Approximation by Piecewise Constants on Convex Partitions

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Abstract

We show that the saturation order of piecewise constant approximation in $L_p$ norm on convex partitions with $N$ cells is $N^{-2/(d+1)}$, where $d$ is the number of variables. This order is achieved for any $f \in W^2_p(\Omega)$ on a partition obtained by a simple algorithm involving an anisotropic subdivision of a uniform partition. This improves considerably the approximation order $N^{-1/d}$ achievable on isotropic partitions. In addition we show that the saturation order of piecewise linear approximation on convex partitions is $N^{-2/d}$, the same as on isotropic partitions.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d \geq 2$. Suppose that $\Delta$ is a partition of $\Omega$ into a finite number of subdomains $\omega \subset \Omega$ called cells, such that $\omega \cap \omega' = \emptyset$ if $\omega \neq \omega'$, and $\sum_{\omega \in \Delta} |\omega| = |\Omega|$, where $|\omega|$ denotes the Lebesgue measure ($d$-dimensional volume) of $\omega$. A partition is said to be convex if each cell $\omega$ is a convex domain. We assume throughout the paper that $\Omega$ admits a convex partition. With a slight abuse of notation, we denote by $|D|$ the cardinality of a finite set $D$, so that $|\Delta|$ stands for the number of cells $\omega$ in $\Delta$.

Given a function $f : \Omega \to \mathbb{R}$, we are interested in the error bounds for its approximation by piecewise polynomials in the space

$$S_n(\Delta) = \left\{ \sum_{\omega \in \Delta} q_\omega \chi_\omega : q_\omega \in \Pi_n^d \right\}, \quad \chi_\omega(x) := \begin{cases} 1, & \text{if } x \in \omega, \\ 0, & \text{otherwise,} \end{cases}$$

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where $\Pi_n^d$, $n \geq 1$, is the space of polynomials of total degree $< n$ in $d$ variables. The best approximation error is measured in the $L_p$-norm $\| \cdot \|_p := \| \cdot \|_{L_p(\Omega)}$:

$$E_n(f, \Delta)_p := \inf_{s \in S_n(\Delta)} \| f - s \|_p, \quad 1 \leq p \leq \infty.$$ 

Clearly,

$$E_n(f, \Delta)_p = \begin{cases} \left( \sum_{\omega \in \Delta} E_n(f)_{L_p(\omega)}^p \right)^{1/p} & \text{if } p < \infty, \\ \max_{\omega \in \Delta} E_n(f)_{L_\infty(\omega)} & \text{if } p = \infty, \end{cases}$$

where

$$E_n(f)_{L_p(\omega)} := \inf_{q \in \Pi_n^d} \| f - q \|_{L_p(\omega)}$$

is the error of the best polynomial approximation of $f$ on $\omega$.

If $\omega$ is a bounded convex domain and $f|\omega$ belongs to the Sobolev space $W_n^p(\omega)$, then the error $E_n(f)_{L_p(\omega)}$ is estimated as

$$E_n(f)_{L_p(\omega)} \leq C_{d,n} \operatorname{diam}(\omega) \| f \|_{W_n^p(\omega)},$$

where $C_{d,n}$ denotes a positive constant depending only on $d$ and $n$ [3], and

$$\| f \|_{W_n^p(\omega)} := \sum_{|\alpha| = n} \left\| \frac{\partial^n f}{\partial x^n} \right\|_{L_p(\omega)}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_d \quad \text{for } \alpha \in \mathbb{Z}_+^d.$$ 

Note that

$$\| f - f_\omega \|_{L_p(\omega)} \leq 2 E_1(f)_{L_p(\omega)}, \quad f_\omega := |\omega|^{-1} \int_\omega f(x) \, dx,$$

see for example [2], and hence (2) implies that the Poincaré inequality

$$\| f - f_\omega \|_{L_p(\omega)} \leq \rho_d \operatorname{diam}(\omega) \| \nabla f \|_{L_p(\omega)}, \quad f \in W_1^1(\omega),$$

holds with a constant $\rho_d$ depending only on $d$ when $\omega$ is bounded and convex, where

$$\| \nabla f \|_{L_p(\omega)} := \left\| \left( \sum_{k=1}^d \left| \frac{\partial f}{\partial x_k} \right|^2 \right)^{1/2} \right\|_{L_p(\omega)}.$$ 

Indeed, it is easy to check that $\| \nabla f \|_{L_p(\omega)}$ is equivalent to the Sobolev seminorm $|f|_{W_1^1(\omega)}$, as

$$\| \nabla f \|_{L_p(\omega)} \leq |f|_{W_1^1(\omega)} \leq d^{\max\{\frac{1}{p}, \frac{1}{p'}\}} \| \nabla f \|_{L_p(\omega)}, \quad 1 \leq p \leq \infty.$$ 

We prefer to use $\| \nabla f \|_{L_p(\omega)}$ in (3) because this seminorm is invariant under orthogonal transformations of the coordinate system, which simplifies some
calculations below. It is important for the proof of Theorem 1 that \( \rho_d \) does not depend on the geometry of the domain.

It follows from (2) that for any convex partition \( \Delta \),

\[
E_n(f, \Delta)_p \leq C_{d,n} \text{diam}^n(\Delta) |f|_{W^p_p(\Omega)}, \quad \text{diam} (\Delta) := \max_{\omega \in \Delta} \text{diam}(\omega).
\]

Obviously, \( \text{diam}(\Delta) \geq C|\Delta|^{-1/d} \), where \( C \) depends only on \( |\Omega| \) and \( d \). Hence, in terms of \( |\Delta| \), the approximation order that can be obtained from the last estimate is not better than

\[
E_n(f, \Delta)_p = \mathcal{O}(|\Delta|^{-n/d}).
\]  

(4)

This order is achieved for example for \( \Omega = (0,1)^d \) on convex partitions \( \Delta_m \), \( m = 1, 2, \ldots \), defined by splitting the cube \( (0,1)^d \) uniformly into \( |\Delta_m| = m^d \) equal subcubes of edge length \( 1/m \).

Asymptotically optimal bounds for the \( L_p \)-error \( e_n(f, \Delta)_p \) of the interpolation by piecewise polynomials of degree \( < n \) on anisotropic triangulations of a polygonal domain in \( \mathbb{R}^2 \) have been studied in [1, 5]. There, for \( n \geq 2, \) an exact constant \( C_n \) is found such that \( \liminf_{|\Delta_N| \to \infty} |\Delta_N|^{n/2} e_n(f, \Delta_N)_p \geq C_n \) as soon as the sequence of triangulations \( \{\Delta_N\} \) satisfies \( \text{diam}(\Delta_N) = \mathcal{O}(|\Delta_N|^{-1/2}) \). Moreover, a sequence \( \{\Delta_N^*\} \) with this property exists such that \( \limsup_{|\Delta_N| \to \infty} |\Delta_N^*|^{n/2} e_n(f, \Delta_N^*)_p \leq C_n \).

In [2, Theorem 2] we have shown that assuming higher smoothness of \( f \) does not help to improve the order \( E_1(f, \Delta_N)_{\infty} = \mathcal{O}(|\Delta_N|^{-1/d}) \) if the sequence of partitions \( \{\Delta_N\} \) is isotropic, that is there is a constant \( c > 0 \) such that \( \text{diam}(\omega) \leq c \rho(\omega) \) for all \( \omega \in \bigcup_N \Delta_N \), where \( \rho(\omega) \) is the maximum diameter of \( d \)-dimensional balls contained in \( \omega \). More precisely, if \( E_1(f, \Delta_N)_{\infty} = o(|\Delta_N|^{-1/d}), N \to \infty \), for a function \( f \in C^1(\Omega) \) and some isotropic sequence of partitions \( \{\Delta_N\} \) with \( \lim \text{diam}(\Delta_N) = 0 \), then \( f \) is a constant. Thus, \( |\Delta|^{-1/d} \) is the saturation order of the piecewise constant approximation on isotropic partitions.

In this paper we show that the order of approximation by piecewise constants can be improved to \( E_1(f, \Delta)_p = \mathcal{O}(|\Delta|^{-2/(d+1)}) \) on suitable anisotropic convex partitions obtained by a simple algorithm if \( f \in W^2_p(\Omega), \Omega = (0,1)^d \) (Algorithm 1 and Theorem 1). Moreover, according to Theorem 2, \( |\Delta|^{-2/(d+1)} \) is the saturation order of piecewise constant approximation in \( L_\infty \)-norm on convex partitions as it cannot be further improved for any \( f \in C^2(\Omega) \) whose Hessian is positive definite at some point. Finally, Theorem 3 shows that the saturation order of piecewise linear approximations on convex partitions is \( |\Delta|^{-2/d} \), that is the same as on isotropic partitions.

In the bivariate case the saturation order \( N^{-2/3} \) has been shown by a different method in [4] for suitable sequences of partitions \( \Delta_N \) of \( (0,1)^2 \) into
polygons with cell boundaries consisting of totally $O(N)$ straight line segments.

## 2 Optimal piecewise constant approximation on convex partitions

In this section we provide a simple algorithm that generates piecewise constant approximations with the approximation order $|\Delta|^{-2/(d+1)}$ on convex polyhedral partitions with totally $O(|\Delta|)$ facets. For the sake of simplicity we only consider $\Omega = (0, 1)^d$.

### Algorithm 1

Assume $f \in W_1^1(\Omega)$, $\Omega = (0, 1)^d$. Split $\Omega$ into $N_1 = m^d$ cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $h = 1/m$. Then split each $\omega_i$ into $N_2$ slices $\omega_{ij}, j = 1, \ldots, N_2$, by equidistant hyperplanes orthogonal to the average gradient $g_i := |\omega_i|^{-1} \int_{\omega_i} \nabla f(x) \, dx$ on $\omega_i$. Set $\Delta = \{ \omega_{ij} : i = 1, \ldots, N_1, \ j = 1, \ldots, N_2 \}$, and define the piecewise constant approximation $s_\Delta(f)$ by

$$s_\Delta(f) := \sum_{\omega \in \Delta} f_\omega \chi_{\omega}, \quad f_\omega := |\omega|^{-1} \int_{\omega} f(x) \, dx.$$  \hfill (5)

Clearly, $|\Delta| = N_1 N_2$ and each $\omega_{ij}$ is a convex polyhedron with at most $2(d+1)$ facets.

This algorithm is illustrated in Fig. 1.

### Theorem 1

Assume that $f \in W_2^p(\Omega)$, $\Omega = (0, 1)^d$, for some $1 \leq p \leq \infty$. For any $m = 1, 2, \ldots$, generate the partition $\Delta_m$ by using Algorithm 1 with $N_1 = m^d$ and $N_2 = m$. Then

$$\| f - s_{\Delta_m}(f) \|_p \leq C_d |\Delta_m|^{-2/(d+1)} (|f|_{W_1^1(\Omega)} + |f|_{W_2^p(\Omega)}),$$  \hfill (6)

where $C_d$ is a constant depending only on $d$.

**Proof.** We only consider the case $p < \infty$ as $p = \infty$ can be derived by obvious modifications of the arguments given here. Note that a different proof in the case $p = \infty$ can be found in [2]. By construction,

$$\| f - s_{\Delta_m}(f) \|_p^p = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \| f - f_{\omega_{ij}} \|_{L_p(\omega_{ij})}^p.$$  

For a fixed $i$, let $\{\sigma_1, \ldots, \sigma_d\}$ be an orthonormal basis of $\mathbb{R}^d$ such that $\sigma_d = \|g_i\|^{-1} g_i$ if $g_i \neq 0$, and let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be the linear mapping defined by
the matrix $\text{diag}(1, \ldots, 1, N_2)$ with respect to the basis $\{\sigma_1, \ldots, \sigma_d\}$. We set $\tilde{\omega}_{ij} = \varphi(\omega_{ij})$, $\tilde{f} = f \circ \varphi^{-1}$. Then $|\tilde{\omega}_{ij}| = N_2|\omega_{ij}|$, $\text{diam}(\tilde{\omega}_{ij}) \leq d/m$, and

$$\|f - f_{\omega_{ij}}\|_{L^p(\omega_{ij})}^p = N_2^{-1}\|\tilde{f} - f_{\omega_{ij}}\|_{L^p(\tilde{\omega}_{ij})}^p.$$ 

Since $f_{\omega_{ij}} = \tilde{f}_{\tilde{\omega}_{ij}}$ and $\tilde{\omega}_{ij}$ is bounded and convex, (3) shows that

$$\|\tilde{f} - f_{\omega_{ij}}\|_{L^p(\tilde{\omega}_{ij})} \leq \rho_d \text{diam}(\tilde{\omega}_{ij}) \|\nabla \tilde{f}\|_{L^p(\tilde{\omega}_{ij})},$$

where $\rho_d$ depends only on $d$. We have

$$\|\nabla \tilde{f}\|_{L^p(\tilde{\omega}_{ij})}^p = \left\| \left( \sum_{k=1}^d |D_{\sigma_k} \tilde{f}|^2 \right)^{1/2} \right\|_{L^p(\tilde{\omega}_{ij})}^p$$

$$= N_2 \left\| \left( N_2^{-2} |D_{\sigma_d} f|^2 + \sum_{k=1}^{d-1} |D_{\sigma_k} f|^2 \right)^{1/2} \right\|_{L^p(\omega_{ij})}^p$$

$$\leq N_2^{1-p} \|D_{\sigma_d} f\|_{L^p(\omega_{ij})}^p + N_2 \sum_{k=1}^{d-1} \|D_{\sigma_k} f\|_{L^p(\omega_{ij})}^p,$$

where $D_{\sigma_k} f = \nabla f^T \sigma_k$ denote the directional derivatives of $f$. Since

$$\int_{\omega_{ji}} D_{\sigma_k} f(x) \, dx = 0, \quad k = 1, \ldots, d - 1,$$
Poincaré inequality (3) also implies
\[ \| D_{\sigma_k} f \|_{L^p(\omega_i)} \leq \rho_d \text{diam}(\omega_i) \| \nabla D_{\sigma_k} f \|_{L^p(\omega_i)}, \quad k = 1, \ldots, d - 1. \]

Hence
\[ \sum_{j=1}^{N_2} \sum_{k=1}^{d-1} \| D_{\sigma_k} f \|_{L^p(\omega_{ij})}^p \leq d \left( \frac{\sqrt{d} \rho_d}{m} \right)^p \| f \|_{W^2_p(\omega)}^p. \]

By combining the above estimates we obtain
\[ \| f - s_{\Delta_m}(f) \|_p^p \leq \left( \frac{d \rho_d}{m} \right)^p N_2^{-1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \| \nabla \tilde{f} \|_{L^p(\omega_{ij})}^p \]
\[ \leq \left( \frac{d \rho_d}{m} \right)^p \sum_{i=1}^{N_1} \left[ d \left( \frac{\sqrt{d} \rho_d}{m} \right)^p \| f \|_{W^2_p(\omega_i)}^p + N_2^{-1} \sum_{j=1}^{N_2} \| D_{\sigma_d} f \|_{L^p(\omega_{ij})}^p \right] \]
\[ \leq \left( \frac{d \rho_d}{m} \right)^p d \left( \frac{\sqrt{d} \rho_d}{m} \right)^p \| f \|_{W^2_p(\Omega)}^p + \left( \frac{d \rho_d}{m N_2} \right)^p \| f \|_{W^2_p(\Omega)}^p. \]

Since \( N_1 = m^d, N_2 = m \), we have \( |\Delta| = m^{d+1} \), and (6) follows with \( C_d = d^{5/2} \rho_d^2 \).

### 3 Saturation Orders

The main result of this section is the following theorem which, together with Theorem 1 shows that the saturation order of piecewise constant approximation on convex partitions is \( |\Delta|^{-2/(d+1)} \).

**Theorem 2.** Assume that \( f \in C^2(\Omega) \) and the Hessian of \( f \) is positive definite at a point \( \hat{x} \in \Omega \). Then there is a constant \( C_{f,d} \) depending only on \( f \) and \( d \) such that for any convex partition \( \Delta \) of \( \Omega \),
\[ E_1(f, \Delta)_\infty \geq C_{f,d} |\Delta|^{-2/(d+1)}. \]

The proof of Theorem 2 will be given at the end of the section.

It turns out that piecewise linear approximations on convex partitions have the saturation order \( |\Delta|^{-2/d} \). Thus, in contrast to piecewise constants, there is no improvement of the order in comparison to isotropic partitions.

**Theorem 3.** Assume that \( f \in C^2(\Omega) \) and the Hessian of \( f \) is positive definite at a point \( \hat{x} \in \Omega \). Then there is a constant \( C_{f,d} \) depending only on \( f \) and \( d \) such that for any convex partition \( \Delta \) of \( \Omega \),
\[ E_2(f, \Delta)_\infty \geq C_{f,d} |\Delta|^{-2/d}. \]
Proof. Since \( f \in C^2(\Omega) \), there is \( \delta > 0 \) and a cube \( Q \subset \Omega \) such that the smallest eigenvalue of the Hessian of \( f \) is at least \( \delta \) everywhere in \( Q \).

Assume that \( \omega \in \Delta \) has nonempty intersection with \( Q \), and let \( x_1, x_2 \in \omega \cap Q \) be such that \( \|x_1 - x_2\| \geq \frac{\delta}{2} \) diam(\( \omega \cap Q \)). Since the univariate function \( g := f|_{[x_1, x_2]} \) is convex with second derivative at least \( \delta \) everywhere in \([x_1, x_2] \), the error of its best \( L_{\infty} \)-approximation by (univariate) linear polynomials is greater or equal \( \frac{\delta}{64} \|x_1 - x_2\|^2 \). Indeed, by parametrising \( g \) with \( t \in [0, 1] \) and assuming without loss of generality that \( g(0) = g(1) = 0 \), we have \( g''(t) \geq \frac{\delta}{2} \|x_1 - x_2\|^2 \) and \( g(t) = \frac{t(t-1)}{2} \int_0^1 g''(\tau) M_t(\tau) d\tau \leq \frac{t(t-1)}{2} \delta \|x_1 - x_2\|^2 \), where \( M_t \) is the Peano kernel of the second divided difference \([0, 1, t] \). Since \( g(\frac{1}{2}) \leq -\frac{\delta}{8} \|x_1 - x_2\|^2 \), Chebyshev theorem implies the claim.

Hence,

\[
E_2(f, \Delta)_{\infty} \geq E_2(f)_{L_{\infty}(\omega \cap Q)} \geq \frac{\delta}{64} \text{diam}^2(\omega \cap Q).
\]

It follows that

\[
|Q| \leq \frac{\mu_d}{2^d} \sum_{\omega \cap Q \neq \emptyset} \text{diam}^d(\omega \cap Q) \leq \mu_d |\Delta| \left( \frac{16}{\delta} \right)^{d/2} E_2(f, \Delta)_{\infty}^{d/2},
\]

where \( \mu_d \) denotes the volume of the \( d \)-dimensional ball of radius 1. Thus,

\[
E_2(f, \Delta)_{\infty} \geq \frac{\delta |Q|^{2/d}}{16 \mu_d^{2/d} |\Delta|^{-2/d}}.
\]

Proof of Theorem 2. We first choose \( \delta > 0 \) and a cube \( Q \subset \Omega \) such that the smallest eigenvalue of the Hessian of \( f \) is at least \( \delta \) everywhere in \( Q \). Clearly, \( \nabla f(\tilde{x}) \neq 0 \) for some \( \tilde{x} \in Q \). Since the gradient of \( f \) is continuous, there is a constant \( \gamma > 0 \) and a cube \( \tilde{Q} \subset Q \) containing \( \tilde{x} \) such that \( D_{\sigma} f(x) \geq \gamma \) for all \( x \in \tilde{Q} \), where \( \sigma = \nabla f(\tilde{x})/\|\nabla f(\tilde{x})\| \). We assume without loss of generality that \( \tilde{Q} = Q \).

The arguments in the proof of Theorem 3 lead to the estimate

\[
E_1(f, \Delta)_{\infty} \geq E_2(f, \Delta)_{\infty} \geq \frac{\delta}{64} \text{diam}^2(\omega \cap Q)
\]

for any \( \omega \in \Delta \) with nonempty intersection with \( Q \).

Moreover, if \([x_1, x_2] \) is an interval in \( \omega \cap Q \) parallel to \( \sigma \), then \( |f(x_2) - f(x_1)| \geq \gamma \|x_2 - x_1\|_2 \), which implies that

\[
E_1(f, \Delta)_{\infty} \geq \frac{\gamma}{2} \|x_2 - x_1\|_2.
\]
Hence $\omega \cap Q$ is contained between two hyperplanes orthogonal to $\sigma$, with distance between them not exceeding $\frac{2}{\gamma} E_1(f, \Delta)$. The penultimate display shows that the intersection of $\omega \cap Q$ with any intermediate hyperplane is contained in a $(d-1)$-dimensional ball of radius $(\frac{64}{\delta} E_1(f, \Delta))^{\frac{1}{2}}$. Therefore, we may estimate the volume of $\omega \cap Q$ as

$$|\omega \cap Q| \leq \frac{2}{\gamma} E_1(f, \Delta) \cdot \mu_{d-1} \left(\frac{64}{\delta} E_1(f, \Delta)\right)^{(d-1)/2},$$

which implies

$$|Q| \leq |\Delta| \frac{2\mu_{d-1}}{\gamma} \left(\frac{64}{\delta}\right)^{(d-1)/2} E_1(f, \Delta)^{(d+1)/2},$$

and Theorem 2 follows.

References


