A note on 3-Steiner intervals and Betweenness\footnote{Work supported by the Ministry of Science of Slovenia and by the Ministry of Science and Technology of India under the bilateral India-Slovenia grants BI-IN/10-12-001 and INT/SLOVENIA-P-17/2009, respectively.}

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Abstract

The geodesic and geodesic interval, namely the set of all vertices lying on geodesics between a pair of vertices in a connected graph, is a part of folklore in metric graph theory. It is also known that Steiner tree of a (multi) set with \( k \) \( (k > 2) \) vertices, generalizes geodesics. In [1] the authors studied the \( k \)-Steiner intervals \( S(u_1, u_2, \ldots, u_k) \) on connected graphs \( (k \geq 3) \) as the \( k \)-ary generalization of the geodesic intervals. The analogous betweenness axiom (\( b_2 \)) and the monotone axiom (\( m \)) were generalized from binary to \( k \)-ary functions as: for any \( u_1, \ldots, u_k, x, x_1, \ldots, x_k \in V(G) \) which are not necessarily distinct,

\( (b_2) \ x \in S(u_1, u_2, \ldots, u_k) \Rightarrow S(x, u_2, \ldots, u_k) \subseteq S(u_1, u_2, \ldots, u_k), \)

\( (m) \ x_1, \ldots, x_k \in S(u_1, \ldots, u_k) \Rightarrow S(x_1, \ldots, x_k) \subseteq S(u_1, \ldots, u_k). \)

The authors conjectured in [1] that the 3-Steiner interval on a connected graph \( G \) satisfies the betweenness axiom (\( b_2 \)) if and only if each block of \( G \) is geodetic of diameter at most 2. In this paper we settle this conjecture. For this we show that there exists an isometric cycle of length \( 2k + 1 \), \( k > 2 \), in every geodetic block of diameter at least 3. We also introduce another axiom (\( b_2(2) \)), which is meaningful only to 3-Steiner intervals and show that this axiom is equivalent to the monotone axiom.

Keywords: Steiner interval, betweenness, geodetic graph

AMS subject classification (2010): 05C12, 05C38, 05C75
1 Introduction and Preliminaries

All graphs considered in this paper are finite, simple and connected. Let $G$ be a connected graph and let $V$ and $E$ denote the vertex set and the edge set of $G$. The notation $(K)$ is used for an induced subgraph of $G$ on vertices in $K \subseteq V$. The $i$-th neighborhood of $v \in V$ is defined as $N_i(v) = \{u \in V | d(u, v) = i\}$ and $E_i(v)$ is the set of edges of $(N_i(v))$. For a vertex $u$ in $N_i(v)$, $w \in N_{i-1}(v)$ with $wu \in E$ is called a predecessor of $u$ with respect to $v$ and a vertex $x \in N_{i-k}(v)$ such that $d(x, u) = k$ is called a $k$-th ancestor of $u$ with respect to $v$. A cycle on $k$ vertices or $k$-cycle for short will be denoted by $C_k$. A vertex of maximal distance on $C_k$ from $u \in V(C_k)$ is called an antipodal vertex of $u$. Paths that lie on a cycle $C$ between vertices $x, y \in V(C)$ are called $x, y$-segments. A subgraph $H$ of a graph $G$ is an isometric subgraph of $G$ if for any pair of vertices $u, v \in V(H)$, there exists a geodesic in $G$ between $u$ and $v$ that lies entirely in $H$. A graph $G$ is geodetic if every pair of its vertices is connected by a unique geodesic (shortest path). Obvious examples of geodetic graphs are odd cycles, trees and complete graphs. A block of a graph is a maximal 2-connected subgraph. If $G$ is geodetic and 2-connected then it is called a geodetic block. A graph is geodetic if and only if each of its block is geodetic [19].

Betweenness is a natural concept which is present in several branches of Mathematics and can be studied axiomatically. Modern axiomatic approach to betweenness is due to Hedlíková, who represented the betweenness relation as a ternary relation and introduced the idea of ternary spaces which unifies the the metric, order and lattice betweenness,[5, 6]. For a latest study on betweenness induced by posets and graphs, refer [18].

The theory of metric betweenness in graphs is developed along with the study of betweenness in general metric spaces. The most natural and well-studied metric in graphs is the “shortest path metric” (geodesic metric) on a connected graph. The geodesic betweenness gives way to look at the set of all “metrically between” vertices defined between two vertices, thus resulting in the notion of “interval” or geodesic interval $I(u, v)$ in a graph, i.e., a set of all vertices that lie on a shortest $u, v$-path. The first systematic study of the interval function $I(u, v)$ of a graph is due to Mulder [10], where the betweenness properties of $I(u, v)$ were formalized.

Betweenness in graphs using the idea of a transit function is introduced in [9], where a transit function on a nonempty set $V$ is a function $R$ from $V \times V$ to $2^V$ such that $R(x, y)$ contains both $x$ and $y$, $R(x, y) = R(y, x)$ and $R(x, x) = \{x\}$. One of the strong betweenness property that the geodesic interval $I$ enjoys is that, if $x$ is between $u$ and $v$, and $y$ is between $u$ and $x$, then $y$ is in between $u$ and $v$. This property of the interval function $I(u, v)$ is defined as the axiom (b2) by Mulder in [9]. The interpretation of the betweenness properties in this sense was first studied by Morgana and Mulder in [8], where it was also proved that the induced path interval $J$ is a betweenness if and only if $G$ is a house, hole, domino-free graph. A
related property (not always satisfied by $I$) is that if $x$ and $y$ are between $u$ and $v$, and $z$ is between $x$ and $y$, then $z$ is between $u$ and $v$. This property is known as the monotone axiom [9] and the graphs in which the monotone axiom is always satisfied are known as the interval monotone graphs [10]. Clearly, the monotone axiom always implies (b2), but the converse need not hold and the characterization of interval monotone graphs is still an open problem. An axiomatic characterization of the transit function $J$ satisfying the betweenness and monotone axioms was recently established in [2].

Note that in a geometric graph vertices on geodesics constitute the corresponding intervals between the pairs of vertices. A $W$-Steiner tree of a (multi)set $W \subseteq V(G)$ is a minimum order tree in $G$ that contains all vertices of $W$. The number of edges in a Steiner tree $T$ of $W$ is called the Steiner distance of $W$ and is denoted by $d(W)$. When $|W| = k$, a $W$-Steiner tree is called a $k$-Steiner tree and it is easy to verify that a 2-Steiner tree is a geodesic. Thus $k$-Steiner trees generalize geodesics. In [1], the authors introduced the k-Steiner intervals ($k \geq 3$) on a connected graph $G = (V, E)$ as the $k$-ary generalization of the geodesic interval $I(u, v)$. More precisely, the $k$-Steiner interval on $G$ is defined as a function $S: V \times V \times \cdots \times V \to 2^V$ such that $S(u_1, \ldots, u_k) = \{v \in V|v$ lies on some Steiner tree of $u_1, \ldots, u_k$ in $G\}$. Steiner intervals in graphs were first introduced in [7] and later studied as a tool for investigating the Steiner number of a graph and other related concepts [4, 11, 12, 13]. Before [1] was published, Steiner intervals were considered for sets of vertices, however, according to the definition above, consideration of $k$-Steiner intervals on multisets is also meaningful.

We have noted that 2-Steiner intervals are precisely the geodesic intervals and $k$-Steiner intervals form a generalization of the geodesic intervals. In this context the analogous concepts of betweenness were considered for $k$-Steiner intervals in [1], where the betweenness axiom (b2) and the monotone axiom (m) were generalized from binary to $k$-ary functions (in particular from geodesic intervals to $k$-Steiner intervals) as follows: for any $u_1, u_2, \ldots, u_k, x, x_1, x_2, \ldots, x_k \in V(G)$ which are not necessarily distinct,

(b2) $x \in S(u_1, u_2, \ldots, u_k) \Rightarrow S(x, u_2, \ldots, u_k) \subseteq S(u_1, u_2, \ldots, u_k),$

(m) $x_1, x_2, \ldots, x_k \in S(u_1, u_2, \ldots, u_k) \Rightarrow S(x_1, x_2, \ldots, x_k) \subseteq S(u_1, u_2, \ldots, u_k).$

We have already observed that the interval function $I(u, v)$ satisfies the (b2) axiom for every connected graph. But note that it is not so in the case of the $k$-Steiner interval, for $k > 2$. In [1], the authors introduced the union property, which says that $S(u_1, \ldots, u_k)$ coincides with the union of geodesic intervals $I(u_i, u_j)$ between all pairs from $\{u_1, \ldots, u_k\}$, and considered its relationship with the betweenness axiom (b2) and the monotone axiom (m) of the $k$-Steiner intervals. They proved that these three conditions are equivalent for $k > 3$ and characterized the graphs
satisfying these axioms as precisely the block graphs (i.e., graphs whose blocks are complete subgraphs). For \( k = 3 \), these three properties are not equivalent and it is further shown that the class of graphs satisfying the union property (which is precisely the class of graphs whose blocks are either complete or \( C_5 \)) is properly contained in the class of graphs satisfying the monotone axiom (which is precisely the class of graphs whose blocks are either complete or isomorphic to the so called graph \( M_n \)) and this class is properly contained in the class of graphs satisfying the betweenness axiom \((b2)\). Graphs \( M_n, n > 1 \) can be constructed from the complete graph \( K_n \) with vertices \( x_1, \ldots, x_n \) and a star \( K_{1,n} \) with \( x \) as its center and \( y_1, \ldots, y_n \) as its leaves, by adding an edge between \( y_i \) and \( x_i \) for \( i = 1, \ldots, n \), see Figure 1. It turned out that all these classes of graphs are geodetic graphs and, moreover, the blocks of these graphs have diameter at most 2. As a matter of fact, in [1], the authors conjectured that graphs whose 3-Steiner interval satisfies the \((b2)\) axiom are precisely the graphs whose blocks are geodetic graphs of diameter at most 2. The main aim of this paper is to settle this conjecture.

\[ \text{Figure 1: Graphs } M_n. \]

An axiomatic approach to betweenness for \( k \)-Steiner intervals in this paper and in [1], motivates to study the \( k \)-ary transit functions and their associated convexities in graphs as generalization of \( k \)-Steiner intervals in a similar way as binary transit functions were introduced as generalizations of the interval function \( I(u, v) \) in [9]. Such an attempt is followed in [3].

In [1], some results on 3-Steiner intervals satisfying the \((b2)\) axiom are given.

**Lemma 1** (Lemma 6, [1]) Let \( G \) be a graph in which the 3-Steiner interval satisfies
the (b2) axiom. Then $G$ does not contain the diamond, $C_4$, and $C_t$, $t \geq 6$, as an isometric subgraph.

Note that a subgraph $H$ of $G$ is geodesic convex if $I(u, v) \subseteq V(H)$ for each $u, v \in V(H)$. The geodesic convex hull of a subgraph $H$ of $G$ is the smallest geodesic convex subgraph of $G$ that contains $H$. We denote $C_{3,5}$, the graph obtained from the cycles $C_3$ and $C_5$ by amalgamating them along an edge of each cycle.

**Lemma 2** (Lemma 7, [1]) Let $G$ be a graph for which the 3-Steiner interval satisfies the (b2) axiom. Suppose $G$ has a subgraph $H$ isomorphic to $C_{3,5}$. Then the geodesic convex hull of $H$ is either the complete graph or $H$ is an induced subgraph in $G$ and its convex hull is isomorphic to $M_3$.

**Lemma 3** (Lemma 8, [1]) Let $G$ be a graph in which the 3-Steiner interval satisfies the (b2) axiom. Suppose $G$ has a 6-cycle as an induced subgraph. Then every pair of antipodal vertices in $C$ has a common neighbor (and all these neighbors are pairwise different).

The following conjecture is proposed in [1].

**Conjecture 4** (Conjecture 12, [1]) Let $G$ be a connected graph. The 3-Steiner interval $S$ on $G$ satisfies the (b2) axiom in $G$ if and only if each block of $G$ is a geodetic graph with diameter at most 2.

In this paper, we settle this conjecture which we postpone until Section 3. In the following section, we search for long cycles in geodetic blocks of diameter at least 3 and prove some interesting results on the existence of such cycles.

## 2 Geodetic blocks of diameter greater than 2

The problem of characterizing geodetic graphs was first proposed by Ore [14] and these graphs have drawn more attention about 30 years ago, see [15, 16, 17, 19]. Constructions of geodetic graphs and some extremal problems are studied by Parathasarathy and Srinivasan in [15] and [17]. See also [19] for geodetic graphs of diameter 2 and [16] for those of diameter 3. We will use in this section some of the results from references cited above, compare them with more recent ones, disprove a conjecture from [16], and prove in some sense a weaker version of it and on the other hand a stronger version of it, that will be later used in connection with graphs in which the 3-Steiner interval satisfies the (b2) axiom.

First note that Lemmas 2 and 3 in graphs where the 3-Steiner interval satisfies the (b2) axiom have the same outcome as the following theorem from [19] that deals with geodetic graphs. For this recall that a chord of a cycle $C$ is an edge between two non-consecutive vertices of $C$ and a bridge of $C$ is a geodesic between two vertices of $C$ that is shorter than the two segments of $C$ that connect them.
Theorem 5 (Theorem 3.4, [19]) If a geodetic graph contains a cycle $C$ of length 6, then $C$ and its chords form one of the following three configurations.

(i) A complete graph.

(ii) Labeling the vertices of $C$ cyclically as $v_1, v_2, \ldots, v_6$, an edge $v_1v_3$ and a bridge between $v_2$ and $v_5$ of length 2 whose intermediate vertex is not on $C$.

(iii) Labeling as in (ii), three bridges of length 2, between $v_1$ and $v_4$, $v_2$ and $v_5$, and $v_3$ and $v_6$, whose intermediate vertices are all different and not on $C$.

To prove Conjecture 4, we need some more results on geodetic graphs and in particular geodetic blocks of sufficiently large diameter ($\geq 3$). In connection with this problem we found a counterexample to the following conjecture from [16] that was, to the best of our knowledge, not yet disproved.

Conjecture 6 (Conjecture 1(a), [16]) In any geodetic block $G$ of diameter $d \geq 2$, there exists an induced cycle of length $2d + 1$.

For this, see the graph depicted in Figure 2 that has diameter 5 (observe that any vertex of degree 2 has eccentricity 5), but has no induced cycle of length 11. Note that in a similar fashion many other counterexamples can be constructed. In contrast to this conjecture we show that there exists an induced cycle in every geodetic block of diameter $\delta \geq 3$ of length at least 7, but not necessarily of length $2\delta + 1$, see the Fact in the proof of Theorem 11 below. Before we prove this proposition we need some further results.

![Figure 2: Geodetic block of diameter 5 having no induced $C_{11}$.](image)

In this section our aim is to explore the nature of cycles in geodetic blocks of diameter greater than 2. We prove that a geodetic block of diameter greater than 2 contains an isometric cycle of length $2k + 1$, where $k \geq 3$. 
First we quote some results from [15] which are used in what follows. The first one is generally known as the Unique Predecessor Theorem (UPT) and we sometimes refer it as such.

**Proposition 7** (Proposition 1, [15]) A graph $G$ is geodetic if and only if for each $v \in V$, every $u \in N_i(v)$ has a unique predecessor with respect to $v$ in $N_{i-1}(v)$, for $2 \leq i \leq e(v)$, where $e(v) = \max\{d(u,v) | u \in V(G)\}$.

**Corollary 8** (Corollary 1, [15]) Each vertex $u$ in a geodetic graph has a unique $k$-th ancestor with respect to $v$ for $1 \leq k \leq d(u,v)$.

**Proposition 9** (Proposition 2, [15]) Let $G$ be a geodetic graph. Let $v \in V(G)$ and $xy \in E_i(v)$. Let $a, b \in N_j(v)$ such that $a \neq b$ and $d(x,a) = d(y,b) = |j - i|$. Then $ab \notin E$.

Note that for $xy \in E(G)$ with $x \in N_i(v)$ and $y \in N_j(v)$ we have $|i - j| \leq 1$. Equipped with these results we will first show that there exists a cycle of length 7 or more in every geodetic block with sufficiently large diameter.

**Lemma 10** Let $B$ be a geodetic block of diameter greater than 2. Then there exist two vertices $x, y$ in $V(B)$ such that

(i) $x$ and $y$ lie on a cycle of length greater than or equal to 7,

(ii) one of the $x,y$-segments of the cycle is the $x,y$-geodesic, and

(iii) $d(x,y) \geq 3$.

**Proof.** Let $B$ be a geodetic block with diameter $\delta \geq 3$. Then there exist two vertices $u, v \in V(B)$ such that the $u,v$-geodesic $P$ is of length $\delta$. Therefore $u \in N_\delta(v)$. Since $B$ is a block, $u$ and $v$ lie on a cycle $C$. Let $P_1 : u_n u_{m-1} \cdots u_1 u_0$ and $P_2 : v_n v_{n-1} \cdots v_1 v_0$ be the two $u,v$-segments of $C$ with $u_m = u = v_n$ and $u_0 = v = v_0$. Since $G$ is a geodetic block, at least one of them, say $P_1$, is not a $u,v$-geodesic. If $P_2 = P$, then the lemma follows with $x = u$ and $y = v$. So assume that $P_2 \neq P$. Choose $P_1$ and $P_2$ so that $C$ is a shortest cycle containing both $u$ and $v$. Since $d_B(v,u_m-1) \leq \delta$ and $d_B(v,v_n-1) \leq \delta$, both $u_{m-1}$ and $v_{n-1}$ cannot be in $N_{\delta-1}(v)$ by UPT. Thus one, say $u_{m-1}$, must be in $N_{\delta}(v)$. We claim that then $u_{n-1}$ must be in $N_{\delta-1}(v)$.

Suppose to the contrary that $v_{n-1} \notin N_\delta(v)$. Then $v_{n-1} \notin P$ and according to the choice of $C$, $P$ must meet $P_1$ and $P_2$ at some vertices other than $u$ and $v$. Let $w$ be the vertex closest to $u$ on $P$ that is also on either $P_1$ or $P_2$. By symmetry we may assume that $w \in P \cap P_1$. Then $u \rightarrow P \rightarrow w \rightarrow P_1 \rightarrow v$ is a $u,v$-path which with $P_2$ forms a cycle that is shorter than $C$. This is a contradiction with the choice of $C$ and hence $v_{n-1} \in N_{\delta-1}(v)$. 

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Let $u'$ be the first vertex on $P$ after $u$ that also lies on $P_1$. Note that $u' \in N_k(v)$ for $1 \leq k \leq \delta - 2$. Thus $u'$ is different from $u_{m-1} \in N_\delta(v)$ and consequently different from $u_{m-2}$.

If $u' \in N_k(v)$, for $1 \leq k \leq \delta - 3$, then the path $u \to P \to u'$ is of length at least 3 and the cycle $u \to P \to u' \to P_1 \to u$ is of length at least 7. By choosing $x = u$ and $y = u'$ the lemma follows.

Now suppose $u' \in N_{\delta-2}(v)$. Then $P$ and $P_2$ split at $v_{n-1}$ and $v_{n-1}$ is adjacent to $u'$ on $P$. Let $m - k$ be the smallest index so that $u_m, u_{m-1}, \ldots, u_{m-k} \in N_\delta(v)$. Clearly, $k \geq 1$. Consider the $u_{m-k}, v'$-geodesic $Q$. Since $u_{m-k-1} \in N_{\delta-1}(v)$, $u_{m-k-1}$ is on $Q$. Let $v'$ be the first vertex of $Q$ after $u$ common to $P_2$ (note that $v'$ can be $v$).

If $u'$ is adjacent to $u_{m-k-1}$, then $v' \in N_k(v)$ for $k \leq \delta - 3$, and if $u'$ is not adjacent to $u_{m-k-1}$, then $v' \in N_k(v)$, $k \leq \delta - 2$.

Suppose $v' \in N_k(v)$, where $k \leq \delta - 3$. In this case, the $u_{m-k}, v'$-geodesic $u_{m-k} \to Q \to v'$ of length at least 3 and the cycle $C$ defined by $u_{m-k} \to Q \to v' \to P_2 \to u \to P_1 \to u_{m-k}$ is of length at least 7. Choosing $x = u_{m-k}$ and $y = v'$ the lemma follows.

Finally suppose $v' \in N_{\delta-2}(v)$. Then the length of the geodesic $u_{m-k} \to Q \to v'$ is at least 2 and furthermore, $v' \neq v$ and $u' \notin Q$. As we traverse from $v'$ to $v$ along $Q$, let $u''$ be the next vertex where $Q$ meets $P_1$ (note that $u''$ can be $v$). Clearly, the length of the geodesic $u_{m-k} \to Q \to u''$ is at least 3 and the cycle $C'$ defined by $u_{m-k} \to Q \to u'' \to P_1 \to u' \to P \to u_{m-1} \to P_1 \to u_{m-k}$ is of length at least 7. By choosing $x = u_{m-k}$ and $y = u''$, we obtain the desired result.

Note that the above lemma can be extended to any nontrivial geodetic block $B$. In this case the cycle of length greater or equal to 7 in item (i) is replaced by a cycle of length at least 3 or 5 when the diameter of $B$ is 1 or 2, respectively.

Next theorem develops the idea of a cycle of “appropriate” length in a geodetic block to an isometric cycle of “appropriate” length.

**Theorem 11** A geodetic block $B$ of diameter $\delta \geq 3$ contains an isometric cycle $C$ of diameter $k$ and length $2k + 1$, where $k \geq 3$.

**Proof.** Let $C$ be the collection of all cycles of length at least 7 that contain a pair $x, y$ of vertices that satisfy conditions (ii) and (iii) of Lemma 10. By Lemma 10, $C$ is nonempty. Let $Z$ be a cycle in $C$ of minimum length. Among all pairs of vertices in $Z$, for which one of the subpaths of $Z$ connecting the pair is a geodesic, let $u, v$ be a pair for which $d(u, v) = r$ is as large as possible. So $r \geq 3$. Let $P_2 : (u =) v_1 v_2 \cdots v_l v_0 (= v)$ be the $u, v$-geodesic in $Z$ and let $P_1 : (u =) u_m u_{m-1} \cdots u_1 u_0 (= v)$ be the other $u, v$-subpath of $Z$. Since $B$ is a geodetic block, $P_1$ has more vertices than $P_2$. We begin by establishing the following:

**Fact:** The cycle $Z$ is an induced cycle and has length $2r + 1$.

**Proof of Fact.** Since $P_2$ is a geodesic and by our choice of $Z$, both $P_1$ and $P_2$ are induced paths. So it suffices to show that $Z$ has no chords that join interior vertices.
of $P_1$ and $P_2$. Since $d(u, v) = r$ and by our choice of $r$, $r - 1 \leq d(u_1, u) \leq r$. If $d(u_1, u) = r - 1$, then both $u_1$ and $v_1$ lie on a $v, u$-geodesic, contrary to the fact that $B$ is a geodetic block. So $d(u_1, u) = r$. Similarly $d(v, u_{m-1}) = r$.

Since $d(u_1, u) = r$, it follows that $u_1$ is not adjacent with any other internal vertex of $P_2$ than possibly $v_1$. If $u_1v_1 \in E(B)$, then for the cycle obtained from $Z$ by deleting $v$ and adding the edge $u_1v_1$ belongs to $C$ whenever $Z$ is longer than 7, contrary to the choice of $Z$. If $Z \cong C_7$, note that this new cycle is a 6-cycle and there are two $u_1, u$-geodesics, contrary to the fact that $B$ is a geodetic block. Similarly $u_{m-1}$ is not adjacent with any internal vertex of $P_2$. Also $v_1u_2 \notin E(B)$, since $B$ is a geodetic block and $v_{r-1}u_{m-2} \notin E(B)$.

We show next that $d(v, u_3) = 3$. If this is not the case, there is a vertex $w$ adjacent with $v$ and $u_3$. If $w$ does not lie on $P_1$ or $P_2$, then the cycle obtained by deleting $u_1$ and $u_2$ from $Z$ and adding the path $u_3wv$ has length less than $Z$ and belongs to $C$ whenever $Z$ is longer than 7, which is not possible. Again for $Z \cong C_7$, we have two $u, v$-geodesics, a contradiction, since $B$ is a geodetic block. So if $d(v, u_3) = 2$, then $u_3v_1 \in E(B)$. Now if $r > 3$, we again obtain a contradiction to our choice of $Z$. Hence $r = 3$. By our choice of $Z$, the cycle obtained from $Z$ by deleting $v, u_1, u_2$ and adding the edge $v_1u_3$ is not in $C$. So either the distance between any two vertices on this cycle is at most 2 or this cycle is a 6-cycle. The latest is not possible by the above discussion, since $B$ is a geodetic block. First means in particular that $d(u_3, v_3) \leq 2$. Note that $u_3v_3 \notin E(B)$, otherwise there exists two $u, v$-geodesics. So $d(v_3, u_2) = 2$. Since $u(= v_3)$ and $v_3$ are not adjacent with $u_3$ either $m = 5$ or $u_3$ and $v_3$ have a common neighbor not on $Z$. The latter case is not possible as it gives a contradiction to the choice of $Z$. So $m = 5$. By the above observation, the $u_1, u$-geodesic has length 3. If this geodesic is disjoint from $P_2$ except for $u$, then we again obtain a contradiction to the choice of $Z$. So the $u, u_1$-geodesic contains exactly one other vertex of $P_2$, namely $v_2$. So the $u_1, v_2$-geodesic has length 2. If the common neighbor of $u_1$ and $v_2$ is not on $Z$, we again produce a contradiction to the choice of $Z$. Hence $v_2u_2 \in E(B)$. But this produces an induced 4-cycle $v_1u_2u_3v_1$, which is not possible. Hence $d(v, u_3) = 3$. Similarly $d(u, u_{m-3}) = 3$. So if $v_1u_j$ is a chord of $Z$ where $j \geq 3$ or $j \leq m - 3$, a cycle of shorter length than $Z$ exists that belongs to $C$, a contradiction. So if $v_1u_j$ is a chord of $Z$, then $m - 2 \leq j \leq 2$. Thus $m \leq 4$. But $m \geq 4$. Hence $m = 4$ and $j = 2$. By the above observations the only possible chord incident with $u_2$, if $m = 4$ is $u_3v_2$. However then, there exists two $u_2, u$-geodesics, which is not possible. So $Z$ has no chords.

Since $d(v, u) = r$ and $B$ is a geodetic block, $m > r$. If $m \neq r + 1$, then the shorter $u, u_1$-segment of $Z$ is not a geodesic. Hence there is a bridge from $u$ to $u_1$. Let $P$ be a $u, u_1$-geodesic. Since $Z$ is chordless, $P$ contains vertices not on $Z$. Let $x$ be the first vertex of $P$ whose successor on $P$ is not on $Z$, and let $y$ be the first vertex after $x$ that is on $Z$ (it must be necessarily be on $P_1$). Then the $x, y$ subpath of $P$ is shorter than both $x, y$-segments of $Z$. Suppose $y = u_j$. Then we can show,
as above, that \( m = 4 \) and \( j = 2 \). But then \( m = r + 1 \), contrary to the assumption. Hence \( Z \) has length \( 2r + 1 \) and the Fact is proved.

We now show that \( Z \) is isometric by showing that \( Z \) has no bridge. Since \( Z \) has no chords, such a bridge must have length at least 2. Since \( P_2, P_1 - u \) and \( P_1 - v \) are geodesics, such a bridge must be from an interior vertex of \( P_1 - u \) to an interior vertex of \( P_2 \). In particular, there is a bridge from an \( x \) on \( P_1 \) to a \( y \) on \( P_2 \) whose internal vertices are not on \( Z \). Suppose \( x = u_j \). As in the proof of the above fact, \( m = 4 \) and \( j = 2 \). If \( y = v_1 \), then the cycle formed from the \( x, y \)-segment of \( Z \) containing \( u \) and the \( x, y \)-subpath of \( P \) has length at least 6. If it exceeds 6, then the \( x, y \)-subpath, i.e., the \( u_2v_1 \)-subpath of \( P \) has length at least 3. But then it is easily seen that a contradiction to the choice of \( Z \) is obtained. So \( d(u_2, v_1) = 2 \).

Let \( w \) be the common neighbor of \( u_3 \) and \( v_1 \). Since \( Z \) has no chords, \( w \) is not on \( Z \). But now there are two \( v, u_3 \)-geodesics which is not possible. So \( y = v_2 \) and we may also assume \( d(v_1, u_3) = 3 \). But then the cycle obtained from the \( x, y \) segment on \( Z \) containing \( v \) together with the \( x, y \)-subpath of \( P \) has length at least 6, but less than that of \( Z \). This cycle cannot have length 6; otherwise, there exist two \( u_2, v_1 \)-geodesics. But if it has length at least 7, it belongs to \( C \), which is not possible. This completes the proof of the theorem. \( \square \)

The above theorem is a powerful tool for studying geodetic blocks, as it will be presented in the following section. However, it seems that the difference between the diameter \( \delta \) of a geodetic block and \( k \), where \( 2k + 1 \) is the size of the longest isometric cycle in the geodetic block, cannot be arbitrarily large. Unfortunately, we did not find any answer in that direction.

3 The class of graphs for which the 3-Steiner interval satisfies the (b2) axiom

In this section, we prove our main theorem which states that in all geodetic graphs where every block has diameter at most 2, the 3-Steiner interval satisfies the (b2) axiom and conversely. For proving the necessary part, we use the results of the previous Section, while for the sufficiency part, we require a description of the structure of Steiner trees in a geodetic block of diameter 2 and some other results. We begin by recalling a lemma from [1].

**Lemma 12** (Lemma 4, [1]) Let \( G \) be a geodetic graph. Let \( u,v,w \) be arbitrary distinct vertices of \( G \). Let \( u' \) be the last vertex that is common to the \( u, v \)- and \( u, w \)-geodesics, \( v' \) the last vertex that is common to the \( v, u \)- and \( v, w \)-geodesics, and \( w' \) the last vertex that is common to the \( w, v \)- and \( w, u \)-geodesics. Then \( u', v', w' \) lie in a block \( B \) of \( G \) and \( S(u,v,w) = I(u,u') \cup I(v,v') \cup I(w,w') \cup S(u',v',w') \).
Let $u, v, w$ be vertices of a connected graph $G$ and $P$ a $u, v$-geodesic in $G$ containing $w$. The assumption, that $P$ is not a $\{u, v, w\}$-Steiner tree, implies the existence of a $\{u, v, w\}$-Steiner tree $T$ (with $d(V(T)) < d(u, v)$), in which a path between $u$ and $v$ is shorter than $P$, a contradiction. Thus we have the following simple observation.

**Remark 13** Let $G$ be a connected graph and $u, v, w \in V(G)$. Then every $u, v$-geodesic containing $w$ is a $\{u, v, w\}$-Steiner tree.

**Lemma 14** Let $G$ be a connected graph and $u, v, w \in V(G)$. Then $w \in I(u, v)$ if and only if every $\{u, v, w\}$-Steiner tree is a $u, v$-geodesic containing $w$.

**Proof.** Let $u, v, w$ be vertices of a connected graph $G$ such that $w \in I(u, v)$. Let $P$ be a $u, v$-geodesic containing $w$. By Remark 13, $P$ is a $\{u, v, w\}$-Steiner tree. Let $T$ be an arbitrary $\{u, v, w\}$-Steiner tree. Clearly, $d(V(T)) = d(u, v)$. Let $Q$ be a $u, v$-path in $T$. Then the length of $Q$ is at most the length of $P$. On the other hand, since $P$ is a $u, v$-geodesic, the length of $Q$ is at least the length of $P$. This is possible if and only if $T$ coincides with $Q$. Proof of the converse is trivial. \[\square\]

**Lemma 15** Let $G$ be a connected graph and $u, v, w \in V(G)$. If $w \in I(u, v)$ then $S(u, v, w) = I(u, w) \cup I(w, v)$.

**Proof.** Let $u, v, w$ be vertices of a connected graph $G$ such that $w \in I(u, v)$. Let $x \in S(u, v, w)$. Then $x$ lies on some $\{u, v, w\}$-Steiner tree. By Lemma 14, $x$ lies on a $u, v$-geodesic $P$ containing $w$. Therefore $x$ lies on a $u, w$-geodesic or a $w, v$-geodesic. Hence $x \in I(u, w) \cup I(v, w)$. Now suppose $x \in I(u, w) \cup I(v, w)$. If $x \in I(u, w)$, then $x$ lies on a $u, w$-geodesic. Hence $x$ lies on a $u, v$-geodesic containing $w$. By Remark 13, $x$ lies on a $\{u, v, w\}$-Steiner tree and therefore $x \in S(u, v, w)$. \[\square\]

**Lemma 16** Let $B$ be a geodetic block of diameter 2 and let $u, v, w$ be distinct vertices of $B$. Let $U = \{u, v, w\}$ and let $x \in S(u, v, w)$. Then $d(U) = 2, 3,$ or 4. Moreover,

(i) If $d(U) = 2$, then $x \in \{u, v, w\}$.

(ii) If $d(U) = 3$, then $x$ lies on a $u, v$-geodesic, or a $u, w$-geodesic, or a $w, v$-geodesic.

(iii) If $d(U) = 4$ and $x$ is not on any of the $u, v$-, $u, w$-, and $w, v$-geodesics then $x$ is adjacent to exactly one of $u, v,$ or $w$. In addition, if $x$ is adjacent to $u$, then any $\{x, v, w\}$-Steiner tree is contained in some $U$-Steiner tree.

**Proof.** Let $B$ be a geodetic block of diameter 2 and let $U = \{u, v, w\}$ where $u, v, w$ are distinct vertices of $B$. Then $d(U) \geq 2$. Moreover, since $d(u, v) \leq 2$, $d(u, w) \leq 2$, and $d(v, w) \leq 2$, we have $d(U) \leq 4$. \[11\]
(i) If $d(U) = 2$, then $\langle U \rangle$ is connected and so $S(u, v, w) = \{u, v, w\}$. Thus $x \in S(u, v, w)$ implies $x \in \{u, v, w\}$.

(ii) Suppose $d(U) = 3$. Then $\langle U \rangle$ is disconnected and each Steiner tree for $U$ contains exactly one vertex that is not in $U$. Let $x$ be a vertex in a Steiner tree for $U$ such that $x \notin U$. Then $x$ is adjacent with vertices from distinct components of $\langle U \rangle$. So $x$ is on a geodesic between two vertices of $U$.

(iii) Suppose $d(U) = 4$. Then $U$ is necessarily an independent set. Let $T$ be a Steiner tree for $U$. If $T$ is a path, say an $u, w$-path, then the two vertices of $V(T) - U$ must be on a $u, v$- and $v, w$-geodesic, respectively. Suppose $T$ is not a path. Then $T$ is isomorphic to the tree of Figure 3. We may assume that the vertices appear in $T$ as labeled in Figure 3. Then the vertex adjacent with $v$ and $w$ is on a $v, w$-geodesic. Suppose $x$ is the neighbor of $u$ in $T$. Then at most one of the edges $xv$, $xw$ belongs to $B$, otherwise $T$ is not a Steiner tree for $U$. So $\langle \{x, v, w\} \rangle$ is disconnected. Hence $d(\{x, v, w\}) \geq 3$. Since $T - u$ is a tree of size 3 containing $x, v$ and $w$ it must be a Steiner tree for $\{x, v, w\}$. The result now follows.

**Lemma 17** Let $B$ be a geodetic block of diameter at most 2 and $u, v, w$ different vertices of $B$. Then $S(x, v, w) \subseteq S(u, v, w)$ for any $x \in \{u, v, w\}$.

**Proof.** If $diam(B) = 1$, then $S(x, v, w) = \{x, v, w\}$ and there is nothing to prove. The same happens if $x = u$. Thus suppose that $diam(B) = 2$ and $x \in \{v, w\}$. We may assume without loss of generality that $x = v$. Then $S(v, v, w) = I(v, w)$ and $I(v, w)$ induces either the edge $vw$ or the path $vyw$. The lemma is again clear if $vw$ is an edge. So let $y$ be the unique common neighbor of $v$ and $w$. By Lemma 16, $d(\{u, v, w\}) \in \{3, 4\}$. By analyzing possible Steiner trees for $\{u, v, w\}$, we see that $y$ must be in $S(u, v, w)$ as well.

Now we are ready to state our main theorem.
Theorem 18 Let $G$ be a connected graph. The 3-Steiner interval $S$ on $G$ satisfies the (b2) axiom if and only if each block of $G$ is a geodetic graph with diameter at most 2.

Proof. Assume that $G$ is a connected graph in which the 3-Steiner interval satisfies the (b2) axiom. First, we prove that each block of $G$ is geodetic. It is enough to prove that $G$ is geodetic. Suppose $G$ is not geodetic. Then for some $u, v \in V$, $u \neq v$, there exist two distinct geodesics, say $P_1$ and $P_2$, connecting $u$ and $v$. Therefore, we can find $x$ in $V(P_1)$ such that $x \notin V(P_2)$ and $w$ in $V(P_2)$ such that $w \notin V(P_1)$. Without loss of generality we may assume that $x$ and $w$ are adjacent to $u$. By Lemma 15, $S(u, x, v) = I(u, x) \cup I(x, v) = \{u, x\} \cup I(x, v)$. Since $w$ is adjacent to $u$ on $P_2$, $w$ is not on an $x, v$-geodesic. Therefore $w \notin S(u, x, v)$. But $w \in I(u, v) = S(u, u, v)$. So we get $S(u, u, v) \notin S(u, x, v)$, which is a contradiction to the assumption. Hence $G$ is geodetic.

Now it remains to prove that each block of $G$ has diameter at most 2. Suppose to the contrary that there exists a block $B$ of $G$ such that $\text{diam}(B) \geq 3$. Then by Theorem 11, $B$ contains an isometric odd cycle of length at least 7. By Lemma 1, we obtain a contradiction with the assumption that the 3-Steiner interval on $G$ satisfies the (b2) axiom.

Conversely, let $G$ be a connected geodetic graph where every block has diameter at most 2. Let $u, v, w \in V(G)$. By Lemma 12, we have $S(u, v, w) = I(u, u') \cup I(v, v') \cup I(w, w') \cup S(u', v', w')$, where $u'$ is the last vertex that is common to the $u, v$- and $u, w$-geodesics, $v'$ is the last vertex common to the $v, u$- and $v, w$-geodesics, and $w'$ the last vertex common to the $w, v$- and $w, u$-geodesics. Recall that $u', v', w'$ lie in the same block $B$ of $G$. By assumption, $\text{diam}(B) \leq 2$. Let $x \in S(u, v, w)$. Then either $x \in I(u, u')$, $x \in I(v, v')$, $x \in I(w, w')$, or $x \in S(u', v', w')$.

**Case 1** If $x \in I(u, u')$, then $I(x, u') \subseteq I(u, u')$. In this case $x'$, which is the last vertex common to the $x, v$- and $x, w$-geodesics, coincides with $u'$. Therefore, using Lemma 12, we derive $S(x, v, w) = I(x, x') \cup I(v, v') \cup I(w, w') \cup S(u', v', w') = I(x, x') \cup I(v, v') \cup I(w, w') \cup S(u', v', w') \subseteq I(u, u') \cup I(v, v') \cup I(w, w') \cup S(u', v', w') = S(u, v, w)$. Similarly we prove that the 3-Steiner interval $S$ on $G$ satisfies the (b2) axiom in cases when $x \in I(v, v')$ or $x \in I(w, w')$.

**Case 2** Let $x \in S(u', v', w')$. Clearly $x \in V(B)$ and $x' = x$. Hence, using Lemma 12, we infer that $S(x, v, w) \subseteq S(u, v, w)$ if and only if $S(x', v', w') \subseteq S(u', v', w')$. Thus we will prove the inclusion

$$S(x', v', w') \subseteq S(u', v', w'). \quad (1)$$
In order to show that the 3-Steiner interval satisfies the \((b2)\) axiom on \(G\), the inclusion (1) is clear for \(x' \in \{u', v', w'\}\), by Lemma 17. Thus let \(x' \notin \{u', v', w'\}\). If \(\text{diam}(B) = 1\), then \(S(u', v', w') = \{u', v', w'\}\) and the inclusion (1) is trivially fulfilled. If \(\text{diam}(B) = 2\) then, by Lemma 16, a \(U\)-Steiner tree (where \(U = \{u', v', w'\}\)) is of length 2, 3, or 4. Moreover, we can ignore the case when \(d(U) = 2\) by Lemma 17.

Suppose \(d(U) = 3\). By Lemma 16 (ii), \(x\) lies on a \(u', v'\)-, a \(u', w'\)-, or a \(w', v'\)-geodesic. Suppose \(x\) lies on a \(u', v'\)-geodesic (in the other two cases the proof follows on similar lines). There are only two possibilities for \(x'\). In both of them, the inclusion (1) easily follows.

Finally suppose \(d(U) = 4\). Again by considering the two possible non-isomorphic Steiner trees for \(U\), it can be shown that (1) holds.

\[ \square \]

4 Yet another axiom

To conclude, we define another axiom which lies in between the axioms \((b2)\) and \((m)\) for the 3-Steiner intervals. For this, note that the axioms \((b2)\) and \((m)\) can be interpreted in the following way. The axiom \((b2)\) implies that by replacing \(u_1\) with an arbitrary vertex of \(S(u_1, u_2, \ldots, u_k)\) we get \(S(x, u_2, \ldots, u_k) \subseteq S(u_1, u_2, \ldots, u_k)\) and the monotone axiom \((m)\) implies that by replacing each \(u_i\) by an arbitrary \(x_i \in S(u_1, u_2, \ldots, u_k)\), we get \(S(x_1, x_2, \ldots, x_k) \subseteq S(u_1, u_2, \ldots, u_k)\).

We denote the axiom \((b2)\) as \((b2(1))\) and \((m)\) as \((b2(k))\). Thus analyzing the axioms \((b2)\) and \((m)\) for the \(k\)-Steiner intervals more closely, we observe that several possible axioms can be defined for \(k\)-Steiner intervals as follows.

Axiom \((b2(i))\), for \(1 \leq i \leq k\):

\[ x_1, x_2, \ldots, x_i \in S(u_1, u_2, \ldots, u_k) \Rightarrow S(x_1, x_2, \ldots, x_i, u_{i+1}, \ldots, u_k) \subseteq S(u_1, u_2, \ldots, u_k), \]

\[ 1 \leq i \leq k. \]

For \(i = 1\) we see that \((b2(i)) = (b2)\) and for \(i = k\) we have \((b2(i)) = (m)\) and also, it can be observed that \((m)\) implies \((b2(i))\) always. Having defined these axioms for the \(k\)-Steiner intervals, we quote the Theorem 3 proved in [1].

**Theorem 19** (Theorem 3, [1]) Let \(G\) be a connected graph and \(k > 3\). The following statements are equivalent:

(i) \(G\) is a block graph,

(ii) the \(k\)-Steiner interval on \(G\) satisfies \((m)\),

(iii) the \(k\)-Steiner interval on \(G\) satisfies \((b2)\),

(iv) the \(k\)-Steiner interval on \(G\) satisfies the union property.

This theorem implies that for \(k > 3\), the class of graphs for which the \(k\)-Steiner interval satisfies the axioms \((b2(i))\), for \(1 \leq i \leq k\), are one and the same and it is
precisely the class of block graphs. When \( k = 3 \), by our main theorem in this paper and the results in [1], it is clear that the class of graphs satisfying (m) is a proper subclass of the class of graphs satisfying (b2) and hence the axioms (b2) and (m) are not equivalent. This observation leads us to the following axiom for 3-Steiner intervals.

Axiom (b2(2)): \( x, y \in S(u, v, w) \Rightarrow S(x, y, w) \subseteq S(u, v, w) \), for every \( u, v, w \) in \( V(G) \).

It is straightforward to check that the Petersen graph in which the 3-Steiner interval satisfies the (b2) axiom, does not satisfy the (b2(2)) axiom (observe that in Figure 4, \( x, y \in S(a, c, e) \), but \( w \in S(x, y, e) \) and \( w \notin S(a, c, e) \)). We will show that this axiom is equivalent to (m) for three vertices.

![Figure 4: The Petersen graph satisfies (b2), but not (b2(2)).](image)

**Theorem 20** Let \( G \) be a connected graph. Then 3-Steiner interval on \( G \) satisfies (m) if and only if 3-Steiner interval on \( G \) satisfies (b2(2)).

**Proof.** As mentioned before (m) always implies (b2(2)) (which implies (b2)). So let \( G \) be a graph that satisfies (b2(2)). Now, consider the Steiner interval \( S(U) \) of \( U = \{u, v, w\} \). By Lemma 12, it is enough to observe the case when \( U \) is contained in one block \( B \) of \( G \), which is a geodetic block of diameter at most 2 by Theorem 18.

Note that if \( |S(U) - U| \leq 2 \) then (m) and (b2(2)) coincide. Hence we may assume that \( |S(U) - U| \geq 2 \). Thus by Lemma 16, we only need to observe cases when \( d(U) \) equals 3 or 4 (namely, \( S(U) = U \) if \( d(U) = 2 \)).

Let \( d(U) = 3 \). Then there exist two nonadjacent vertices in \( U \), say \( u \) and \( v \). Clearly \( d(u, v) = 2 \). Let \( x \) be their common neighbor. Another common neighbor
would yield a 4-cycle, which is not possible. If \( uv, vw \not\in E(G) \) then \( d(u, w) = d(w, v) = 2 \) and since \( d(U) = 3 \) we have \( wx \in E(G) \). Again there is no other common neighbor except \( x \) of \( w \) and \( u \), and \( w \) and \( v \). Hence \( S(U) = U \cup \{x\} \) and we are done in this case since \( |S(U) - U| = 1 \). Now let \( uw \in E(G) \). Then \( vw \not\in E(G) \).

Then there is a common neighbor \( y \) of \( v \) and \( w \), which again must be unique. (Note that \( y \) can be equal to \( x \).) Then \((m)\) and \((b2(2))\) coincide since \( |S(U) - U| \leq 2 \).

Figure 5: The graph from the proof of Theorem 20.

Assume now that \( d(U) = 4 \). Suppose first that there exists a Steiner tree for \( U \) that is isomorphic to a tree on Figure 3. Let this tree be \( T \). Let \( x \) be the vertex of degree 3 and \( y \) the vertex of degree 2 of that tree. Then there is no other Steiner tree for \( U \) that is isomorphic to \( T \) and contains \( x \), otherwise the additional vertex would form a 4-cycle with \( x, y, u \) and then \( ux \in E(B) \), since \( B \) is geodetic, contrary to \( d(U) = 4 \). Since \( B \) has diameter 2, there exists a common neighbor \( t \) of \( v \) and \( u \) and a common neighbor \( z \) of \( u \) and \( w \). Note that \( t \) and \( z \) can be equal to \( y \) (not both at the same time) and that they are unique, since otherwise we again have a 4-cycle, which yields the same contradiction as before. Hence \( uzwxtu \) is a 6-cycle and by Theorem 5, we have only three possibilities. Clearly \((i)\) is not possible since \( d(U) = 4 \). Case\((ii)\) yields that \( S(U) \) induce graph \( M_3 \), for which \((m)\) is fulfilled by Theorem 11 from [1]. (Note that \((ii)\) covers the case when one of \( t \) or \( z \) equals \( y \).) For \((iii)\) note that we obtain one vertex deleted Petersen graph, see Figure 5.

Let \( s \) be a common neighbor of \( v \) and \( z \) and \( r \) a common neighbor of \( t \) and \( w \). Since \( B \) has diameter 2, there is a common neighbor \( q \) of \( y \) and \( s \). It is easy to see that \( q \notin \{u, z, w, v, t, r\} \), since otherwise we get two 4-cycles that force step by step the complete graph. Note that all vertices with a possible exception of \( q \) have degree greater than 2 and that they do not induce a complete graph. Hence \( B \) is not isomorphic to \( M_n \). Also \( q \notin S(U) \) and \( q \in S(\{y, s, u\}) \). Hence \((b2(2))\) is not fulfilled in this case.

Suppose finally that there is no Steiner tree for \( U \) isomorphic to \( T \). Then a
path $uxvyw$ is a Steiner tree. Clearly $uw$ is not an edge, since $d(U) = 4$. But then $u$ and $w$ must have a common neighbor $z$, since $B$ has diameter 2. It is easy to see that 6-cycle $uxvywzu$ is induced. Namely, every edge would force either a tree isomorphic to $T$ or a contradiction with $d(U) = 4$. Since $B$ is geodetic $u$ and $y$ must have a common neighbor, that yields a tree isomorphic to $T$, a final contradiction. □

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