 Let $G$ be a graph, let $\delta \leq \alpha \leq \Delta$ and let $4\alpha \Delta \leq n - 1$. Then

$$h(G) \leq 2(\alpha \Delta (n - 1))^{\frac{1}{2}} - \frac{7}{8} \alpha \Delta + (\Delta - \alpha) (\delta - 1) + 1.$$ 

Before we prove this lemma let us see how it yields theorems 1 and 2.

Note first that the case $\Delta = 1$ (and so also $\delta = 1$) is trivial. For then $h(G)$ is the least integer $k$ such that $\binom{k}{2}$ is at least the number of edges. So

$$h(G) \leq 1 + n^{\frac{1}{2}} \leq 2(n - 1)^{\frac{1}{2}}$$

for $n \geq 3$.

So it suffices to assume that $\Delta \geq 2$.

Next suppose that $4\alpha \Delta \geq n - 1$ (where $\alpha = \Delta$ or $\delta$). Then $2(\alpha \Delta (n - 1))^{\frac{1}{2}} \geq n - 1$ and so we need worry only if $h(G) = n$. So suppose that $h(G) = n$ and thus $G$ has diameter at most 2. Then

$$1 + \delta + \delta(\Delta - 1) \geq n$$

and so $\alpha \Delta \geq \delta \Delta \geq n - 1$. But now $2(\alpha \Delta (n - 1))^{\frac{1}{2}} \geq 2(n - 1) \geq n$ (for $n \geq 2$).

Hence, in order to prove theorem 1, it suffices to assume that $\Delta \geq 2$ and $4\Delta^2 < n - 1$; and the result now follows immediately from lemma 2.1. Also, in order to prove theorem 2, it suffices to assume that $\Delta \geq 2$ and $4\delta \Delta < n - 1$. By lemma 2.1, $h(G) \leq 2(\delta \Delta (n - 1))^{\frac{1}{2}} + x$, where $x = \frac{1}{8} \delta \Delta - \delta (\delta - 1) - \Delta + 1$. Now if $\delta = 1$ then $x = 1 - \frac{7}{8} \Delta \leq 0$ for $\Delta \geq 2$. If $2 \leq \delta \leq 8$ then $x \leq \frac{1}{8} \delta \Delta - \Delta \leq 0$. Further, if $\delta > 8$ then $x \leq \frac{1}{8} \delta \Delta \leq \frac{1}{16} (\delta \Delta (n - 1))^{\frac{1}{2}}$, since $4\delta \Delta \leq n - 1$.

In order to prove lemma 2.1 we will use:
Lemma 2.2  For any $a > 0$ there is an integer $t > 0$ such that
\[ \frac{a^2}{t} + t < 2a + \frac{1}{4a}. \]

Proof of lemma 2.2  The function $f(x) = \frac{a^2}{x} + x$ for $x > 0$ is convex, with minimum at $x = a$ where $f(a) = 2a$. For $c > 2a$ the line $y = c$ meets the curve $y = f(x)$ at the two roots $x_1 < x_2$ of $\frac{a^2}{x} + x = c$. Now $x_2 - x_1 = (c^2 - 4a^2)^{\frac{1}{2}}$. Hence if $c^2 - 4a^2 = 1$ then some integer $t$ satisfies $x_1 < t \leq x_2$, and $f(t) \leq c$. So
\[ f(t) \leq c = (4a^2 + 1)^{\frac{1}{2}} \]
\[ < 2a + \frac{1}{4a}. \]

Proof of lemma 2.1. A partial harmonious colouring of $G$ is a harmonious colouring of an induced subgraph of $G$ such that no uncoloured vertex has neighbours already coloured with the same colour. Order the vertices $v_1, v_2, \ldots, v_n$ so that $v_j$ is a vertex of least degree in the subgraph induced by $\{v_1, \ldots, v_j\}$, for $j = 1, \ldots, n$. Let $k$ be the right hand side in the lemma rounded down to the nearest integer, and let $t$ be a positive integer to be chosen below. Suppose that for some $j$, $1 \leq j < n$ we have a partial harmonious colouring with $k$ colours which colours vertices $v_1, \ldots, v_j$ and is such that each colour set has size at most $t$. We show that we can extend this colouring to $v_{j+1}$.

Suppose that $f$ colour sets are ’full’, that is, of size $t$. Let $x$ be the number of colours unavailable for $v_{j+1}$. We must show that $x \leq k - 1$, and then we are done. Clearly the number $x - f$ of unavailable ’non-full’ colours is at most $n - 1 - ft$. We now give another bound on $x - f$. 

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Let $A$ be the set of coloured neighbours of vertex $v_{j+1}$, and let $B$ be the set of uncoloured neighbours of $v_{j+1}$. Thus

$$|A| \leq \hat{\delta} \quad \text{and} \quad |B| \leq \Delta - |A|.$$  

Let $A'$ be the set of coloured vertices with the same colour as some vertex in $A$, so that $|A'| \leq |A| t$. Let $A''$ be the set of coloured vertices adjacent to a vertex in $A'$. Thus

$$|A''| \leq |A'| \Delta - |A| \leq |A| t \Delta - |A|.$$  

Let $B'$ be the set of coloured vertices adjacent to a vertex in $B$. Thus

$$|B'| \leq |B| (\hat{\delta} - 1).$$  

Now we have

$$x - f \leq |A| + |A''| + |B'| \leq |A| t \Delta + (\Delta - |A|)(\hat{\delta} - 1) \leq \hat{\delta} t \Delta + (\Delta - \hat{\delta})(\hat{\delta} - 1) \leq \alpha t \Delta + (\Delta - \alpha)(\hat{\delta} - 1),$$  

since $t \Delta \geq \hat{\delta} - 1$.

We have thus shown that

$$x \leq f + \min \{n - 1 - ft, \alpha t \Delta + (\Delta - \alpha)(\hat{\delta} - 1)\}.$$  

Thus $x \leq g(f)$ where

$$g(y) = y + \min \{n - 1 - yt, \alpha t \Delta + (\Delta - \alpha)(\hat{\delta} - 1)\}.$$  

Now $g(y) \leq g(y_0)$ where $n - 1 - y_0 t = \alpha t \Delta + (\Delta - \alpha)(\hat{\delta} - 1)$, and

$$g(y_0) = \frac{n - 1}{t} + \alpha \Delta t - \alpha \Delta + (\Delta - \alpha)(\hat{\delta} - 1)(1 - \frac{1}{t}) \leq \frac{n - 1}{t} + \alpha \Delta t - \alpha \Delta + (\Delta - \alpha)(\hat{\delta} - 1).$$
Now let $a = \left(\frac{n-1}{\alpha \triangle}\right)^{\frac{1}{2}}$ so that $a \geq 2$, and choose $t$ as the positive integer supplied by lemma 2.2. Then,

$$\frac{n-1}{t} + \alpha \triangle t = \alpha \triangle \left(\frac{a^2}{t} + t\right)$$

$$< \alpha \triangle \left(2a + \frac{1}{4a}\right)$$

$$\leq 2(\alpha \triangle (n-1))^{\frac{1}{2}} + \frac{1}{8} \alpha \triangle .$$

Hence

$$x \leq g(y_0) < 2(\alpha \triangle (n-1))^{\frac{1}{2}} - \frac{7}{8} \alpha \triangle + (\triangle - \alpha)(\delta - 1).$$

Thus $x \leq k - 1$, as required. \qed

3 A generalisation of harmonious colourings

Let us call a proper colouring of a graph $G$ $k$-harmonious if for each pair of colours the set of edges between the two colour sets forms a matching with at most $k$ edges. The $k$-harmonious chromatic number $h_k(G)$ is the least number of colours in such a colouring.

Clearly $h_1(G) = h(G)$ and $h_k(G) \geq h_{k+1}(G)$ for all $k = 1, 2, \ldots$. Also for $k \geq n/2$, $h_k(G) = h_{k+1}(G) = h_\infty(G)$ say. Thus $h_\infty(G)$ is the least number of colour in a proper colouring such that no path $P_3$ (with 3 vertices) is 2-coloured. As with $h(G)$ we have $h_\infty(G) \leq n$ and $h_\infty(G) = n$ if and only if $G$ has diameter at most 2.

It is noted in [1] that by sequentially colouring the square of $G$ (in which two vertices are adjacent if and only if they are at distance 1 or 2 in $G$) we obtain $h_\infty(G) \leq \triangle^2 + 1$. Also the example at the end of section 1 shows that we cannot improve much on this bound.
Now consider for example $h_2(G)$. It is not clear how to give an algorithmic proof of a good upper bound on $h_2(G)$, for example to show that $h_2(G) = o(\Delta n^{1/2})$ as $n, \Delta \to \infty$ with $\Delta = o(n^{1/2})$. However we can give a probabilistic proof. (Our constants below can be improved.)

**Theorem 4** Let $G$ be any non-trivial graph. Then, for each $k = 1, 2, \ldots$

$$h_k(G) \leq \max \left\{ 21(k + 1)\Delta^2, \ 18 \Delta^{1 + k} \left( \frac{n}{k} \right)^{1/2} \right\}.$$  

In particular $h_2(G) = o(\Delta n^{1/2})$ as $n, \Delta \to \infty$ with $\Delta = o(n^{1/2})$. Another interesting case is when $k = \Delta$.

**Corollary 5** For any graph $G$,

$$h_\Delta(G) \leq 23 \max \left\{ \Delta^3, \ n^{1/2} \right\}.$$  

In particular for the d-cube $Q_d$ with $n = 2^d$ and $\Delta = d$, $h_d(Q_d) = O(2^{d/2})$. Might such a result be of interest to designers of VLSI circuits?

## 4 Proof of theorem 4

In order to prove Theorem 4 we use the Erdős-Lovász Local Lemma (in its non-symmetric form), which is the following ([4], or see [3] or [7]).

**Lemma 4.1**

Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Let the graph $D$ with set of nodes $\{1, 2, \ldots, n\}$ and set of edges $E$ be a dependency graph for the events $A_i$; that is, assume that for each $i$, $A_i$ is independent
of the family of events \( \{ A_j : \{ i, j \} \notin E \} \). If there are reals \( 0 \leq y_i < 1 \) such that for all \( i \)

\[
Pr(A_i) \leq y_i \prod_{\{ i, j \} \in E} (1 - y_i)
\]

then

\[
Pr(\bigcap_{i=1}^{n} \bar{A}_i) \geq \prod_{i=1}^{n} (1 - y_i) > 0,
\]

so that with positive probability no event \( A_i \) occurs. \( \square \)

**Proof of theorem 4** Let \( G = (V, E) \) be a graph with \( n \) vertices. Let \( k \) be a positive integer, and let \( x \) be a positive integer at least the right hand side in theorem 4. By theorem 1 we may assume that \( k \geq 2 : \) this will save on a constant later. Let \( f : V \to \{1, \ldots, x\} \) be a random vertex colouring of \( G \), where, for each \( v \in V \) independently the colour \( f(v) \) is chosen uniformly from \( \{1, \ldots, x\} \). We shall show that with positive probability \( f \) is \( k \)-harmonious.

We define three types of 'bad' events.

**Type A.** For each edge \( \{u, v\} \) of \( G \) let \( A_{\{u,v\}} \) be the event that \( f(u) = f(v) \).

**Type B.** For each set \( \{u, v, w\} \) of vertices such that \( \{u, v\} \) and \( \{v, w\} \) are edges but \( \{u, w\} \) is not, let \( B_{\{u,v,w\}} \) be the event that \( f(u) = f(w) \).

**Type C.** For each unordered set \( S \) of \( (k + 1) \) ordered pairs \( (u_i, v_i) \) \( i = 1, \ldots, k+1 \) of distinct vertices, such that the corresponding induced subgraph on these \( 2k + 2 \) vertices has exactly the edges \( \{u_i, v_i\} \) \( i = 1, \ldots, k+1 \), let \( C_S \) be the event that \( f(u_1) = \cdots = f(u_{k+1}) \) and \( f(v_1) = \cdots = f(v_{k+1}) \).

It is easy to see that if none of these events occurs then \( f \) is a \( k \)-
harmonious colouring of $G$. We define the natural dependency graph $D$ for these events. This graph has a node for each event and two nodes are adjacent in $D$ if and only if the corresponding events involved a common vertex of $G$. We must bound degrees in $D$. The following lemma is immediate.

**Lemma 4.2** For each vertex $v$ of $G$,

1. $v$ is in at most $\triangle$ edges of $G$;
2. $v$ is in at most $\frac{3}{2} \triangle (\triangle - 1)$ sets $\{u, v, w\}$ corresponding to events of type B; and
3. $v$ is in at most $\binom{n}{k} \triangle k^2 + 1$ sets $S$ corresponding to events of type C.

We may now write down the following matrix, in which for $i, j \in \{A, B, C\}$ the $i, j$ entry is an upper bound on the number of type $j$ nodes adjacent to a given type $i$ node.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$2\triangle$</td>
<td>$3 \triangle (\triangle - 1)$</td>
<td>$2 \binom{n}{k} \triangle k^2 + 1$</td>
</tr>
<tr>
<td>B</td>
<td>$3\triangle$</td>
<td>$\frac{9}{2} \triangle (\triangle - 1)$</td>
<td>$3 \binom{n}{k} \triangle k^2 + 1$</td>
</tr>
<tr>
<td>C</td>
<td>$2(k + 1)\triangle$</td>
<td>$3(k + 1) \triangle (\triangle - 1)$</td>
<td>$2(k + 1) \binom{n}{k} \triangle k^2 + 1$</td>
</tr>
</tbody>
</table>

Now each type A event has probability $p_A = \frac{1}{x}$. Similarly we have $p_B = \frac{1}{x}$ and $p_C = \frac{1}{x^2}$. In order to apply lemma 4.1 (the local lemma) it suffices to show that for suitable $0 \leq y_A, y_B, y_C < 1$ the following three inequalities hold.

- (a) $\frac{1}{x} = p_A \leq y_A (1 - y_A)^{2\triangle} (1 - y_B)^{3\triangle (\triangle - 1)} (1 - y_C)^{2 \binom{n}{k} \triangle k^2 + 1}$
- (b) $\frac{1}{x} = p_B \leq y_B (1 - y_A)^{3\triangle} (1 - y_B)^{2\triangle (\triangle - 1)} (1 - y_C)^{3 \binom{n}{k} \triangle k^2 + 1}$
- (c) $\frac{1}{x^2} = p_C \leq y_C (1 - y_A)^{2(k + 1)\triangle} (1 - y_B)^{3(k + 1) \triangle (\triangle - 1)} (1 - y_C)^{2(k + 1) \binom{n}{k} \triangle k^2 + 1}$

Let us set $y_A = 2p_A = y_B = 2p_B$ and $y_C = 2p_C$. Now it suffices to check

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only inequality (c), and this will follow if we establish

\[(c)' \quad \frac{1}{x} \leq (1 - \frac{2}{x})^{3(k+1)} \Delta^2 (1 - \frac{2}{x^{2k}}) 2^{(k+1)}(\frac{n}{k})\Delta^{k+1}.
\]

But, since \(x \geq 21(k+1)\triangle^2\),

\((1 - \frac{2}{x})^{3(k+1)}\Delta^2 \geq 1 - \frac{2}{x} 3(k+1)\triangle^2 \geq \frac{5}{7}.
\]

Also note that \(18^k > 14(k+1)e^k \quad \text{since} \quad k \geq 2.
\]

Hence, since \(x \geq 18 \triangle^k + \frac{2}{\sqrt{k}} (\frac{n}{k})^{\frac{1}{2}}\) we have

\[x^{2k} \geq 18^k \triangle^{k+1} (\frac{n}{k})^k \geq 14(k+1)(\frac{ne}{k})^k \triangle^{k+1};
\]

and so

\[\left(1 - \frac{2}{x^{2k}}\right)^{2(k+1)}(\frac{n}{k})\Delta^{k+1} \geq 1 - \frac{2}{x^{2k}} 2(k+1)\left(\frac{ne}{k}\right)^k \triangle^{k+1} \geq \frac{5}{7}.
\]

Hence the right hand side in (c)' is at least \((\frac{5}{7})^2 > \frac{1}{2}\), as required. \(\square\)

It remains only to deduce corollary 5. Observe first that the maximum value of \(k^{\frac{1}{2k}}\) over positive integers \(k\) equals \(3^{\frac{1}{2}} \approx 1.2\). So if \(k = \triangle\) then

\[18 \triangle^\frac{1}{k} + \frac{1}{\sqrt{k}} (\frac{n}{k})^{\frac{1}{2}} = 18k^{\frac{1}{2k}} n^{\frac{1}{2}} < 22n^{\frac{1}{2}}.
\]

Also, by theorem 1,

\[h_{\triangle}(G) \leq h_1(G) \leq 22n^\frac{1}{2} \quad \text{if} \quad \triangle \leq 11.
\]

And, if \(k = \triangle \geq 12\) then

\[21(k+1)\Delta^3 = 21\frac{k+1}{k} \Delta^3 < 23 \Delta^3.
\]

Hence by theorem 4

\[h_{\triangle}(G) \leq \max \left\{ 23\Delta^3, 22n^\frac{1}{2} \right\}.
\]
References


