Convex $p$-partitions of bipartite graphs

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Abstract

A set of vertices $X$ of a graph $G$ is convex if it contains all vertices on shortest paths between vertices of $X$. We prove that for fixed $p \geq 1$, all partitions of the vertex set of a bipartite graph into $p$ convex sets can be found in polynomial time.

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1. Introduction

Given a graph $G = (V, E)$, a set $X$ of vertices is called convex if $G[X]$, the graph induced by $X$, contains all shortest paths between any two of its vertices. (All graphs here are undirected and simple.) The notion probably first appeared in [8], see also [10], and later became also known as geodesic convexity, or $d$-convexity, in order to distinguish it from different notions of convexity in graphs and other combinatorial structures (see [7] for an early overview). The book [11] gives an up-to-date survey of results on convexity in graphs.

One of the approaches to convexity in graphs comes from the viewpoint of computational complexity. Clearly, computing the distances between all
pairs, one can decide in polynomial time if a given set of vertices is convex. To determine the size of a largest convex set not covering the whole graph, however, is an NP-complete problem, even for bipartite graphs, albeit linear for cographs \[5\]. The same phenomenon occurs (NP-completeness even for bipartite graphs, but linearity for cographs) if we wish to determine related invariants such as the hull number and the geodetic number of a graph \[1,4,6\].

We focus here on the notion of a convex \(p\)-partition of a graph, that is, a partition of the vertex set into \(p\) convex sets. For instance, any graph on \(n\) vertices containing a matching of size \(m\) has a convex \((n-m)\)-partition, and trivially, any graph has a convex 1-partition. Deciding whether a graph has a convex \(p\)-partition, for fixed \(p \geq 2\), is NP-complete for arbitrary graphs, and linear time solvable for cographs \[2\]. Also, chordal graphs have convex \(p\)-partitions for all \(p \geq 1\) \[2\].

In view of the above described panorama, it was conjectured in \[11\] that also for bipartite graphs, it should be NP-complete to decide whether they have a convex \(p\)-partition. We show that, for any fixed \(p \geq 1\), this is not the case. More precisely, we prove that for \(p \geq 1\), all convex \(p\)-partitions of a bipartite graph can be enumerated in polynomial time. This extends a recent result of Glantz and Mayerhenke \[9\], who prove the same for the case \(p = 2\). They also showed that all convex 2-partitions of a planar graph can be found in polynomial time.

2. Bipartite graphs with convex \(p\)-partitions

We start by reproving the result for bipartite graphs from \[9\] in a slightly different way. At the same time, this will serve as a base for the general case. We denote the distance between two vertices \(u\) and \(v\) in a graph \(G\) by \(d_G(u,v)\).

**Lemma 1.** Given a convex set \(C\) in a connected bipartite graph \(G\), and an edge \(uv\) with \(u \in C\), \(v \notin C\) we have that \(d_G(u',u) < d_G(u',v)\), for each \(u' \in C\).

**Proof.** Suppose otherwise. Observe that since \(G\) is bipartite, \(d_G(u',u) \neq d_G(u',v)\), and thus we may assume \(d_G(u',u) > d_G(u',v)\). Then there is a shortest path \(P\) from \(u'\) to \(v\) not containing \(u\). Extending \(P\) to \(u\) through the edge \(vu\), gives a shortest path from \(u'\) to \(u\), a contradiction, as \(u\) and \(u'\) lie in the convex set \(C\), but \(v \notin C\). \(\square\)
Let \( e = uv \) be an edge of \( G \) and denote by \( X_{uv} \) the set of vertices that are closer to \( u \) than to \( v \). If \( G \) is a connected bipartite graph, then \( V \) is the disjoint union \( X_{uv} \cup X_{vu} \). From Lemma 1 we get the following corollaries.

**Corollary 2.** Let \( uv \) be an edge of a connected bipartite graph \( G \). If \( C \) is a convex set containing \( u \) and not containing \( v \), then \( C \subseteq X_{uv} \).

**Corollary 3.** Let \( G = (V, E) \) be a connected bipartite graph, with a partition of \( V \) into convex sets \( X_1, X_2 \). Let \( uv \in E \), with \( u \in X_1 \) and \( v \in X_2 \). Then \( X_1 = X_{uv} \) and \( X_2 = X_{vu} \).

From the previous corollary it is direct that there are at most \( |E| \) convex 2-partitions and that we can enumerate all convex 2-partitions in polynomial time.

**Proposition 4.** We can enumerate in polynomial time all convex 2-partitions of a connected bipartite graph.

We now prove that for fixed \( p \geq 3 \) we can enumerate in polynomial time all convex \( p \)-partitions of a connected bipartite graph. To this purpose we extend the idea present in Corollary 3.

We write \([p]\) for the set \( \{1, \ldots, p\} \). For a set \( F \) of edges, let \( V(F) \) denote the set of all endvertices of edges of \( F \).

Given a convex \( p \)-partition \( \mathcal{X} = \{X_1, X_2, \ldots, X_p\} \) of a graph \( G = (V, E) \), we call a pair \((F, \phi)\) an \( \mathcal{X} \)-skeleton, if \( F \subseteq E \) and \( \phi : V(F) \rightarrow [p] \) satisfy the following:

- all edges of \( F \) go between distinct parts of \( \mathcal{X} \);
- if there is at least one edge in \( E \) between \( X_i \) and \( X_j \), then there is exactly one edge of \( F \) between \( X_i \) and \( X_j \);
- \( \phi(v) = i \) if \( v \in X_i \).

Note that the first two conditions might be equivalently expressed by saying that after contracting the sets \( X_i \) and deleting all remaining edges that are not in \( F \), we are left with a (simple) graph \( H_{(F,\phi)} \) whose edges represent the edges of \( G \) that cross the partition. The last condition says \( \phi \) assigns the same colour to all vertices of \( V(F) \) that become identified in \( H_{(F,\phi)} \).

Note that for a connected graph \( G \) the second condition implies that \( V(F) \cap X_j \neq \emptyset \), for each \( j \in [p] \). Then, the third condition implies that \( \phi \) is a surjective function.
Theorem 5. Let $G = (V, E)$ be a connected bipartite graph, let $F \subseteq E$ and let $\phi : V(F) \rightarrow \mathbb{N}$. If $G$ has a convex $p$-partition with skeleton $(F, \phi)$, then this partition is unique. We can find such partition or show it does not exist in polynomial time.

Proof. Define lists $L(u)$ for each vertex $u \in V$ by setting

$$L(u) := \mathbb{N} - \{\phi(w) : u \in X_{vw} \text{ for some } vw \in F\}.$$ 

For each pair of vertices $u$ and $w$ define $I[u, w]$ as the set of vertices in shortest paths between $u$ and $w$. For each vertex $u$, define

$$L'(u) := L(u) - \{\phi(w) : w \in V(F) \text{ and } \phi(w) \notin L(v) \text{ for some } v \in I[u, w]\}.$$ 

We will prove that if $G$ has a convex $p$-partition $\mathcal{X} = \{X_1, \ldots, X_p\}$ with skeleton $(F, \phi)$, then, for each $i \in \mathbb{N}$,

$$L'(u) = \{i\} \text{ for every } u \in X_i. \quad (1)$$

We first observe that

$$i \in L(u) \text{ for every } u \in X_i. \quad (2)$$

Otherwise, there are $u \in X_i$ and $vw \in F$ such that $\phi(w) = i$ and $u \in X_{vw}$. Hence, $u, w \in X_i$ and $v \in I[u, w]$. Since $X_i$ is convex, $v \in X_i$, contradicting the fact that the edge $vw$ of $F$ must join distinct parts of $\mathcal{X}$. This contradiction proves (2).

Moreover,

$$i \in L'(u) \text{ for every } u \in X_i. \quad (3)$$

Otherwise, by (2), there are $u \in X_i$, $w \in V(F)$ and $v \in I[u, w]$ such that $\phi(w) = i$ and $i \notin L(v)$. Now, on the one hand, since $u, w \in X_i$ and $v \in I[u, w]$, the convexity of $X_i$ implies that $v \in X_i$. On the other hand, since $i \notin L(v)$, we know by (2) that $v \notin X_i$. This contradiction proves (3).

Next, we now show that, for each $j \in \mathbb{N}$,

$$\text{if } v'w' \in E, \text{ with } w' \in X_j \text{ and } v' \notin X_j, \text{ then } j \notin L(v'). \quad (4)$$

This is immediate if $v'w' \in F$, by the definition of $L(v')$. Otherwise, there is $vw \in F$ such that $w \in X_j$, and $v, v' \in X_i$ for some $i \neq j$. Lemma 1 applied
to the convex set $X_i$ and the edge $vw$ yields that $d(v', v) < d(v', w)$; i.e., $v' \in X_{vw}$. Thus, the definition of $L(v')$ gives that $j \notin L(v')$, proving (4).

We now prove (1). Consider $u \in X_i$ and $j \in [p] - \{i\}$. Let $w \in V(F) \cap X_j$ (as $G$ is connected, this set is non-empty) and let $P$ be a shortest path between $u$ and $w$. By construction, $P$ has some edge $vw'$ such that $v \notin X_j$ and $w' \in X_j$. By (4), we have that $j \notin L(v)$. As $v \in I[u, w]$, and as $\phi(w) = j$, the definition of $L'(u)$ implies that $j \notin L'(u)$. This completes the proof of (1).

Therefore, a convex $p$-partition with skeleton $(F, \phi)$ exists if and only if the following conditions hold: (i) $|L'(u)| = 1$ for each vertex $u$ of $G$; (ii) the parts of the corresponding partition are convex. The time needed to find a convex $p$-partition with skeleton $(F, \phi)$ is dominated by the time needed to compute the distance function of the graph. Indeed, once the distance function is known, the construction of each list $L(u)$ takes constant time and the construction of the each list $L'(u)$ takes linear time.

When given a connected bipartite graph $G$ and an integer $p$, we can decide whether $G$ has a convex $p$-partition as follows. We first guess a candidate skeleton $(F, \phi)$ and then, by using Theorem 5, we compute in polynomial time the unique (if any) partition $\{X_1, \ldots, X_p\}$ associated to $(F, \phi)$. The choices for $(F, \phi)$ are bounded from above by a function that depends only on $p$. In fact, if $(F, \phi)$ is a skeleton of some partition, then it must satisfy the following properties.

- The size of $F$ satisfies $|F| \in \{p - 1, \ldots, \binom{p}{2}\}$.
- The function $\phi$ is surjective.
- Identifying all vertices $v \in V(F)$ of the same colour under $\phi$ yields a connected simple graph.

From the first condition we know that there are roughly at most $p^2\left(\binom{p}{2}\right)$ choices for $F$. From the second condition we know that there are roughly $\binom{|F|}{2} \leq p^{2p}$ functions $\phi$. Since the problem of determining the convex $p$-partitions of a graph can be reduced in polynomial time to computing the convex $p'$-partitions of its components for $p' \in \{1, \ldots, p\}$ [2, 3], we conclude the following.

**Corollary 6.** For each fixed $p \geq 1$, all convex $p$-partitions of a bipartite graph can be enumerated in polynomial time.
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