STOCHASTIC STABILITY OF NETWORKED CONTROL SYSTEMS WITH TIME-VARYING SAMPLING PERIODS

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Abstract. This paper studies the problem of stochastic stability for networked control systems with both network-induced delay and transmitted data dropout by using time-varying sampling period method, where the number of data packet dropout is driven by a finite state Markov chain. Based on Lyapunov approach, sufficient conditions for the stochastic stability of the networked control system are derived and stabilization controller is constructed in terms of linear matrix inequalities correspondingly. A simulation example is worked out to illustrate the effectiveness of our result.

Key Words. Networked control systems, stochastic stability, time-varying sampling period, Lyapunov method, and linear matrix inequalities.

1. Introduction

Networked control systems (NCSs) are feedback control systems whose feedback paths are implemented by a real-time network. Recently, much attention has been paid to the study of stability analysis and controller design of networked control systems, due to their low cost, reduced weight and power requirements, simple installation and maintenance, high reliability, and so on [1].

A basic problem in NCSs is the stability of the system. Network-induced delay and data packet dropout are the main characteristic in NCSs. In real-time control systems, delay and packet dropouts will degrade the performance of control systems and even make systems unstable, so it is significant to overcome the adverse influences of time delay and packet dropout. There are many studies on stochastic stability with network-induced delay and packet dropout [2-6], in which the sampling period is assumed to be constant.

In NCSs, constant sampling period is usually adopted, if constant sampling period is adopted, sampling period should be large enough to avoid network congestion when the network is occupied by the most users, so network bandwidth cannot be sufficiently used when the network is idle. Recently, there are a number of papers considering the problem of varying sampling period of control systems [7-11]. In [7], the problem of designing $H_\infty$ controllers for NCSs with both network-induced time delay and packet dropout by using active varying sampling period method is studied. In the work of [8,9], the stability problem of digital feedback control systems with time-varying sampling periods is discussed. In [10], the authors present an interval model of networked control systems with time-varying sampling periods and time-varying network-induced delays and discuss the problem of stability of networked control systems. But [7-10] did not study the problems of stochastic
stochastic stability of networked control systems. In [11], the stochastic stability of NCSs with time-varying sampling periods and delays driven by two Markov chains are discussed, but [11] did not consider the problem of data packet dropout.

In [12], an iterative approach is proposed to model networked control linear systems with arbitrary but finite data packet dropout as switched linear systems. [6] extends the results in [12] to a stochastic setting. In this paper, we present an extension of the result in [6] to time-varying sampling period. We deal with the problem of stochastic stability analysis for networked control systems with time-varying sampling periods and data packet dropout driven by a finite state Markov chain. For a given state-feedback controller, the stochastic stabilization of NCSs is investigated by stochastic Lyapunov function together with linear matrix inequality method. The paper is organized as follows: In Section 2, we present the model of NCSs with data dropout and network-induced delay by using time-varying sampling period method. Section 3 provides sufficient conditions for stochastic stability of the NCSs. In order to illustrate the results, an example is presented in Section 4. Finally, the conclusion is presented.

2. Preliminaries and problem statement

The structure of the considered NCS is shown in Fig.1, where the plant is described by the following linear time-invariant system model

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

where \( x(t), u(t) \) are the state vector and control input vector respectively; \( A, B \) are known constant matrices. Throughout this paper, matrices, if not explicitly stated, are assumed to have appropriate dimensions.

For NCSs, the shorter the sampling period, the better the system performance, however, short sampling period will increase the possibility of network congestion. If constant sampling period is adopted, sampling period should be large enough to avoid network congestion, so network bandwidth cannot be sufficiently used when the network is idle.

In the following, we will choose the time-varying sampling period method to make model networked control linear systems with finite data packet dropout by using an iterative approach [12]. Thus, not only the network bandwidth can be made full use, but also the analysis and design of NCSs with time delay and packet dropout can be simplified.
Suppose $t_k$ is the $k$th sampling instant, $h_k$ is the length of the $k$th sampling period, the instant that the control input $u_k$ reaches actuator is $\tilde{t}_k$. When the network is idle, denote the smallest sampling period as $h_{\text{min}}$, and when the network is occupied by the most users, define the biggest as $h_{\text{max}}$, then $h_k = t_{k+1} - t_k \in [h_{\text{min}}, h_{\text{max}}]$.

Partition $[h_{\text{min}}, h_{\text{max}}]$ into $l$ equidistant small intervals ($l$ is a positive integer), then the next sampling instant $t_{k+1}$ can be chosen as

$$t_{k+1} = \begin{cases} t_k + d_1 & \tilde{k} \in [d_1, d_2) \\ t_k + h_{\text{max}} & \tilde{k} \geq t_k + h_{\text{max}} \end{cases}$$

where $d_1 = h_{\text{min}} + a(h_{\text{max}} - h_{\text{min}})/l$, $d_2 = h_{\text{min}} + (a+1)(h_{\text{max}} - h_{\text{min}})/l$, $a = 0, 1, \ldots, l-1$.

Then $h_k = h_{\text{min}} + b(h_{\text{max}} - h_{\text{min}})/l$ that is the sampling period $h_k$ switches in the finite set $\mathcal{d} = \{ h_{\text{min}}, (h_{\text{max}} - h_{\text{min}})/l, \ldots, h_{\text{max}} \}$.

Suppose the controller and the actuator are combined into one event-driven unit, the sensor is both clock-driven and event-driven. At the same time, we suppose constant delay between the sensor and the combined unit is $\tau_k = h_k$, which makes sure there is only one step delay between the sensor and the combined unit.

The discrete time representation of (1) can be described as follows

$$x(k+1) = \Phi_k x(k) + \Gamma_k u(k)$$

where

$$\Phi_k = e^{Ah_k}, \Gamma_k = \int_0^{h_k} e^{As}dA B.$$ 

In this paper the static state feedback controller is described as

$$u(k) = K \bar{x}(k)$$

where $K$ is the state feedback gain matrix to be designed, $\bar{x}(k)$ is the state measurement that is successfully transmitted over the network, that is

$$\bar{x}(k) = \begin{cases} x_{k-1} & \text{the packet } x_{k-1} \text{ is successfully transmitted,} \\ \tilde{x}_{k-1} & \text{the packet } x_{k-1} \text{ is dropped.} \end{cases}$$

Then, the closed-loop system from (2) and (3) can be expressed as

$$x(k+1) = \Phi_k x(k) + \Gamma_k K \bar{x}(k).$$

When a sensor data is successfully sent to the controller through the communication link, it will be put into a single register and substitute the old data. The controller reads out the content of the register $\bar{x}(k)$ and utilizes the data to compute the new control input, which will be applied to the plant.

We assume that the packet containing $x(0)$ with one step delay is transmitted to the controller successfully, then

$$x(1) = \Phi_0 x(0), x(2) = \Phi_1 x(1) + \Gamma_1 K x(0).$$

In the next step, if the data packet containing $x(1)$ is transmitted to the controller successfully, then $x(3) = \Phi_2 x(2) + \Gamma_2 K x(1)$, otherwise $x(3) = \Phi_2 x(2) + \Gamma_2 K x(0)$. 

Suppose that the successive successfully transmitted instants of \( k \) are \( 0 = k_0 < k_1 < \ldots < k_j < \ldots \), then the closed loop system of the NCSs may be described by

\[
\begin{align*}
x(k_1) & = (\Phi_{b}^{k_1-k_0-1}\Phi_{h_{b0}} + \Phi_{b}^{k_1-k_0-2}\Gamma_{b}K + \cdots + \Gamma_{b}K)x(k_0), \\
x(k_2) & = (\Phi_{b}^{k_2-k_1-1}\Phi_{h_{b1}} + \Phi_{b}^{k_2-k_1-2}\Gamma_{b}K + \cdots + \Gamma_{b}K)x(k_1) \\
& + \Phi_{b}^{k_2-k_1-1}\Gamma_{h_{b1}}Kx(k_0), \\
& \vdots \\
x(k_{j+1}) & = (\Phi_{b}^{k_{j+1}-k_{j-1}}\Phi_{h_{b{j-1}}} + \Phi_{b}^{k_{j+1}-k_{j-2}}\Gamma_{b}K + \cdots + \Gamma_{b}K)x(k_j) \\
& + \Phi_{b}^{k_{j+1}-k_{j-1}}\Gamma_{h_{b{j-1}}}Kx(k_{j-1})
\end{align*}
\]

Let \( d_{j+1} = k_{j+1} - k_j \), which is referred to as one successive transmission period, in other words, the number of data packet dropout is \( d_{j+1} - 1 \), then

\[
x(k_{j+1}) = A(d_{j+1})x(k_j) + B_1(d_{j+1})Kx(k_j) + B_2(d_{j+1})Kx(k_{j-1})
\]

where

\[
\begin{align*}
\Phi_{h_{b{j}}} & = e^{A_{h_{b{j}}}}, \quad \Phi_{b} = e^{A_{h_{b\text{max}}}}; \\
\Gamma_{h_{b{j}}} & = \int_{0}^{h_{b{j}}} e^{As}ds, \quad \Gamma_{b} = \int_{0}^{h_{b\text{max}}} e^{As}dsB, \\
A(d_{j+1}) & = \Phi_{b}^{k_{j+1}-k_{j-1}}\Phi_{h_{b{j-1}}}B_1(d_{j+1}) = \Phi_{b}^{k_{j+1}-k_{j-2}}\Gamma_{b}B + \cdots + \Gamma_{b}, \\
B_2(d_{j+1}) & = \Phi_{b}^{k_{j+1}-k_{j-1}}\Gamma_{h_{b{j-1}}}
\end{align*}
\]

Define \( x(k_{j+1}), x(k_j), x(k_{j-1}) \) as \( \xi(j+1), \xi(j), \xi(j-1) \) respectively, then

\[
\xi(j+1) = A(d_{j+1})\xi(j) + B_1(d_{j+1})K\xi(j) + B_2(d_{j+1})K\xi(j-1).
\]

Let \( z(j) = [\xi^T(j), \xi^T(j-1)K^T]^T \) be the augmented state vector, the system (6) can be written as

\[
z(j+1) = \tilde{A}(d_{j+1})z(j), \quad j = 0, 1, 2, \ldots.
\]

where

\[
\tilde{A}(d_{j+1}) = \left( \begin{array}{cc} A(d_{j+1}) + B_1(d_{j+1})K & B_2(d_{j+1}) \\ 0 & K \end{array} \right),
\]

let

\[
E(d_{j+1}) = \left( \begin{array}{cc} A(d_{j+1}) & B_2(d_{j+1}) \\ 0 & 0 \end{array} \right), \quad F(d_{j+1}) = \left( \begin{array}{c} B_1(d_{j+1}) \\ I \end{array} \right), \quad \tilde{K} = \left( \begin{array}{cc} K & 0 \end{array} \right),
\]

then

\[
\tilde{A}(d_{j+1}) = E(d_{j+1}) + F(d_{j+1})\tilde{K}.
\]

It is assumed that the bound of data packet dropout is \( M - 1 \) and \( d_j \in S = \{1, 2, \ldots, M\} \). For a given \( d_j = \alpha \in S \), we can write

\[
A(\alpha) = \Phi_{b}^{\alpha-1}\Phi_{h_{b\alpha}}, \quad B_1(\alpha) = \Phi_{b}^{\alpha-2}\Gamma_{b} + \cdots + \Gamma_{b}, \quad B_2(\alpha) = \Phi_{b}^{\alpha-1}\Gamma_{h_{b\alpha}}.
\]
3. Stochastic stability of NCS under state feedback

In this section, sufficient conditions will be presented for the stochastic stability of NCS (2) with (3).

In this paper, we suppose \( \{ d_j, j = 0, 1, 2 \cdots \} \) is a finite state Markov chain with transition probability \( r_{\alpha \beta} = P(d_{j+1} = \beta | d_j = \alpha), \forall \alpha, \beta \in S \) and \( \sum_{\beta=1}^{M} r_{\alpha \beta} = 1 \).

Definition 1. The unforced stochastic parameter system (7) is stochastic stable if for every initial state \( z_0 = z(0) \) and initial distribution \( d_0 \in S \), there is a finite matrix \( Q > 0 \) such that

\[
\varepsilon(\sum_{j=0}^{\infty} \| z(j) \|^2 | d_0) < z_0^T Q z(0).
\]

The following lemma will be needed in establishing our main result.

Lemma 1. [13] Consider the linear matrix equation \( D + AXB = 0 \).

1) This equation has a solution \( X \) if and only if \( D - A^+ DB^+ B = 0 \).
2) All solutions are parameterized by \( X = -A^+ DB^+ + R - A^+ DBB^+ \),

where \( R \) is an arbitrary matrix with compatible size, \( A^+ \) denotes the Moore-Penrose pseudo-inverse of matrix \( A \).

In Lemma 1, if \( A = I \) and \( B \) is of full row rank, then the solution of the linear matrix equation \( D + XB = 0 \) is given by \( X = -DB^+ = -DB^T (BB^T)^{-1} \).

Theorem 1. If there exist positive symmetric matrices \( X(1) > 0, \ldots, X(M) > 0, G > 0 \), a matrix \( Y \) and \( G - X(\alpha), \forall \alpha \in S \) is of full rank such that the following LMI holds for every feasible values of \( d_j \) and \( h_k \)

\[
\begin{pmatrix}
X(\alpha) - G & G^T \\
\Xi & -\Lambda
\end{pmatrix} < 0
\]

where

\[
\Xi = \begin{bmatrix}
\sqrt{\alpha_1}(G^T E^T(1) + Y^T F^T(1)) & \cdots & \sqrt{\alpha_M}(G^T E^T(M) + Y^T F^T(M)) \end{bmatrix}^T,
\]

\[
\Lambda = \text{diag} \{X(1), \ldots, X(M)\},
\]

\[
G_{11} \in R^{n \times n}, G_{12} \in R^{n \times m}, G_{22} \in R^{m \times m},
\]

then NCS (2) with (3) is stochastic stable and if the LMI (8) is solvable, the controller parameter \( K \) is given by

\[
K = Y \left( G_{11}^T G_{12}^T + G_{12} G_{11}^T \right)^{-1}.
\]

Proof: We first show that the system (7) is stochastic stable. To this end, let us consider the stochastic Lyapunov function

\[
V(z(j), d_j) = z^T(j) P(d_j) z(j)
\]

where

\[
P(\alpha) = X^{-1}(\alpha), d_j = \alpha \in S.
\]
then
\[
\begin{align*}
\epsilon \{[V(z(j + 1), d_{j+1}) - V(z(j), d_j)]d_j = \alpha & \\
= & \epsilon \{V(z(j + 1), d_{j+1})|d_j = \alpha\} - V(z(j), d_j) \\
= & \sum_{\beta=1}^{M} r_{\alpha\beta}(z^T(j + 1)P(\beta)z(j + 1)) - z^T(j)P(\alpha)z(j) \\
= & z^T(j)(\sum_{\beta=1}^{M} r_{\alpha\beta} \tilde{A}^T(\beta)P(\beta)\tilde{A}(\beta) - P(\alpha))z(j) \\
= & z^T(j)\Omega_\alpha z(j),
\end{align*}
\]
where
\[
\Omega_\alpha = \sum_{\beta=1}^{M} r_{\alpha\beta} \tilde{A}^T(\beta)P(\beta)\tilde{A}(\beta) - P(\alpha).
\]
Since \(G - X(\alpha)\) is of full rank, so
\[(G - X(\alpha))^TX^{-1}(\alpha)(G - X(\alpha)) = G^TX^{-1}(\alpha)G - G^T + X(\alpha) > 0.\]
We have
\[(11)\]
\[X(\alpha) - G - G^T > -G^TX^{-1}(\alpha)G.\]
Furthermore, noting that \((G_{11} \ G_{12}) \in R^{n\times(n+m)}\) is of full row rank and
\[(G_{11} \ G_{12})^+ = (G_{11}^T G_{12}^T (G_{11} G_{11}^T + G_{12} G_{12}^T)^{-1},\]
then from (9) and Lemma 1 we get \(K \tilde{G} = K( G_{11} \ G_{12})^+ = Y,\) from which (8)
and (11) it follows that
\[(12)\]
\[
\begin{pmatrix}
-G^TX^{-1}(\alpha)G & \Pi^T \\
\Pi & -Y
\end{pmatrix} < 0
\]
where
\[
\Pi = [\sqrt{\alpha_1}G^T\tilde{A}^T(1) \ldots \sqrt{\alpha_M}G^T\tilde{A}^T(M)], Y = diag\{P^{-1}(1), \ldots, P^{-1}(M)\}.
\]
By Schur complement, (12) is equivalent to \(G^T\Omega_\alpha G < 0,\) which implies that
\[(13)\]
\[\Omega_\alpha < 0.
\]
Therefore, we have
\[(14)\]
\[\epsilon \{V(z(j + 1), d_{j+1}) - V(z(j), d_j)\} \leq -\lambda_{\min}(-\Omega_\alpha)\|z(j)\|^2 \leq -\eta\|z(j)\|^2
\]
where \(\eta = \inf\{\lambda_{\min}(-\Omega_\alpha), \alpha \in S\} > 0.\) For any integer \(N \geq 1,\) we have
\[(15)\]
\[\epsilon \{V(z(N + 1), d_{N+1}) - V(z(0), d_0)\} < -\eta \epsilon \{\sum_{j=0}^{N} \|z(j)\|\}
\]
then
\[(16)\]
\[\epsilon \{\sum_{j=0}^{N} \|z(j)\|^2\} < 1/\eta[\epsilon \{V(z(0), d_0)\} - \epsilon \{V(z(N + 1), d_{N+1})\}] \leq 1/\eta[\epsilon \{V(z(0), d_0)\}].\]
Let $N \to \infty$ and $Q = (1/\eta)I$, then 
\[
\varepsilon \left\{ \sum_{j=0}^{\infty} \| z(j) \|^2 \right\} < 1/\eta \varepsilon \left\{ V(z(0), d_0) \right\} = z^T(0) Q z(0).
\]

From Definition 1, the system (7) is stochastic stable.

Now we show that the system (2) with (3) is also stochastic stable. In fact, note that the state $x(k)$ of the system (2) with (3) at the instant $k$, $k_j \leq k < k_{j+1}$, can be expressed by $x(k) = \hat{A}(\alpha) z(j)$, where for $\alpha \in S$,
\[
\hat{A}(\alpha) = \left( \Phi_b^{-1} \Phi_{h_{j+1}} + \Phi_b^{2-1} \Gamma_b + \cdots + \Gamma_b \Phi_b^{-1} \Gamma_h_{j+1} \right).
\]

Denote $\rho = \max_{\alpha \in S} \| \hat{A}(\alpha) \|$, then $\| x(k) \| \leq \rho \| z(j) \|$, from which and (16) it follows that
\[
\varepsilon \left\{ \sum_{j=0}^{N} \| x(k) \|^2 \right\} \leq \rho^N \varepsilon \left\{ \sum_{j=0}^{N} \| z(j) \|^2 \right\} < 1/\eta \rho^N \varepsilon \left\{ (z(0), d_0) \right\},
\]

Let $N \to \infty$ and $\bar{Q} = (\rho^N/\eta)I$, then
\[
\varepsilon \left\{ \sum_{j=0}^{N} \| x(k) \|^2 \right\} < \bar{Q} \varepsilon \left\{ V(z(0), d_0) \right\} = z^T(0) \bar{Q} z(0) = \bar{x}^T(0) \bar{Q} \bar{x}(0).
\]

This completes the proof.

4. Numerical example

To illustrate the theoretical results in previous section, we present an open loop unstable system as follows[7]
\[
\dot{z}(t) = \begin{pmatrix} -0.3901 & 0.8855 \\ 1.4900 & -0.9821 \end{pmatrix} z(t) + \begin{pmatrix} -0.5359 \\ 1.0727 \end{pmatrix} u(t)
\]

Suppose the minimum sampling period of sensor is 0.05$s$, the maximum sampling period is 0.2$s$ and suppose the feasible values of sampling period are $h_1 = 0.05$s, $h_2 = 0.1$s, $h_3 = 0.2$s. For simplicity, suppose the maximum transmission period $M = 3$ and $\{ d_j = 0, 1, \cdots \}$ is a Markov chain(Fig. 2) with transition probability described by
\[
R = \begin{pmatrix} 0.15 & 0.49 & 0.36 \\ 0.16 & 0.81 & 0.03 \\ 0.64 & 0.09 & 0.27 \end{pmatrix}.
\]
Solving the linear matrix inequality (8), we obtain the solution given by

\[ X(1) = \begin{pmatrix} 11.9200 & -17.9600 & -5.5293 \\ -17.9600 & 28.8711 & 2.2776 \\ -5.5293 & 2.2776 & 52.1498 \end{pmatrix} \]

\[ X(2) = \begin{pmatrix} 11.7887 & -17.4545 & -6.2377 \\ -17.4545 & 27.6423 & 2.8506 \\ -6.2377 & 2.8506 & 52.9251 \end{pmatrix} \]

\[ X(3) = \begin{pmatrix} 10.5961 & -15.7199 & -5.4330 \\ -15.7199 & 24.9373 & 2.5603 \\ -5.4330 & 2.5603 & 50.1099 \end{pmatrix} \]

\[ G = \begin{pmatrix} 14.2711 & -21.7468 & -5.8865 \\ -21.7468 & 35.1971 & 2.4332 \\ -5.8865 & 2.4332 & 64.0816 \end{pmatrix} \]

\[ Y = \begin{pmatrix} -4.3812 & -1.1508 & 15.6079 \end{pmatrix} \]

According to Theorem 1, we know that system (7) is stochastic stable, where the gain matrix is given by

\[ K = \begin{pmatrix} -3.6300 & -2.2497 \end{pmatrix} \]

If constant sampling period is adopted, \( h_3 \) should be chosen as the sampling period to avoid network congestion when the network is occupied by the most users, the controller gain obtained is \( K = \begin{pmatrix} -4.3507 & -2.6098 \end{pmatrix} \), the plant state response and controlled output are pictured in Fig.3(a). If the initial sampling period is \( h_1 \), then at the instant \( 2s \), the sampling period switches to \( h_3 \), and the controller gain \( K = \begin{pmatrix} -3.6300 & -2.2497 \end{pmatrix} \) is adopted, the plant state response and controlled output are pictured in Fig.3(b). Fig.3 illustrates the NCS can reach stabilization even if there exist time delays and packet dropouts.

5. Conclusion

The stochastic stability of NCSs with network-induced delay and data packet dropout by using time-varying sampling period method is studied, where the number of data packet dropout is modeled as a finite state Markov chain. Sufficient conditions for stochastic stability of NCSs with data dropout and network-induced delay are given. The simulation results have illustrated the effectiveness of the method.

References


Figure 3. The trajectory of system states


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