Hamiltonian circuit and linear array embeddings in faulty $k$-ary $n$-cubes

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Abstract

In this paper, we investigate the fault-tolerant capabilities of the $k$-ary $n$-cubes for even integer $k$ with respect to the hamiltonian and hamiltonian-connected properties. The $k$-ary $n$-cube is a bipartite graph if and only if $k$ is an even integer. Let $F$ be a faulty set with nodes and/or links, and let $k \geq 3$ be an odd integer. When $|F| \leq 2n - 2$, we show that there exists a hamiltonian cycle in a wounded $k$-ary $n$-cube. In addition, when $|F| \leq 2n - 3$, we prove that, for two arbitrary nodes, there exists a hamiltonian path connecting these two nodes in a wounded $k$-ary $n$-cube. Since the $k$-ary $n$-cube is regular of degree $2n$, the degrees of fault-tolerance $2n - 3$ and $2n - 2$ respectively, are optimal in the worst case.

Keywords: Cycle embeddings; Hamiltonian; $k$-ary $n$-cube; Fault tolerance; Linear array embeddings

1. Introduction

In many parallel computer systems, processors are connected based on an interconnection network. Such networks usually have a regular degree, i.e., every node is incident with the same number of links. Popular instances of interconnection networks include hypercubes, star graphs, meshes, the $n$-ary $k$-cubes, etc.

The $k$-ary $n$-cube, denoted by $Q^k_n$, is regular of degree $2n$, edge symmetric, and vertex symmetric. Several properties of it has been studied in the literature. For example, in [3,4], meshes and hamiltonian cycles are embedded into healthy $k$-ary $n$-cubes, and the connectivity of $Q^k_n$ is shown to be $2n$, which equals the degree of each vertex. Furthermore, message routing and single-node broadcasting algorithms are given in [4]. The problem of conditional node connectivity on $Q^k_n$ is investigated in [6]. Cycles are said to be disjoint if they share no edges. In [2], $n$ edge disjoint hamiltonian cycles are found in $Q^k_n$. In [1], Ashir and Stewart studied the problem of hamiltonian cycle embeddings in $Q^k_n$ with a possibility of link failures.

Hamiltion circuit and linear array embeddings are desired properties in an interconnection network [5,9,14]. Many works related to embeddings of longest cycles and paths in various interconnection networks have been studied previously, including hypercubes [5,11], $k$-ary $n$-cubes [1], stars [8,14], arrangement graphs [9,12], etc.

Ashir and Stewart [1] showed that, with only edge faults and under the condition that every node is incident with at least two fault-free edges, a wounded $k$-ary $n$-cube still has a hamiltonian circuit, provided that there are no more than $4n - 5$ faulty edges. The situation of having both faulty nodes and faulty links remains unanswered, and the hamiltonian linear array embeddings in $Q^k_n$ have not been discussed yet even in a healthy $Q^k_n$.

Since failures are inevitable, fault-tolerance is an important issue in multiprocessor systems. In this paper, we consider a possibility of both node and link failures, and discuss the fault-tolerant capabilities of the $k$-ary $n$-cubes with respect to the hamiltonian and hamiltonian-connected properties. Let $F$ be a faulty set with nodes and/or links. We observe that $Q^k_n$ is bipartite if and only if $k$ is even. When $k$ is even and there is a faulty node, there exists neither a hamiltonian cycle nor a hamiltonian path between two vertices in different particle sets in a wounded $Q^k_n$. Therefore, throughout this paper, we suppose that $k$ is an odd integer with $k \geq 3$. Then, a ring of maximum length, or a hamiltonian cycle, in a wounded $Q^k_n$ can be constructed, provided that $|F| \leq 2n - 2$ for $n \geq 2$. On the other hand, if $|F| \leq 2n - 3$ for $n \geq 2$, we provide a construction of a linear array of maximum length, or a hamiltonian path, connecting two arbitrary vertices in a wounded $Q^k_n$. In both cases, we have achieved optimal solutions.
The reason is as follows. First, any hamiltonian cycles cannot be found in a wounded $Q_n^k$ when there are $2n-1$ faulty edges incident to a single node. Second, suppose that there are $2n-2$ edge faults incident to a node $x$. Let $y$ and $z$ be two nodes of $Q_n^k$ incident to $x$. Then, there is no hamiltonian path connecting $y$ and $z$ when all the edges incident to $x$ are faulty except $(x, y)$ and $(x, z)$.

The rest of this paper is organized as follows. We give some definitions, notation, and terminology in Section 2. Using the recursive structure of the $k$-ary $n$-cubes, we construct rings and linear arrays, respectively, traversing all the nodes in wounded $k$-ary $n$-cubes in Section 3. Finally, in Section 4, we present the conclusion.

2. Preliminaries

Throughout this paper, an interconnection network is represented by an undirected simple graph $G$. Given a graph $G$, we denote the vertex set and the edge set as $V(G)$ and $E(G)$, respectively. A path, denoted by $(v_1, v_2, \ldots, v_k)$, is a sequence of adjacent vertices where all the vertices are distinct except possibly $v_1 = v_k$. We say that a path is a hamiltonian path if it traverses all the vertices of $G$ exactly once. A cycle is a path that begins and ends with the same vertex. A hamiltonian cycle is a cycle which includes all the vertices of $G$. A graph is hamiltonian if it has a hamiltonian cycle. A graph $G$ is hamiltonian connected if, for any two arbitrary vertices $x$ and $y$ in $G$, there is a hamiltonian path connecting $x$ and $y$.

We consider the fault-tolerance of a graph $G$ in the following. Let $F$ be a faulty set which may contain both vertices and edges. Let $F_v = F \cap V(G)$ and $F_e = F \cap E(G)$. $G - F$ denotes the subgraph of $G$ induced by $V(G) - F_v$. Let $k$ be a positive integer. A graph $G$ is $k$-fault-tolerant hamiltonian (abbreviated as $k$-hamiltonian) if $G - F$ is hamiltonian for every $F$ with $|F| \leq k$. A graph $G$ is $k$-fault-tolerant hamiltonian connected (abbreviated as $k$-hamiltonian connected) if $G - F$ is hamiltonian connected for every $F$ with $|F| \leq k$.

The $k$-ary $n$-cube $Q_n^k$ is a graph consisting of $k^n$ vertices labeled by the integers from $0$ to $k^n-1$ for $k \geq 3$ and $n \geq 1$. Two vertices are adjacent if and only if the representations of their labels in base $k$ differ by one (modulo $k$) in exactly one position. We refer to $(x, y) \in E(Q_n^k)$ where $x$ differs from $y$ in the $d$th position, for $0 \leq d \leq n-1$, as an edge of dimension $d$. We say that $Q_n^k$ is divided into $Q_n^0, Q_n^1, \ldots, Q_n^{k-1}$ (abbreviated as $Q_n^i$, $i \leq k-1$, if there are no ambiguities) along dimension $d$ for some $0 \leq d \leq n-1$ if $Q_i^f$, for every $0 \leq i \leq n-1$, is a subgraph of $Q_n^k$ induced by the vertices labeled by $x_{d-1} \ldots x_0$ (see Fig. 1). It is clear that each $Q_n^i$ is isomorphic to $Q_n^{i-1}$ for $0 \leq i \leq k-1$. Note that $Q_n^i$ can be divided into $k$ copies of $Q_n^{i-1}$ along different dimensions. For $0 \leq i, j \leq k-1$, we use $[i, j]$ to denote a set of integers: $[i, j] = \{i \mid i \leq j \}$ if $i \leq j$, and $[i, j] = \{i \mid i \leq k-1 \text{ or } 0 \leq i \leq j \}$ if $i > j$. $Q_n^i$ (abbreviated as $Q_n^{i, j}$) denotes the subgraph of $Q_n^i$ which is induced by $\{u \mid u \in V(Q_i^f) \cap \{i \mid i \leq j \}\}$.

3. Hamiltonian path and cycle embeddings

Let $k$ be an odd integer with $k \geq 3$, and let $n \geq 2$ be an integer. Let $F \subseteq V(Q_n^k) \cup E(Q_n^k)$ be the set of faulty vertices and/or edges in $Q_n^k$. Let $Q_n^k$ be divided into $Q[0], Q[1], \ldots, Q[k-1]$ along some dimension, and let $F' = F \cap (V(Q[l]) \cup E(Q[l]))$ for every $0 \leq l \leq k-1$. We refer to an edge $(x, y) \in E(Q_n^k)$ where all of $x, y$, and $(x, y)$ are fault-free, as a safe crossing-edge.

In the following lemmas, namely Lemmas 1–3, we shall construct hamiltonian paths in faulty $Q[l, j]$ for every $l, j \in [0, k-1]$ when each faulty $Q[l]$ is hamiltonian connected for $l \in [i, j]$. These preliminaries will be useful for further discussions.

As a first step, we shall construct a hamiltonian path between two arbitrary vertices belonging to $Q[l]$ in a faulty $Q[i, j]$ (see Fig. 2(a)).

**Lemma 1.** Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i, j]) \cup E(Q[i, j])$ be a faulty set with $|F| \leq 2n-3$. If $Q[i] - F$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting every two vertices $u_i$ and $v_j$ in $V(Q[i] - F)$ in $Q[i, j] - F$ for every $n \geq 3$ and odd $k \geq 3$.

**Proof.** If $i = j$, this lemma holds. So we suppose that $i \neq j$. We may assume without loss of generality that $i = 0$ in the following discussion. Since $Q[0] - F$ is hamiltonian connected for every $l \in [0, j]$, there is a hamiltonian path, say $Q_0(u_0, v_0)$ ($u_0 = u_i$ and $v_0 = v_j$), in $Q[0] - F$ (see Fig. 3(a)). The length of $P_0(u_0, v_0) = |V(Q[0] - F)| - 1 \geq k^{n-1} - |F| - 1$, and the number of faults outside $Q[0]$ is at most $(2n-3) - |F|$. When $n$ and $k \geq 3$, $\frac{k^{n-1} - |F| - 1}{2} \geq \frac{3^{n-1} - |F| - 1}{2} > (2n-3) - |F|$. Hence, we can find two consecutive vertices, say $w_0$ and $z_0$, on $P_0(u_0, v_0)$ such that $(u_0, w_1)$ and $(z_0, z_1)$ are safe crossing-edges where $w_1$ and $z_1$ are the neighbors of $w_0$ and $z_0$ in $Q[1]$, respectively. Let $(u_0, P_0, z_0, v_0)$, where $P_0 = (u_0, v_0)$, and let $P_1(w_0, z_1)$ be a hamiltonian path in $Q[1] - F$. $(u_0, P_0, z_0, v_0)$, $w_0, w_1, P_1(w_1, z_1), z_1, z_0, P_0, z_0, v_0)$ forms a hamiltonian path in $Q[0, 1] - F$. Repeating the above construction, we have a hamiltonian path in $Q[0, 1] - F$. □
In the following lemma, we shall construct a hamiltonian path between two arbitrary vertices $u_i \in V(Q[i] - F^j)$ and $u_j \in V(Q[j] - F^j)$ in a faulty $Q[i, j]$ (see Fig. 2(b)). Note that $Q[i, j]$ can tolerate $2n - 2$ faults in this lemma, which is the maximum degree of the fault-tolerance of hamiltonian cycle embeddings. In addition, we want all the vertices in $Q[j] - F^j$ to form a subpath on this hamiltonian path for proving Lemma 3.

**Lemma 2.** Let $i, j \in [0, k - 1]$, and let $F \subseteq V(Q[i, j]) \cup E(Q[i, j])$ be a faulty set with $|F| \leq 2n - 2$. If $Q[l] - F^l$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting two arbitrary vertices $u_i \in V(Q[i] - F^j)$ and $u_j \in V(Q[j] - F^j)$ in $Q[i, j] - F$ such that all the vertices in $Q[j] - F^j$ form a subpath on this hamiltonian path for every $n \geq 3$ and odd $k \geq 3$.

**Proof.** If $i = j$, the statement follows. Hence, we suppose that $i \neq j$. Without loss of generality, we may assume that $i = 0$ (see Fig. 3(b)). Note that $|F| = (2n - 2)$ and $|V(Q[0])| = k^{n-1}$. Since $k^{n-1} - (2n - 2) \geq 9 - 4 = 5$ for every $n \geq 3$ and odd $k \geq 3$, there exists a safe crossing-edge, say $(v_0, v_1)$, where $v_0 \neq u_0$, $v_0 \in V(Q[0] - F^0)$, $v_1 \neq u_j$, and $v_1 \in V(Q[1] - F^1)$. By assumption, $Q[l] - F^l$ is hamiltonian connected for every $l \in [0, j]$, so we have a hamiltonian path, say $P_0(u_0, v_0)$, in $Q[0] - F^0$. Continuing this process, we can join all hamiltonian paths in $Q[l] - F^l$, for all $l \in [0, j - 1]$, to form a hamiltonian path, namely $R(v_0, v_{j-1})$, in $Q[0, j - 1] - F$ such that $(v_{j-1}, v_j)$ is a safe crossing-edge where $v_{j-1} \neq u_0$, $v_{j-1} \in V(Q[j - 1] - F^{j-1})$, $v_j \neq u_j$, and $v_j \in V(Q[j] - F^j)$. Let $S(v_j, u_j)$ be a hamiltonian path in $Q[j]$. $(u_0, R(u_0, v_{j-1}), v_{j-1}, v_j, S(v_j, u_j))$ is a hamiltonian path in $Q[0, j] - F$, and $S(v_j, u_j)$ contains all vertices in $Q[j] - F^j$. □

In the following lemma, we construct a hamiltonian path between two arbitrary vertices $u_i \in V(Q[i] - F^j)$ and $u_k \in V(Q[s] - F^s)$ with $s \in [i, j]$ in a faulty $Q[i, j]$ (see Fig. 2(c)). Note that $Q[i, j]$ can tolerate $2n - 3$ faults in this lemma, which is the maximum degree of the fault-tolerance of hamiltonian path embeddings.

**Lemma 3.** Let $i, j \in [0, k - 1]$, and let $F \subseteq V(Q[i, j]) \cup E(Q[i, j])$ be a faulty set with $|F| \leq 2n - 3$. If $Q[l] - F^l$ is hamiltonian connected for every $l \in [i, j]$), there exists a hamiltonian path connecting every two vertices $u_i \in V(Q[i] - F^j)$ and $u_k \in V(Q[s] - F^s)$ in $Q[i, j] - F$ with $s \in [i, j]$ for every $n \geq 3$ and odd $k \geq 3$.

**Proof.** If $i = j$, the statement is true. Therefore, we assume that $i \neq j$. By Lemma 2, there exists a hamiltonian path, say $R(u_i, u_s)$, in $Q[i, s] - F$ such that all the vertices in $Q[s] - F^s$ form a subpath on $R(u_i, u_s)$. Using the counting argument in the proof of Lemma 1, we can find two consecutive vertices, say $u_s$ and $v_s \in V(Q[s])$, on $R(u_i, u_s)$ such that $(u_s, u_{s+1})$ and $(v_s, v_{s+1})$ are safe crossing-edges where $u_{s+1}$ and $v_{s+1} \in V(Q[s+1])$. By Lemma 1, there is a hamiltonian path, namely $S(u_{s+1}, v_{s+1})$, in $Q[s + 1, j] - F$. Let $(u_i, R(u_i, u_s), u_s, v_s, \ldots)$
We discuss the existence of a hamiltonian cycle in the following three cases. Since \( k \) is odd with \( n \geq 3 \), we may assume that \( P_{0}(u_{0}, v_{0}) \) and \( P_{0}(u_{1}, v_{1}) \) are safe crossing-edges in \( Q_{n}^{k} \). Without loss of generality, we may assume that \( k \geq 3 \). Furthermore, let \( u_{1}, v_{1} \) be the neighbors of \( u_{0} \) and \( v_{0} \) in \( Q_{1} \), respectively, and let \( u_{k-1} \) and \( v_{k-1} \) be the neighbors of \( u_{0} \) and \( v_{0} \) in \( Q_{k} \), respectively. Since there is at most one fault outside \( Q_{0} \), either the two edges \( (u_{0}, u_{1}) \) and \( (v_{0}, v_{1}) \) are safe crossing-edges or the two edges \( (u_{0}, u_{1-1}) \) and \( (v_{0}, v_{k-1}) \) are safe crossing-edges. By assumption, \( Q_{n-1}^{k} \) is \((2n-4)\)-hamiltonian. Therefore, there is a hamiltonian path, namely \( P_{0}(u_{0}, v_{0}) \), \( P_{0}(u_{1}, v_{1}) \), \( R(u_{1}, v_{1}) \), \( u_{1}, u_{0} \) forms a hamiltonian cycle in \( Q_{n}^{k} \). 

Case 2: \( |F| = 2n - 3 \).

By assumption, \( Q_{n-1}^{k} \) is \((2n-4)\)-hamiltonian. Therefore, there is a hamiltonian path, namely \( P_{0}(u_{0}, v_{0}) \), in \( Q_{0}^{k} \) - \( F^{0} \). Let \( u_{1} \) and \( v_{1} \) be the neighbors of \( u_{0} \) and \( v_{0} \) in \( Q_{1} \), respectively, and let \( u_{k-1} \) and \( v_{k-1} \) be the neighbors of \( u_{0} \) and \( v_{0} \) in \( Q_{k} \), respectively. Since there is at most one fault outside \( Q_{0} \), either the two edges \( (u_{0}, u_{1}) \) and \( (v_{0}, v_{1}) \) are safe crossing-edges or the two edges \( (u_{0}, u_{1-1}) \) and \( (v_{0}, v_{k-1}) \) are safe crossing-edges. Without loss of generality, we may assume that \( (u_{0}, u_{1}) \) and \( (v_{0}, v_{1}) \) are safe crossing-edges. By assumption, \( Q_{n-1}^{k} \) is \((2n-5)\)-hamiltonian and \( 2n - 5 \geq 1 \) for \( n \geq 3 \), so \( |F| - F^{i} \) is hamiltonian connected for every \( l \in [1, k - 1] \) and \( n \geq 3 \). Since \( 1 < 2n - 3 \) for \( n \geq 3 \), by Lemma 3, there is a hamiltonian path, namely \( R(u_{1}, v_{1}) \), in \( Q_{1} \), \( k - 1 \) - \( F \). Therefore, \( u_{0}, P_{0}(u_{0}, v_{0}), v_{0}, v_{1}, R(v_{1}, u_{1}), u_{1}, u_{0} \) forms a hamiltonian cycle in \( Q_{n}^{k} \) - \( F^{i} \).

Case 3: \( |F| = 2n - 4 \).

By assumption, \( Q_{n-1}^{k} \) is \((2n-4)\)-hamiltonian. Therefore, there is a hamiltonian cycle, say \( C_{0} \), in \( Q_{0}^{k} \) - \( F^{0} \). Since there are at most two faults outside \( Q_{0} \), we can find two consecutive vertices, namely \( u_{0} \) and \( v_{0} \), on \( C_{0} \) for \( n \geq 3 \) such that \( (u_{0}, u_{1}) \) and \( (v_{0}, v_{1}) \) are safe crossing-edges, where \( u_{1} \) and \( v_{1} \) are the neighbors of \( u_{0} \) and \( v_{0} \) in \( Q_{1} \) respectively. Note that \( Q_{l} \) - \( F^{i} \) is hamiltonian-connected for every \( l \in [1, k - 1] \) and \( n \geq 4 \). In this situation, the proof is similar to Case 1.

When \( n = 3 \), it is possible that in addition to \( Q_{0} \), there exists another copy of \( Q_{n-1}^{k} \), say \( Q_{i} \), which contains two faults (if all other copies contain at most 1 fault then by proceeding as above we are done). Hence, both of \( Q_{0} \) - \( F^{0} \) and \( Q_{i} \) - \( F^{i} \) are not necessarily hamiltonian connected, but both are hamiltonian. There is a hamiltonian cycle, say \( C_{i} \), in \( Q_{i} \) (see Fig. 4(b)). Note that there is no fault outside \( Q_{0} \) and \( Q_{i} \), and \( |F| - F^{i} \) is hamiltonian connected for every \( l \not\in [0, i] \). We may assume without loss of generality that \( i \neq k - 1 \). We can find a safe crossing-edge, say \( (u_{i-1}, u_{i}) \), where \( u_{i-1} \in Q_{i} \) and \( u_{i} \in Q_{i} \). By Lemma 3, there is a hamiltonian path, namely \( R(u_{1}, u_{i-1}) \), in \( Q_{i} \), \( i - 1 \) - \( F \) (if \( i = 1 \), then \( u_{i-1}, u_{1} \) = \( u_{0}, u_{1} \), and there is no \( R(u_{1}, u_{i-1}) \)). Let \( v_{k-1} \in V(Q_{i} - F^{i}) \) be a neighbor of \( v_{0} \). Let \( v_{0} \) be adjacent to \( u_{0} \) on \( C_{i} \) such that \( v_{0} \) is not hamiltonian connected in \( Q_{i} \), \( i + 1 \), \( k - 1 \). By Lemma 3, there is a hamiltonian path, namely \( S(v_{i-1}, v_{k-1}) \), in \( Q_{i} \), \( i + 1 \), \( k - 1 \). Furthermore, let \( (u_{0}, P_{0}(u_{0}, v_{0}), v_{0}) = C_{0} \) and

![Fig. 4. Cases of Theorem 6. (a) Case 1; (b) Case 2 (when \( Q[1] \) - \( F^{i} \) is not hamiltonian connected).](image)
Theorem 7. Let $k$ be an odd integer with $k \geq 3$. Then, $(u_0, u_1, R(u_1, u_i), u_{i-1}, u_i, P_i(u_i, v_i), v_i, v_i+1, S(v_i+1, v_k-1), v_k-1, v_0, P_0(v_0, u_0), u_0) = C_i$. Then, $(u_0, u_1, R(u_1, u_i), u_{i-1}, u_i, P_i(u_i, v_i), v_i, v_i+1, S(v_i+1, v_k-1), v_k-1, v_0, P_0(v_0, u_0), u_0)$ is a hamiltonian cycle in $Q^k_n - F$.

Proof. We want to prove that there exists a hamiltonian path connecting every two vertices $x$ and $y$ in $Q^k_n - F$ for every $F$ with $|F| \leq 2n - 3$. Since $x \neq y$, we can divide $Q^k_n$ into $Q[0], Q[1], \ldots, Q[k-1]$ along some dimension such that $x$ and $y$ are in different $Q^k_{n-1}$'s. Furthermore, without loss of generality, we may assume that $|F| \geq |F_i|$ for every $0 \leq i \leq k-1$. We discuss the existence of a hamiltonian path connecting $x$ and $y$ in the following three cases.

Case 1: $|F| = 2n - 3$.

By assumption, $Q^k_{n-1}$ is $(2n-4)$-hamiltonian. Hence, there is a hamiltonian path, namely $P_0(u_0, v_0)$, in $Q[0] - F$. Note that there is no fault outside $Q[0]$. So $Q[i]$ is hamiltonian connected for every $l \in [1, k-1]$. We divide this case further into two subcases, Case 1.1 and Case 1.2, as follows.

Case 1.1: $x \in V(Q[0] - F)$ and $y \in V(Q[i] - F)$ where $i \neq 0$ (see Fig. 5(a)).

We may assume that the distance from $x$ to $u_0$ is at least as far as the distance from $x$ to $v_0$ on $P_0(u_0, v_0)$. Let $(u_0, P_0(x, v_0), v_0, x, P_0, y, v_0) = P_0(u_0, v_0)$. Since there is no fault in $Q[0] - F$, we can divide this case further into two subcases, Case 1.1 and Case 1.2, as follows.

Case 1.2: $x \in V(Q[i] - F')$ and $y \in V(Q[j] - F')$ where $i, j \neq 0$.

We may assume that $i > j$. Suppose that both of $x$ and $y$ are neighbors of $u_0$ (or $v_0$). So, $x \in Q[k-1]$ and $y \in Q[1]$ (see Fig. 5(b)). Let $v_{k-1}$ be the neighbor of $v_0$ in $Q[k-1]$. Since there is no fault in $Q[k-1]$, by assumption, there exists a hamiltonian path, say $P_{k-1}(x, v_{k-1})$, in $Q[k-1]$. Let $u_0$ and $z_0$ be two consecutive vertices on $P_0(u_0, v_0)$. Also, let $w_1$ and $z_1$ be the neighbors of $u_0$ and $z_0$ in $Q[1]$, respectively. By Lemma 3, there is a hamiltonian path, namely $R(w_1, z_1)$, in $Q[1, k-2] - y$. Let $(u_0, P_0(x, v_0), v_0, z_0, P_0, z_0, v_0, v_0) = P_0(u_0, v_0)$. Then, $(y, u_0, P_0, y, v_0, w_1, R(w_1, z_1), z_1, z_0, P_0, z_0, v_0, v_0, P_{k-1}(v_{k-1}, x), x)$ is a hamiltonian path connecting $x$ and $y$ in $Q^k_n - F$. Otherwise, suppose that...
either $x$ or $y$ is not a neighbor of $u_0 (or v_0)$. Let $u_1 \in Q[1]$ and $v_{k-1} \in Q[k-1]$ be neighbors of $u_0$ and $v_0$, respectively (see Fig. 5(c)). We may assume without loss of generality that $u_1 \neq y$ and $v_{k-1} \neq x$. By Lemma 3, there exist hamiltonian paths, say $S(x, v_{k-1})$ and $T(u_1, y)$, in $Q[i, k-1]$ and $Q[1, i-1]$, respectively. As a result, $(x, S(x, v_{k-1}), v_{k-1}, v_0, P_0(v_0, u_0), u_0, u_1, T(u_1, y), y)$ is a hamiltonian path connecting $x$ and $y$ in $Q_n^k - F$.

Case 2: $|F^0| = 2n - 4$.

By assumption, $Q[0]$ is $(2n-4)$-hamiltonian. So there is a hamiltonian cycle, namely $C_0$, in $Q[0] - F_0$. Note that there is at most one fault outside $Q[0]$. Therefore, $Q[1] - F^1$ is hamiltonian connected for every $l \in [1, k-1]$. We divide this case further into two subcases Case 2.1 and Case 2.2 as follows.

Case 2.1: $x \in V(Q[0] - F^0)$ and $y \in V(Q[i] - F^1)$ where $i \neq 0$ (see Fig. 6(a)).

Let $u_0 \in V(C_0)$ be adjacent to $x$ on $C_0$ such that $u_0$ is not a neighbor of $y$. Let $u_1 \in V(Q[1] - F^1)$ be a neighbor of $u_0$. Since there is at most one fault outside $Q[0]$, we may assume without loss of generality that $(u_0, u_1)$ is a safe crossing-edge. By Lemma 3, there is a hamiltonian path, namely $R(u_1, y)$, in $Q[1, k-1] - F$. Let $(x, P_0(x, u_0), u_0, x) = C_0$. $(x, P_0(x, u_0), u_0, u_1, R(u_1, y), y)$ forms a hamiltonian path connecting $x$ and $y$ in $Q[n]^k - F$.

Case 2.2: $x \in V(Q[i] - F^1)$ and $y \in V(Q[j] - F^1)$ where $i, j \neq 0$ (see Fig. 6(b)).

We may assume that $i > j$. Since there is at most one fault outside $Q[0]$, we can choose two adjacent vertices, say $u_0$ and $v_0$, on $C_0$ such that $(u_0, u_{k-1})$ and $(v_0, v_1)$ are safe crossing-edges, $u_{k-1} \neq x$, and $v_1 \neq y$ where $u_{k-1} \in Q[k-1]$ and $v_1 \in Q[1]$ are neighbors of $u_0$ and $v_0$, respectively. By Lemma 3, there exists a hamiltonian path, namely $R(v_1, y)$, in $Q[1, i-1] - F$, and also, a hamiltonian path, namely $S(x, u_{k-1})$, in $Q[i, k-1] - F$. Let $(u_0, P_0(u_0, v_0), v_0, u_{k-1}, u_{k-1}, u_0, v_0, R(v_1, y), y)$ forms a hamiltonian path in $Q^n_k - F$.

Case 3: $|F^0| \leq 2n - 5$.

As a result, $Q[l] - F^1$ is hamiltonian connected for every $l \in [0, k-1]$. We may assume without loss of generality that $x \in V(Q[0] - F^0)$. Since $|F| \leq 2n - 3$, by Lemma 3, there is a hamiltonian path connecting $x$ and $y$ in $Q[0, k-1] - F$. Hence there exists a hamiltonian path connecting $x$ and $y$ in $Q[n]^k - F$.

In conclusion, the fault-tolerant hamiltonicity of $Q[n]^k$ is given in the following theorem.

**Theorem 8.** If $k$ is odd with $k \geq 3$ and $n \geq 2$, $Q[n]^k$ is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected.

**Proof.** By Corollary 5, Theorems 6, 7, and a simple mathematical induction, this theorem is proved. □

4. Conclusion

We have shown how to find a hamiltonian cycle and a hamiltonian path joining two arbitrary vertices in a wounded $k$-ary $n$-cube. When $k$ is an odd integer, $Q[n]^k$ is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected. Furthermore, our results are optimal (explained in Section 1). For even integer $k$, $Q[n]^k$ is a bipartite graph. It is easy to see that $Q[n]^k$ contains a hamiltonian cycle. However, with one single fault vertex, the remaining network does not contain any hamiltonian cycle. Therefore, for the fault-tolerant hamiltonian and hamiltonian-connected properties of $Q[n]^k$, with $k$ even, we can only consider edge faults. Let $F_e \subseteq E(Q[n]^k)$ be the set of faulty edges in $Q[n]^k$ with $|F_e| \leq 2n-2$ (not $2n-3$). For even integer $k$, we intend in future to show that $Q[n]^k - F_e$ has a hamiltonian path connecting two arbitrary vertices belonging to different partite sets and a path of maximum length, $k^n-2$, connecting two arbitrary vertices in the same partite set for every $n \geq 2$ and even $k \geq 4$. This problem has not yet been resolved.

The fault-tolerant hamiltonian and hamiltonian-connected properties are fundamental tools for exploring further properties concerning cycle or path embedding problems. For example, a graph $G$ is pancyclic if a cycle of length $l$ can be embedded into $G$ for $4 \leq l \leq |V(G)|$. In [7], fault-tolerant pancyclicity of Möbius cubes was studied by using the fault-tolerant hamiltonian and hamiltonian-connected properties of Möbius cubes. In addition, by employing hamiltonian cycles and paths in faulty hypercubes, linear array and cycle embeddings in conditional faulty hypercubes were investigated [13].

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