On Some Hermite-Hadamard Type Inequalities for Convex Functions via Hadamard Fractional Integrals

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Abstract: We explore a new Hadamard type inequality for Hadamard fractional integrals and derive a new fractional integral identity. We use the newly established fractional integral identity to obtain new Hadamard fractional version Hermite-Hadamard inequalities for twice differentiable functions. Then, we derive new inequality results by applying these identities.

Keywords: Hermite-Hadamard type inequality, Convex function, Hadamard fractional integrals

1 Introduction

The well-known Hermite-Hadamard integral inequality was established by Hermite at the end of 19th century (see [1]). There are many recent contributions to improve this inequality, please refer to [2, 3, 4, 5, 6] and references therein. It is worth noting that there are some interesting results about Hermite-Hadamard inequalities via fractional integrals according to the corresponding integral equalities involving a given class of differential convex functions. For more details on such new, important and interesting mathematical branch, one can refer to papers [7, 8, 9, 10, 11, 12, 13, 14].

For a well defined convex function \( g \) on \([c, d]\), the left (right) Hadamard fractional integral of order \( v > 0 \) is defined by (see [15])

\[
(H^v_J)_c g(z) = \frac{1}{\Gamma(v)} \int_c^z \left( \ln \frac{s}{c} \right)^{v-1} g(s) ds,
\]

and

\[
(H^v_J)_d g(z) = \frac{1}{\Gamma(v)} \int_z^d \left( \ln \frac{s}{z} \right)^{v-1} g(s) ds.
\]

Throughout this paper, we denote

\[
I_f(z, \mu, v, c, d) = (1 - \mu)g(z)[(d - z)^v + (z - c)^v] + \mu [g(c)(z - c)^v + g(d)(d - z)^v] - \Gamma(v + 1) \left[ H^v_J g(\ln)(e^d) + H^v_J g(\ln)(e^c) \right],
\]

where \( \mu \in [0, 1] \) and \( \mu > 0 \).

In [10, Theorem 2.1], the authors obtained an interesting Hadamard type inequality for Hadamard fractional integrals via nondecreasing and convex function. Here, we establish a new Hadamard type inequality for Hadamard fractional integrals via convex function (see Theorem 2.1):

\[
f \left( \frac{c + d}{2} \right) \leq \frac{\Gamma(v + 1)}{2(d - c)^v} \left[ H^v_J g(\ln)(e^d) + H^v_J g(\ln)(e^c) \right] \leq \frac{g(c) + g(d)}{2}.
\]

In [10, Lemma 3.1], the authors obtained an Hadamard fractional integrals identity involving once differentiable mapping. Here, we give a new Hadamard fractional integrals identity involving once differentiable mapping as follows.

\[
L_g(z, \mu, v; c, d) = (z - c)^v \int_0^1 (s^v - \mu)g'(sz + (1 - s)c)ds - \int_0^1 (s^v - \mu)g'(sz + (1 - s)d)ds.
\]

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Moreover, we also establish new Hadamard fractional integrals identity involving twice differentiable mapping (see Theorem 3.6).

The next aim of this paper is to establish some new Hermite-Hadamard type inequalities for once and twice convex functions via Hadamard fractional integrals (see Theorems 3.3,3.8) which improves [10, Theorems 3.3]. These results have some relationships with [10], however, we point that our current results are different from the previous results in [10] and generalize [10] in some sense.

2 A new Hadamard type inequalities

**Theorem 2.1.** Assume that \( v > 0 \) and the function \( g : [c, d] \rightarrow R \) is convex. Then we have

\[
f \left( \frac{c + d}{2} \right) \leq \frac{\Gamma(v + 1)}{2(d-c)^v} \left[ HJ^v_{(c^+)}(g \circ \ln)(e^d) + HJ^v_{(c^-)}(g \circ \ln)(e^c) \right] \leq \frac{g(c) + g(d)}{2},
\]

(1)

**Proof.** It follows from the convexity of the function \( f \) that

\[
f \left( \frac{x+y}{2} \right) \leq \frac{f(x) + f(y)}{2}.
\]

For \( 0 \leq s \leq 1 \), let \( y = sc + (1-s)d, z = sd + (1-s)c \) and multiply by \( s^{v-1} \) in each side, then

\[
2^{v-1}f \left( \frac{c + d}{2} \right) \leq t^{v-1} [f(sc + (1-s)d) + g(sd + (1-s)c)].
\]

(2)

Integrating (2) over \([0,1]\), we obtain:

\[
\frac{2}{v} g \left( \frac{c + d}{2} \right) \leq \frac{\Gamma(v)}{(d-c)^v} \left[ HJ^v_{(c^+)}(g \circ \ln)(e^d) + HJ^v_{(c^-)}(g \circ \ln)(e^c) \right],
\]

which implies that

\[
\frac{2}{v} g \left( \frac{c + d}{2} \right) \leq \frac{\Gamma(v)}{(d-c)^v} \left[ HJ^v_{(c^+)}(g \circ \ln)(e^d) + HJ^v_{(c^-)}(g \circ \ln)(e^c) \right].
\]

Now using the convexity of \( g \) again, for \( s \in [0,1] \), we have

\[
g(sc + (1-s)d) \leq sg(c) + (1-s)g(d), \quad g(sd + (1-s)c) \leq sg(d) + (1-s)g(c),
\]

which yields:

\[
s^{v-1}[g(sc + (1-s)d) + g(sd + (1-s)c)] \leq s^{v-1}[g(c) + g(d)].
\]

(3)

Integrating (3) over \([0,1]\), we have

\[
\int_0^1 s^{v-1}f(sv + (1-s)d)ds + \int_0^1 s^{v-1}f(sd + (1-s)c)ds \leq \int_0^1 [g(c) + g(d)]s^{v-1}ds.
\]

So we can get the following result

\[
\frac{\Gamma(v)}{(d-c)^v} \left[ HJ^v_{(c^+)}(g \circ \ln)(e^d) + HJ^v_{(c^-)}(g \circ \ln)(e^c) \right] \leq \frac{[g(c) + g(d)]}{v}.
\]

Then the proof is well completed.
3 New Hadamard fractional Hermite-Hadamard type inequalities

We begin to establish a new fractional integral identity which will be used in what follows.

Lemma 3.1. Assume that \( g : [c, d] \rightarrow R \) and \( g' \in L[c, d] \), \( g \) is a convex function, then we have

\[
I_g(z, \mu, v; c, d) = (z - c)^{v+1} \int_0^1 (s^\nu - \mu) g'(sz + (1 - s)c)ds - (d - z)^{v+1} \int_0^1 (s^\nu - \mu) g'(sz + (1 - s)d)ds,
\]

for all \( z \in [c, d], v > 0 \) and \( \mu \in [0, 1] \).

Proof. For \( z \neq c \) we have

\[
(z - c) \int_0^1 (s^\nu - \mu) g'(sz + (1 - s)c)ds = (s^\nu - \mu)g(sz + (1 - s)c) \frac{1}{0} - v \int_0^1 s^{\nu-1} g(sz + (1 - s)c)ds = (1 - \mu)g(z) + \mu g(c) - \frac{v}{(z - c)^\nu} \int_c^e (ln t - c)^\nu-1 g(ln t) \frac{dt}{t} = (1 - \mu)g(z) + \mu g(c) - \frac{\Gamma(v+1)}{(z - c)^\nu} H_f^\nu(g \circ ln)(e^\nu).
\]

For \( z \neq d \), we get

\[
-(d - z) \int_0^1 (s^\nu - \mu) g'(sz + (1 - s)d)ds = (s^\nu - \mu)g(sz + (1 - s)d) \frac{1}{0} - v \int_0^1 s^{\nu-1} g(sz + (1 - s)d)ds = (1 - \mu)g(z) + \mu g(d) - \frac{v}{(d - z)^\nu} \int_d^e (d - ln t)^\nu-1 g(ln t) \frac{dt}{t} = (1 - \mu)g(z) + \mu g(d) - \frac{\Gamma(v+1)}{(d - z)^\nu} H_f^\nu(g \circ ln)(e^d).
\]

Then multiplying both sides of (5) and (6) by \( (z - c)^\nu \) and \( (d - z)^\nu \), respectively, we obtain the desired result immediately.

Concerning (4), if we set \( z = d \) and \( z = c \), one has

\[
I_g(d, \mu, v; c, d) = (d - c)^{v+1} \int_0^1 (s^\nu - \mu) g'(sd + (1 - s)c)ds,
\]

and

\[
I_g(c, \mu, v; c, d) = -(d - c)^{v+1} \int_0^1 (s^\nu - \mu) g'(sc + (1 - s)d)ds.
\]

Using the right-sided inequality result of (4), we have the following conclusion.

Remark 3.2. Assume that \( v > 0 \) and the function \( g : [c, d] \rightarrow R \) is convex. Then

\[
\frac{1}{2} \left[ I_{g_{[\mu, v; c, d]}}(g_{[\mu, v; c, d]}) + I_{g_{[\mu, v; c, d]}}(g_{[\mu, v; c, d]}) \right] = \frac{g(c) + g(d)}{2} - \frac{\Gamma(v+1)}{2(d - c)^\nu} \left[ H_f^\nu(g \circ ln)(e^\nu) + H_f^\nu(g \circ ln)(e^d) \right] = \frac{(d - c)^{v+1}}{2} \int_0^1 (s^\nu - \mu) [g'(sd + (1 - s)c) - g'(sc + (1 - s)d)]ds.
\]

Theorem 3.3. Assume that \( g : [c, d] \rightarrow R \) and \( g' \in L[c, d] \). Suppose that \( |g'|^p \) is a convex function for some fixed \( p \geq 1 \), then

\[
|I_g(z, \mu, v; c, d)| \leq A \left\{ \frac{1}{2} \left( A_1(\nu, \mu) \left[ (z - c)^{v+1} \left( A_2(\nu, \mu) |g'(z)|^p + A_3(\nu, \mu) |g'(c)|^p \right)^\frac{1}{p} \right] \right) + (d - z)^{v+1} \left( A_2(\nu, \mu) |g'(c)|^p + A_3(\nu, \mu) |g'(d)|^p \right)^\frac{1}{p} \right\}.
\]
for all \(z \in [c, d]\), \(\mu \in [0, 1]\) and \(v > 0\), where

\[
A_1(v, \mu) = \frac{2v\mu^{1+\frac{1}{v}} + 1}{v+1} - \mu
\]

\[
A_2(v, \mu) = \frac{v}{v+1}\mu^{1+\frac{1}{v}} + \frac{1}{v+2} - \frac{\mu}{2}
\]

\[
A_3(v, \mu) = \frac{2v}{v+1}\mu^{1+\frac{1}{v}} - \frac{2}{v+2}\mu^{1+\frac{1}{v}} + \frac{1}{(v+2)(v+1)} - \frac{\mu}{2}.
\]

**Proof.** Using Lemma 3.1, we obtain

\[
|I_g(z, \mu, v; c, d)| \leq I_{g_1}(z, \mu, v; c, d) + I_{g_2}(z, \mu, v; c, d),
\]

where

\[
I_{g_1}(z, \mu, v; c, d) := (z - c)^{v+1} \int_0^1 |s^v - \mu| |g'(sz + (1-s)c)| ds,
\]

\[
I_{g_2}(z, \mu, v; c, d) := (d - z)^{v+1} \int_0^1 |s^v - \mu| |g'(sz + (1-s)d)| ds.
\]

Following Hölder inequality, we derive

\[
I_{g_1}(z, \mu, v; c, d) \leq (z - c)^{v+1} \left( \int_0^1 |s^v - \mu| |dt| \right)^{1 - \frac{1}{p}}
\]

\[
\left\{ \int_0^1 (\mu - s^v) \left[ s|g'(z)|^p + (1-s)|g'(c)|^p \right] ds
\]

\[
+ \frac{1}{\mu^{1+\frac{1}{p}}}(s^v - \mu) \left[ s|g'(z)|^p + (1-s)|g'(c)|^p ds \right] \right\}^{\frac{1}{p}}
\]

\[
= (z - c)^{v+1} \left( \frac{2v\mu^{1+\frac{1}{v}} + 1}{v+1} - \mu \right)^{1 - \frac{1}{p}} \left[ \left( \frac{v}{v+1}\mu^{1+\frac{1}{v}} + \frac{1}{v+2} - \frac{\mu}{2} \right) |g'(z)|^p
\]

\[
+ \left( \frac{2v}{v+1}\mu^{1+\frac{1}{v}} - \frac{2}{v+2}\mu^{1+\frac{1}{v}} + \frac{1}{(v+2)(v+1)} - \frac{\mu}{2} \right) |g'(c)|^p \right]^{\frac{1}{p}}.
\]

Similarly, we obtain

\[
I_{g_2}(z, \mu, v; c, d) \leq (d - z)^{v+1} \left( \frac{2v\mu^{1+\frac{1}{v}} + 1}{v+1} - \mu \right)^{1 - \frac{1}{p}} \left[ \left( \frac{v}{v+1}\mu^{1+\frac{1}{v}} + \frac{1}{v+2} - \frac{\mu}{2} \right) |g'(z)|^p
\]

\[
+ \left( \frac{2v}{v+1}\mu^{1+\frac{1}{v}} - \frac{2}{v+2}\mu^{1+\frac{1}{v}} + \frac{1}{(v+2)(v+1)} - \frac{\mu}{2} \right) |g'(d)|^p \right]^{\frac{1}{p}}.
\]

From above, one can submitting (10) and (11) to (9) to derive the result.

**Remark 3.4.** In Theorem 3.3, we set \(z = \frac{c+d}{2}\), \(\mu = 0\). It follows from the inequality (8) that

\[
\left| I_g \left( \frac{c+d}{2}, 0, v; c, d \right) \right|
\]

\[
\leq \left( \frac{1}{v+1} \right)^{1 - \frac{1}{p}} \left[ \frac{1}{v+2} \left( \frac{d-c}{2} \right)^{v+1} \right]^{\frac{1}{p}} \left\{ \left[ g'(\frac{c+d}{2}) \right]^p + \left[ \frac{1}{v+1} |g'(c)|^p \right]^{\frac{1}{p}} + \left[ \frac{1}{v+1} |g'(d)|^p \right]^{\frac{1}{p}} \right\}.
\]

**Remark 3.5.** Let \(p = 1\). Then for all \(z \in [c, d]\), \(\mu \in [0, 1]\) and \(v > 0\),

\[
\left| \frac{1}{2} I_g(c, \mu, v; c, d) + I_g(d, \mu, v; c, d) \right| \leq \frac{1}{2} (d-c)^{v+1} \left[ |g'(c)| + |g'(d)| \right] |A_2(v, \mu) + A_3(v, \mu)|
\]

In what follows, we give the corresponding results for twice differential functions.
Theorem 3.6. Assume that $g'' \in L[c, d]$. Then

$$I_g(z, \mu, v; c, d) = \left(\frac{1}{v+1} - \mu\right)g'(z)[(z-c)^{v+1} - (d-z)^{v+1}]$$

$$- \left[(z-c)^{v+2} \int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} g''(sz + (1-s)c)ds\right]$$

$$+(d-z)^{v+2} \int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} g''(sz + (1-s)d)ds. \quad (12)$$

Proof. Note that

$$\int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} dg'(sz + (1-s)c)$$

$$= (z-c) \int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} g''(sz + (1-s)c)ds$$

$$= \frac{s^{v+1} - \mu(v+1)s}{v+1} g'(sz + (1-s)c)|^1_0 - \int_0^1 g'(sz + (1-s)c)(s' - \mu)ds$$

$$= \left(\frac{1}{v+1} - \mu\right)g'(z) - \int_0^1 g'(sz + (1-s)c)(s' - \mu)ds, \quad (13)$$

and

$$\int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} dg'(sz + (1-s)d)$$

$$= -(d-z) \int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} g''(sz + (1-s)d)ds$$

$$= \frac{s^{v+1} - \mu(v+1)s}{v+1} g'(sz + (1-s)d)|^1_0 - \int_0^1 g'(sz + (1-s)d)(s' - \mu)ds$$

$$= \left(\frac{1}{v+1} - \mu\right)g'(z) - \int_0^1 g'(sz + (1-s)d)(s' - \mu)ds, \quad (14)$$

Multiplying both sides of (13) and (14) by $(z-c)^{v+1}$ and $-(d-z)^{v+1}$, respectively, we have

$$(z-c)^{v+1} \int_0^1 g'(sz + (1-s)c)(s' - \mu)ds$$

$$= (z-c)^{v+2} \int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} g''(sz + (1-s)c)ds. \quad (15)$$

and

$$-(d-z)^{v+1} \int_0^1 g'(sz + (1-s)d)(s' - \mu)ds$$

$$= -(d-z)^{v+2} \int_0^1 \frac{s^{v+1} - \mu(v+1)s}{v+1} g''(sz + (1-s)d)ds. \quad (16)$$

By adding the results of (15) and (16), we complete the proof.

Further, we have

Remark 3.7.

$$\frac{1}{2}[I_g(d, \mu, v; c, d) + I_g(c, \mu, v; c, d)]$$

$$= \frac{(d-c)^{v+1}}{2} \left[\left(\frac{1}{v+1} - \mu\right)[g'(d) - g'(c)]\right]$$

$$- \frac{(d-c)^{v+2}}{v+1} \int_0^1 (s^{v+1} - \lambda(v+1)s)[g''(sz + (1-s)c) + g''(sz + (1-s)d)]ds.$$

The conditions are just like Theorem 3.6.
Theorem 3.8. Assume that \( g'' \in L[c,d] \) and the function \( |g''|^p \) is convex for some fixed \( p \geq 1 \) on \([c,d]\), then
\[
|J_g(z, \mu, v; c, d)| \leq \left| \left( \frac{1}{v + 1} - \mu \right) g'(z)[(z - c)^{v+1} - (d - z)^{v+1}] \right|
+ A_4^{-\frac{1}{p}}(v, \mu) \left\{ \left( \frac{z - c}{v + 1} \right)^{v+2} \left[ A_5(v, \mu)|g''(z)|^p + A_6(v, \mu)|g''(c)|^p \right] \right\}^{\frac{1}{p}}
+ \left( \frac{(d - z)^{v+2}}{v + 1} \left[ A_5(v, \mu)|g''(z)|^p + A_6(v, \mu)|g''(d)|^p \right] \right\}^{\frac{1}{p}},
\]
where
\[
A_4(v, \mu) = \frac{v}{v + 2} \left[ \mu(v + 1) \right]^{1 + \frac{1}{p}} \frac{\mu(v + 1)}{2} + \frac{1}{v + 2},
A_5(v, \mu) = \frac{2v}{3(v + 3)} \mu(v + 1) \left[ \mu(v + 1) \right]^{1 + \frac{1}{p}} \frac{\mu(v + 1)}{3} + \frac{1}{(v + 2)(v + 3)} - \mu(v + 1),
A_6(v, \mu) = \frac{v}{v + 2} \left[ \mu(1 + v) \right]^{1 + \frac{1}{p}} \frac{2v}{3(v + 3)} \mu(1 + v) \left[ \mu(1 + v) \right]^{1 + \frac{1}{p}} \frac{\mu(v + 1)}{6}.
\]

Proof. It follows from (12) that:
\[
|J_g(z, \mu, v; c, d)| \leq \left| \left( \frac{1}{v + 1} - \mu \right) g'(z)[(z - c)^{v+1} - (d - z)^{v+1}] \right|
+ \left( \frac{(z - c)^{v+2}}{v + 1} |J_{g_1}(z, \mu, v; c, d)| + \left( \frac{(d - z)^{v+2}}{v + 1} |J_{g_2}(z, \mu, v; c, d)| \right) \right),
\]
where
\[
J_{g_1}(z, \mu, v; c, d) = \int_0^1 \left[ s^{v+1} - \mu(v + 1)s \right] g''(sz + (1 - s)c)ds,
J_{g_2}(z, \mu, v; c, d) = \int_0^1 \left[ s^{v+1} - \mu(v + 1)s \right] g''(sz + (1 - s)d)ds.
\]

By using the Hölder inequality we can get
\[
|J_{g_1}(z, \mu, v; c, d)| \leq \int_0^1 \left| s^{v+1} - \mu(v + 1)s \right| |g''(sz + (1 - s)c)| ds
\leq \left( \int_0^1 \left| s^{v+1} - \mu(v + 1)s \right| ds \right)^{1 - \frac{1}{p}} \times \left\{ \int_0^1 \left[ \mu(v + 1) s - s^{v+1} \right] (1 - s)|g''(c)|^p ds \right\}^{\frac{1}{p}}
+ \left( \int_0^1 \left| s^{v+1} - \mu(v + 1)s \right| ds \right)^{1 - \frac{1}{p}} \times \left\{ \int_0^1 \left[ \mu(v + 1) s - s^{v+1} \right] |g''(z)|^p ds \right\}^{\frac{1}{p}}
\leq \left( \int_0^1 \left| s^{v+1} - \mu(v + 1)s \right| ds \right)^{1 - \frac{1}{p}} \times \left\{ \int_0^1 \left[ \mu(v + 1) s - s^{v+1} \right] ds \right\}^{\frac{1}{p}} \times \left\{ \int_0^1 \left[ s^{v+2} - \mu(v + 1)s \right] ds \right\}^{\frac{1}{p}}
+ |g''(c)|^p \left[ \int_0^1 \left[ \mu(v + 1) s - \mu(v + 1)s - s^{v+1} + s^{v+2} \right] ds \right]^{\frac{1}{p}} \times \left\{ \int_0^1 \left[ \mu(v + 1) s - s^{v+2} - \mu(v + 1)s + \mu(v + 1)s \right] ds \right\}^{\frac{1}{p}}
= A_4^{-\frac{1}{p}}(v, \mu) \left[ A_5(v, \mu)|g''(c)|^p + A_6(v, \mu)|g''(c)|^p \right]^{\frac{1}{p}}.
\]

Using the same method, we get
\[
|J_{g_2}(z, \mu, v; c, d)| \leq A_4^{-\frac{1}{p}}(v, \mu) \left[ A_5(v, \mu)|g''(d)|^p + A_6(v, \mu)|g''(d)|^p \right]^{\frac{1}{p}}.
\]
On the other hand,
\[
\int_0^1 |s^{v+1} - \mu (v+1)s|ds = \int_0^{\mu(v+1)} \left[ \frac{1}{\mu(v+1)} \right]^{\frac{1}{v+2}} |s - s^{v+1}|ds + \int_{\mu(v+1)}^1 \left[ s^{v+1} - \mu (v+1)s \right]ds
= \frac{v}{v+2} |\mu(v+1)|^{\frac{1}{v+2}} - \frac{\mu(v+1)}{2} + \frac{1}{v+2}.
\]
(21)

By submitting (19), (20) and (21) into (18), the proof is completed.

Remark 3.11. Assume that
\[
\int_0^1 \left| I_s(c, \mu, v; c, d) \right|^2 ds \leq \frac{(d-c)^{v+1}}{2} \left( \frac{1}{v+1} - \mu \right) \left| g'(d) - g'(c) \right| + \frac{(d-c)^{v+2}}{2(v+1)} \left[ |g''(c)| + |g''(d)| \right] |[A_5(v, \mu) + A_6(v, \mu)] |.
\]

Remark 3.9. If \( q = 1 \), then
\[
\left| \frac{1}{2} I_{s} (c, \mu, v; c, d) + I_{s} (d, \mu, v; c, d) \right| \leq \frac{(d-c)^{v+1}}{2} \left( \frac{1}{v+1} - \mu \right) \left| g'(d) - g'(c) \right| + \frac{(d-c)^{v+2}}{2(v+1)} \left[ |g''(c)| + |g''(d)| \right] |[A_5(v, \mu) + A_6(v, \mu)] |.
\]

Remark 3.10. In Theorem 3.8, we set \( z = \frac{c+d}{2} \), \( \mu = 0 \). It follows the inequality (17) that
\[
\left| I_{s} (c, \mu, v; c, d) \right| \leq \frac{1}{v+1} \left( \frac{1}{v+2} \right)^{\frac{1}{v+2}} \left( \frac{1}{v+3} \right)^{\frac{1}{v+2}} \left( \frac{d-c}{2} \right)^{v+2} \left\{ \left[ g'' \left( \frac{c+d}{2} \right) \right]^{p} + \frac{1}{v+2} |g''(c)|^{p} + \frac{1}{v+2} |g''(d)|^{p} \right\}.
\]

Remark 3.11. Assume that \( |g''(z)| \leq M \), \( z \in [c, d] \) and \( M > 0 \). Then
\[
\left| \frac{1}{2} I_{s} (d, \mu, v; c, d) + I_{s} (c, \mu, v; c, d) \right| \leq \frac{vM(d-c)^{v+2}}{2(v+1)(v+2)}.
\]

Proof. By using mean value theorem for \( g' \), we have
\[
\left| \frac{d-c)^{v+1}}{2} \int_0^1 (s^v - \mu) |g'(sd + (1-s)c) - g'(sc + (1-s)d)|ds \right|
= \frac{(d-c)^{v+1}}{2} \left| \int_0^1 (s^v - \mu) |g'(sd + (1-s)c) - g'(sc + (1-s)d)|ds \right|
\leq \frac{(d-c)^{v+1}}{2} \left| \int_0^1 (s^v - \mu) M |(sd + (1-s)c) - (sc + (1-s)d)|ds \right|
= \frac{(d-c)^{v+1}}{2} \left| \int_0^1 (s^v - \mu) (2s - 1)ds \right|
= \frac{vM(d-c)^{v+2}}{2(v+1)(v+2)}.
\]
The proof is ok.

4 Applications

For \( v, \omega, v \neq \omega \), we consider special means of real numbers as follows:
\[
H(v, \omega) = \frac{2}{\frac{1}{v} + \frac{1}{\omega}}, \quad v, \omega \in R \setminus \{0\}
\]
\[
A(v, \omega) = \frac{v + \omega}{2}, \quad v, \omega \in R,
\]
\[
L(v, \omega) = \frac{\omega - v}{\ln |\omega| - \ln |v|}, \quad |\omega| \neq |v|, v \omega \neq 0.
\]
\[
L_n(v, \omega) = \left[ \frac{\omega^{n+1} - v^{n+1}}{(n+1)(\omega - v)} \right]^\frac{1}{n}, \quad n \in Z \setminus \{-1, 0\}, v, \omega \in R, v \neq \omega
\]

Let \( c, d \in R^+ \) and \( c < d \).
Proposition 1.
\[ |A(e^c, e^d) - L(e^c, e^d)| \leq \frac{1}{6}(d - c)^2(e^c + e^d). \]

**Proof.** Choose \( g'(\ln z) = z, \mu = 1 \) and \( v = 1 \). One can apply Remark 3.5 to complete the proof.

Proposition 2.
\[ |H^{-1}(e^c, e^d) - L(e^{-c}, e^{-d})| \leq \frac{1}{6}(d - c)^2(e^{-c} + e^{-d}). \]

**Proof.** Choose \( c^{-1} > d^{-1}, f'(\ln z) = \frac{1}{z}, \mu = 1 \) and \( v = 1 \), one can apply Remark 3.5 to obtain the results.

Proposition 3.
\[ |A(e^{(n+1)c}, e^{(n+1)d}) - \frac{e^d - e^c}{d - c} L_n^m(e^c, e^d)| \leq \frac{1}{6}(d - c)^2(n + 1)[e^{(n+1)c} + e^{(n+1)d}]. \]

**Proof.** Choose \( f'(\ln z) = z, \mu = 1 \) and \( v = 1 \), one can apply Remark 3.5 to obtain the results.

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**References**