Entropy numbers of embeddings of some 2-microlocal Besov spaces

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Abstract

We investigate compactness and asymptotic behaviour of the entropy numbers of embeddings

\[ B_{p_1,q_1}^{s_1,s_1'}(\mathbb{R}^n, U) \hookrightarrow B_{p_2,q_2}^{s_2,s_2'}(\mathbb{R}^n, U). \]

Here \( B_{p,q}^{s,s'}(\mathbb{R}^n, U) \) denotes a 2-microlocal Besov space with a weight given by the distance to a fixed set \( U \subset \mathbb{R}^n \).

Keywords: 2-microlocal spaces, compact embeddings, \( d \)-sets, entropy numbers.

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1 Introduction

Let $X$ and $Y$ be Banach spaces and $P : X \to Y$ a linear and continuous operator. The entropy numbers $e_k(P : X \to Y), k = 1, 2, \ldots$ describe the compactness of $P$ in a qualitative way (cf. Section 3 for the definition). In particular, $P$ is compact if and only if

$$\lim_{k \to \infty} e_k(P : X \to Y) = 0.$$ 

Let $X = Y$ and let $P \in \mathcal{L}(X)$ be a compact map. Let $(\lambda_k)_{k=1}^{\infty}$ be the sequence of all non-zero eigenvalues of $P$, repeated according to algebraic multiplicity and ordered so that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq 0$. Then Carl’s inequality

$$|\lambda_k(P : X \to X)| \leq \sqrt{2} e_k(P : X \to X)$$

links the behaviour of entropy numbers and eigenvalues. It opens the door to several applications of the following type: given a pseudodifferential operator $P$ with known mapping properties, i.e. $P \in \mathcal{L}(X)$, check the asymptotic behaviour of its entropy numbers and derive by means of Carl’s inequality information about the distribution of eigenvalues of $P$. Many times the behaviour of $e_k(P : X \to X)$ can be traced to the behaviour of $e_k(id : \tilde{X} \to \tilde{Y})$, where $id$ denotes the identity operator and $\tilde{X}$ and $\tilde{Y}$ are appropriate weighted Besov or Lizorkin-Triebel spaces, cf. [ET, Chapt. 5].

Here we concentrate on some 2-microlocal Besov spaces $B_{s,mloc}^{s,p,q}(\mathbb{R}^n, w)$ with one special type of weights - $w_j(x) = (1 + 2^j \text{dist}(x,U))^{s'}$ and denoted by $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$. The asymptotic behaviour of the entropy numbers will depend on the ratio of the weights instead of the weights itself. In a first step we shall investigate weights of the same type with respect to some bounded subset of $\mathbb{R}^n$ such that

$$\frac{w_{1,j}(x)}{w_{2,j}(x)} = (1 + 2^j \text{dist}(x,U))^{s'}, \quad x \in \mathbb{R}^n, \quad s' > 0.$$ 

The definition of such spaces was firstly given by Peetre 1975 [Pe] $B_{p,q}^s(a)$ with $(1 + 2^j |x|)^a$. Later Bony 1984 [Bo] introduced spaces $H_{x_0}^{s,s'}$ to describe local regularity of functions near a singularity in $x_0$. In connection with wavelet methods Jaffard 1991 [Ja] introduced and studied also spaces $C_{x_0}^{s,s'}$ again with weight $(1 + 2^j |x - x_0|)^{s'}$, that means with $U = \{x_0\}$. Some more consideration can be found also in [JaMe], [Xu] and [MeXu]. Later Morito and Yamada [MoYa] extended the definition for homogeneous spaces to weights $(1 + 2^j \text{dist}(x,U))^{s'}$.

Systematically such spaces were studied in the thesis of Kempka in 2008 [Ke 1]. He introduced a large class of admissible weight-sequences and proved a lot of properties for these spaces and in particular a characterization by wavelets. This characterization is the starting point of our considerations. It allows to 'transfer' the problem to weighted sequence spaces. Then the techniques, developed and used in [KLSS 2] and [KLSS 3], can be applied.
The paper is organized as follows. In Section 2 we introduce 2-microlocal Besov spaces and discuss a few properties. In particular we recall a characterization by wavelets - proved by Kempka 2008 [Ke 1]. This enables us to deal with weighted sequence spaces instead of weighted function spaces. First we can show in Section 3 that the restriction to a bounded set $U$ is a natural condition if we are interested in compact embeddings. Section 4 deals with the investigation of compactness and the entropy numbers of embeddings of weighted sequence spaces. Finally, in Section 5 we shift these results from the sequence space level to the function space.

**Notations.** The symbol $id$ refers always to the identity operator. Sometimes we do not indicate the spaces where $id$ is considered and similarly for other operators. Let $T$ be a linear operator which maps the Banach space $A$ into the Banach space $B$. If no confusion is possible, we feel free to write $\| T \|$ instead of the more exact versions $\| T \| L(A,B)$ or $\| T |A \to B \|$ . We denote $a \sim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1} a \leq b \leq c a.$$ 

All unimportant constants will be denoted by $c$, sometimes with additional indices.

## 2 2-microlocal Besov spaces

In this section we recall the definition and a few of the properties of 2-microlocal Besov spaces.

### 2.1 Definition and preliminaries

The classical two-microlocal spaces were defined by Peetre in 1975 [Pe] and J.M. Bony in 1984 [Bo]. Later the space have been studied in connection with wavelet methods by S.Jaffard in 1991 [Ja] and S. Jaffard and Y. Meyer, see [JaMe] for details and references. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1 \quad \text{and} \quad supp \varphi_0 \subset \{\xi : |\xi| \leq 2\}. \quad (1)$$

For $j \geq 1$ define

$$\varphi_j(\xi) := \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi). \quad (2)$$

Then we have

$$f = \varphi_0(D)f + \sum_{j=1}^{\infty} \varphi_j(D)f \quad f \in S'(\mathbb{R}^n)$$

where

$$\varphi_j(D)f(x) := (2\pi)^{-n} \int \int e^{i\xi(x-y)}\varphi_j(\xi)f(y) \, dy \, d\xi \quad .$$
The 2-microlocal space $H_{x_0}^{s,s'}(\mathbb{R}^n)$ was introduced in [Bo] as the set of all tempered distributions such that
\[
||2^{js}(1 + 2^j|x - x_0|)^{s'}\varphi_j(D)f||_{L_2} \leq c_j \quad \text{with} \quad \sum_{j=0}^{\infty} c_j^2 < \infty.
\]
Moreover, the space $C_{x_0}^{s,s'}(\mathbb{R}^n)$ in [Ja] is the collection of tempered distributions such that
\[
||2^{js}(1 + 2^j|x - x_0|)^{s'}\varphi_j(D)f||_{L_2} \leq C \quad \text{holds for all } j.
\]

**Remark 1** Please note that, if $s > 0$ and $s' > 0$ then $f \in C_{x_0}^{s,s'}$ if and only if $f \in C^s(\mathbb{R}^n)$ and
\[
||f||_{C^{s+s'}(\Gamma_\rho)} \leq C \rho^{-s'}
\]
where $\Gamma_\rho = \{ x : \rho \leq |x - x_0| \leq 3\rho \}$ . This means that we have not only $f$ belongs to $C^{s+s'}$ outside of $x_0$, but also some decrease of the corresponding norms with respect to the annuli.

For our purposes it will be convenient to work with the following class of weight-sequences defined in [Ke 1].

**Definition 1** Let $w = (w_j)_{j \in \mathbb{N}_0}$ be a sequence of non-negative measurable functions. The sequence is called an admissible weight sequence belonging to $W^{\alpha}_{a_1a_2}$ if there is a constant $C > 0$ with:
\[
0 < w_j(x) \leq C w_j(y)(1 + 2^j|x - y|)^{\alpha}
\]
and
\[
2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{holds for all } j \in \mathbb{N}_0.
\]

For each fixed $j$ the first condition coincides exactly with the usual one for weighted Besov spaces - see for example [ScTr], [ET] or [KLSS 2]. About single weight which fulfills this condition we say that it belongs to $W^{\alpha}_{\text{scalar}}$. The second condition compares weights for different indices.

By $W^{\alpha}_{\text{scalar}}$ we will denote here the union of $W^{\alpha}_{\text{scalar}}$ and by $W$ the union of $W^{\alpha}_{a_1a_2}$ over all parameters. This is now a set of weight sequences instead of single weights as in [ScTr] or [KLSS 2].

**Remark 2** Typical examples for such weight sequences are:
\[
\begin{align*}
    w_j(x) & = (1 + 2^j|x - x_0|)^{s'} & x_0 & \in \mathbb{R}^n \\
    w_j(x) & = 2^{js}(1 + 2^j|x - x_0|)^{s'} & x_0 & \in \mathbb{R}^n \\
    w_j(x) & = (1 + 2^j\text{dist}(x,U))^{s'} & U & \subset \mathbb{R}^n \\
    w_j(x) & = 2^{js}w(x) & \text{for fixed } w & \in W^{\alpha}_{\text{scalar}} \\
    w_j(x) & = \beta_j & \text{where } (\beta_j)_{j \in \mathbb{N}_0} & \text{fulfills } d_0 \beta_j \leq \beta_{j+1} \leq d_1 \beta_j \quad \text{for all } j.
\end{align*}
\]
It is known that for the last example, embeddings are compact only if the function spaces are defined on bounded domains as for the classical Besov spaces $B_{p,q}$. For some results about entropy numbers of the embeddings we refer to [Leo 2] and [CoKu]. In the case of one scalar weight, in many different situations, the behavior of the entropy numbers is known - see for example [ET], [HT 1], [HT 2], [HaSk], [KLSS 1], [KLSS 2], [KLSS 3]. So we will concentrate on the first three examples which are included in some sense in the third one.

Some elementary properties of the admissible weight sequences were considered in [Ke 1].

**Proposition 1** Let $w = (w_j)_{j \in \mathbb{N}_0} \in \mathcal{W}_{\alpha_1,\alpha_2}$, $v = (v_j)_{j \in \mathbb{N}_0} \in \mathcal{W}_{\beta_1,\beta_2}$ and $\sigma \geq 0$ then

$$w^{-1} := (w_j^{-1})_{j \in \mathbb{N}_0} \in \mathcal{W}_{\alpha_2,\alpha_1}$$

$$w^\sigma := (w_j^\sigma)_{j \in \mathbb{N}_0} \in \mathcal{W}_{\sigma\alpha_1,\sigma\alpha_2}$$

Furthermore $w \cdot v$ and $w + v$ belong again to a suitable class $\mathcal{W}_{\gamma_1,\gamma_2}$.

Let $\varphi_0 \in S(\mathbb{R}^n)$ with (1) and $\varphi_j$ be defined by dilation as in (2). Then once again we have a suitable resolution of unity so

$$f = \varphi_0(D)f + \sum_{j=1}^{\infty} \varphi_j(D)f \quad f \in S'(\mathbb{R}^n).$$

Using such resolution Kempka defined in 2008, see [Ke 1] and also [Ke 3], generalized 2-microlocal spaces as follows.

**Definition 2** Let $w = (w_j(x))_{j \in \mathbb{N}_0} \in \mathcal{W}_{\alpha_1,\alpha_2}, 0 < p, q \leq \infty$ and $s$ be real. We put

$$B^{s, mloc}_{p,q}(\mathbb{R}^n, w) = \{ f \in S'(\mathbb{R}^n) : ||f|B^{s, mloc}_{p,q}(\mathbb{R}^n, w)||_\varphi < \infty \},$$

with

$$||f|B^{s, mloc}_{p,q}(\mathbb{R}^n, w)||_\varphi = \left( \sum_{j=0}^{\infty} 2^{jsq} ||w_j \varphi_j(D)f|L_p(\mathbb{R}^n)||^q \right)^{1/q}.$$

There is an analog definition for $F^{s, mloc}_{pq}(\mathbb{R}^n, w)$. In [Ke 2] these definitions were extended even to variable parameters $p$.

We mention some obvious relations of these spaces to some well-known definitions.

- In case of $w_j(x) = 1$ they are the usual Besov spaces.

- In case of $w_j(x) = (1 + |x|^\alpha)$ independent of $j$ we obtain the usual weighted Besov spaces with polynomial weights.
In case of \( w_j(x) = \beta_j \) where \( d_0 \beta_j \leq \beta_{j+1} \leq d_1 \beta_j \) \( \beta_{j+1} \) is an admissible sequence of real numbers we obtain Besov spaces of generalized smoothness, see for example [FL].

With \( w_j(x) = (1 + 2^j|x - x_0|)^{s'} \) we have \( C^s,\mathcal{S}'(\mathbb{R}^n) = B_{\infty,\infty}^{s,m_{loc}}(\mathbb{R}^n, w) \) and \( H_{\mathcal{S}'}(\mathbb{R}^n) = B_{2,2}^{s,m_{loc}}(\mathbb{R}^n, w) \).

In case of \( w_j(x) = (1 + 2^j \text{dist}(x,U))^s \) the corresponding spaces were denoted by \( B_{p,q}^{s,s'}(\mathbb{R}^n)(U) \). The homogeneous version of the spaces were considered by Morito and Yamada in [MoYa].

A lot of properties of these spaces were proved in [Ke 1]. For example the independence of the spaces from the choice of the function \( \phi \), theorems about Fourier multipliers, the existence of lift-operators, embeddings, maximal functions and also some far reaching properties as characterizations by local means, by atoms and molecules and by wavelets were considered there.

### 2.2 Wavelet decomposition of 2-microlocal Besov spaces

We follow the notation in [Tr 2] and [Ke 1]. Let \( \psi_M \in C^k(\mathbb{R}) \) and \( \psi_F \in C^k(\mathbb{R}) \) be real compactly supported Daubechies wavelets with

\[
\int_{\mathbb{R}} x^\beta \psi_M(x) \, dx = 0 \quad \text{for } |\beta| < k .
\]

By a tensor product procedure these can be generalized to the \( n \)-dimensional case - see [Tr 2, 4.2.1.].

Let \( G = (G_1, \cdots, G_n) \in G^j = \{F, M\}^{n^*} \) where \( n^* \) indicates that at least one of the components of \( G \) must be an \( M \) and let \( G^0 = \{F, M\}^n \). The cardinality of \( \{F, M\}^{n^*} \) is \( 2^n - 1 \) and the cardinality of \( \{F, M\}^n \) is \( 2^n \), respectively. We define

\[
\psi_{G, m}^j(x) := 2^{j2} \prod_{r=1}^n \psi_{G_r}(2^j x_r - m_r)
\]

where \( j \in \mathbb{N}_0, G \in G^j \) and \( m \in \mathbb{Z}^n \). Then \( \{\psi_{G, m}^j(x) : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\} \) is an orthonormal basis in \( L_2(\mathbb{R}^n) \) - see [Tr 2, 4.2.1.]. For a characterization of the 2-microlocal Besov spaces we need the following sequence spaces - see [Ke 1, Definition 5.21].

**Definition 3** Let \( w \in \mathcal{W}, s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \).

\[
\tilde{b}_{p,q}^{s,m_{loc}}(w) = \{(\lambda_{G, m}^j)_j, G, m : ||\lambda||_{\tilde{b}_{p,q}^{s,m_{loc}}(w)} < \infty \}
\]

and

\[
||\lambda||_{\tilde{b}_{p,q}^{s,m_{loc}}(w)} = \left( \sum_{j=0}^{\infty} 2^{j(s-\frac{n}{2})} \sum_{G \in G^j} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{G, m}^j|^p w_j^p(2^{-j} m) \right)^{q/p} \right)^{1/q}.
\]
Also the following proposition can be found there, see [Ke 1, Theorem 5.30].

**Proposition 2** Let \( f \in B^{s,\text{loc}}_{pq}(\mathbb{R}^n, w) \), \( k \) large enough and

\[
\lambda^j_{G,m}(f) := 2^{j^2} < f, \psi^j_{G,m} > = 2^{j^2} \int f(x) \psi^j_{G,m}(x) \, dx.
\]

Then

\[
I : f \rightarrow 2^{j^2} < f, \psi^j_{G,m} >
\]

is an isomorphic map from \( B^{s,\text{loc}}_{pq}(\mathbb{R}^n, w) \) onto \( \tilde{B}^{s,\text{loc}}_{p,q}(w) \).

This characterization was specified to the spaces \( B^{s,s'}_{p,q}(\mathbb{R}^n, U) \), see [Ke 1, Corollary 5.33], in the following way:

A function \( f \) belongs to \( B^{s,s'}_{p,q}(\mathbb{R}^n, U) \) if, and only if, it can be represented as

\[
f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_{G,m}(f) 2^{-j^2} \psi^j_{G,m} \quad \text{with} \quad \lambda \in \tilde{b}^{s,s'}_{p,q}(U).
\]

The representation is unique

\[
\lambda^j_{G,m}(f) := 2^{j^2} < f, \psi^j_{G,m} > = 2^{j^2} \int f(x) \psi^j_{G,m}(x) \, dx
\]

and

\[
I : f \rightarrow 2^{j^2} < f, \psi^j_{G,m} >
\]

is an isomorphic map from \( B^{s,s'}_{p,q}(\mathbb{R}^n, U) \) onto \( \tilde{b}^{s,s'}_{p,q}(U) \).

Here

\[
||| \lambda |||_{\tilde{b}^{s,s'}_{p,q}(U)} := \left( \sum_{j=0}^{\infty} 2^{j(s - \frac{nq}{p})} \sum_{G \in G^j} \left( \sum_{m \in \mathbb{Z}^n} |\lambda^j_{G,m}| (1 + 2^j \text{dist}(2^{-j} m, U))^{s'p} \right)^{q/p} \right)^{1/q}.
\]

### 3 The characterization of continuous and compact embeddings of weighted sequence spaces

The aim of this section is to characterize all possible situations where either a continuous embedding or a compact embedding holds true. By means of the Proposition 2 we can shift the problem to the side of the sequence spaces.

#### 3.1 Some preliminaries

Motivated by Definition 3 and Proposition 2 we introduce the following sequence spaces. For a given weight-sequence \((w_j)_j \in \mathcal{W}\) we define

\[
\ell_q(2^j \ell_p(w)) := \left\{ \lambda = (\lambda_{j,m})_{j,m} : \lambda_{j,m} \in \mathbb{C}, \right. \left. \left\| \lambda \ell_q(2^j \ell_p(w)) \right\| = \left( \sum_{j=0}^{\infty} 2^{j^2q} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}| w_j(2^{-j} m)^q \right)^{q/p} \right)^{1/q} \right\} < \infty.
\]
(usual modifications if \( p = \infty \) and/or \( q = \infty \)).

It should be clear that we can work with the spaces \( \ell_q(2^{js} \ell_p(w)) \) instead of \( \tilde{b}_{p,q}^{m,loc}(w) \) when we ask about boundedness and compactness of the embeddings. Similar consideration, with one weight function instead of a sequence of weights can be found in [KLSS 1],[KLSS 2], [KLSS 3], [HaPi], [HaSk] or [CoKu].

Observe further that also
\[
\ell_q\left(2^{j(s_1-\frac{n}{p_1})} \ell_{p_1}(w_1)\right) \hookrightarrow \ell_q\left(2^{j(s_2-\frac{n}{p_2})} \ell_{p_2}(w_2)\right)
\]
is equivalent to
\[
\ell_q\left(2^{j(s_1-s_2-n(\frac{1}{p_1}-\frac{1}{p_2}))} \ell_{p_1}(w_1/w_2)\right) \hookrightarrow \ell_q(\ell_{p_2}).
\]
So it will be again sufficient to consider the unweighted space as the target space. Moreover, it will be convenient to use the following well known abbreviation
\[
\delta := s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right).
\]

### 3.2 Continuous and compact embeddings of general weighted sequence spaces

We switch to a general point of view picked up from Leopold [Leo 3], see also [KLSS 2]. Let \( \beta = (\beta_j)_{j=0}^\infty \) and \( w = (w_{j,m})_{j,m} \) be sequences of positive numbers. Then we put
\[
\ell_q(\ell_p(w)) := \left\{ \lambda = (\lambda_{j,m})_{j,m} : \lambda_{j,m} \in \mathbb{C} , \right. \\
\left. \| \lambda \ell_p(w) \| = \left( \sum_{j=0}^\infty \beta_j^q \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m} w_{j,m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}
\]
(usual modifications if \( p = \infty \) and/or \( q = \infty \)). For a real number \( a \) we define \( a_+ := \max(a,0) \).

**Theorem 1** (i) The embedding \( \ell_q(\ell_p(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2}) \) holds if and only if
\[
\left( \beta_j^{-1} \left( \| (w_{j,m}^{-1})_{m} \|_{p_2} \right)_j \right)_j \in \ell_{q^*}
\]
where
\[
\frac{1}{p^*} := \left( \frac{1}{p_2} - \frac{1}{p_1} \right)_+ \quad \text{and} \quad \frac{1}{q^*} := \left( \frac{1}{q_2} - \frac{1}{q_1} \right)_+.
\]
Moreover, it holds
\[
\| \text{id} \ell_q(\ell_p(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2}) \| = \left\| \left( \beta_j^{-1} \left( \| (w_{j,m}^{-1})_{m} \|_{p_2} \right)_j \right)_j \ell_{q^*} \right\|.
\]

(ii) The embedding \( \ell_q(\ell_p(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2}) \) is compact if and only if
\[
\left( \beta_j^{-1} w_{j,m}^{-1} \right)_{j,m} \in \ell_{q^*}(\ell_{p^*}),
\]
7
and in addition
\[ \lim_{j \to \infty} \beta_j^{-1} \| (w_{j,m})_m \|_{\ell^q_p} = 0 \quad \text{if} \quad q^* = \infty, \quad (3) \]
and
\[ \lim_{|m| \to \infty} w_{j,m} = \infty \quad \text{for all} \quad j \in \mathbb{N}_0 \quad \text{if} \quad p^* = \infty. \quad (4) \]

We refer to [KLSS 2] for the proof.

**Corollary 1** Let \( U \) be unbounded and \( w_{j,m} = w_j(2^{-j}m) = (1 + 2^j \operatorname{dist}(2^{-j}m, U))^{s'} \). Then the embedding
\[ \ell_{q_1}(2^{j^0} \ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2}). \]
can never become compact.

**Proof** For each fixed \( j_0 \) there exists a sequence \( m_i \) with \( |m_i| \to \infty \) and \( \operatorname{dist}(2^{-j_0}m_i, U) \leq \sqrt{n} \). Consequently we have
\[ 1 \leq w_{j_0,m_i} = w_{j_0}(2^{j_0}m_i) = (1 + 2^{j_0} \operatorname{dist}(2^{-j_0}m_i, U))^{s'} \leq (1 + \sqrt{n})^{s'} \quad s' \geq 0. \]

It follows from the last corollary that only bounded subsets of \( \mathbb{R}^n \) are of interest for us. It should be obvious that we can restrict the attention to compact subsets. We are able to give the precise description for so called \( d \)-sets, so we recall the notation now. Let \( U \) be a compact set and \( \mu \) a Radon measure with \( \operatorname{supp} \mu = U \). The set \( U \) is called a \( d \)-set, \( 0 \leq d \leq n \), if for each ball of radius \( r \) and centered in \( \gamma \in U \) holds
\[ \mu(B(\gamma, r)) \sim r^d, \quad \text{for} \quad 0 < r < 1 \]
cf. [Tr 1].

Let a \( d \)-set \( U \) be covered by balls of radius \( 2^{-k} \), centered on \( U \), such that the balls with the same center but with radius \( 2^{-k-\kappa} \) for fixed \( \kappa \in \mathbb{N} \) are disjoint. Then it is well-known, that the number of balls in such a covering is always equivalent to \( 2^{kd} \) where the equivalence constants depend on \( \kappa \) but are independent of the radius \( 2^{-k} \).

We are interested in estimation of a number of dyadic cubes of a fixed side length that are in a predetermined distance to the set \( U \).

For \( j \in \mathbb{N}_0 \) and \( i \in \mathbb{N} \) we denote by \( N_{j,i} \) the number of cubes \( Q_{j,m} \) of side length \( 2^{-j} \), centered in \( 2^{-j}m \) with
\[ \sqrt{n} \cdot 2^{-j+i} < \operatorname{dist}(Q_{j,m}, U) \leq 4 \sqrt{n} \cdot 2^{-j+i}. \quad (5) \]
Lemma 1 Let \( U \) be a \( d \)-set, then

\[
N_{j,i} \sim \begin{cases} 2^{i \cdot 2^{(j-i)d}} & 0 \leq i < j, \\ 2^{i} & j \leq i. \end{cases}
\]

Proof Let us assume that the \( d \)-set \( U \) is contained in the unit ball \( \{ x : |x| \leq 1 \} \).

Step 1. If \( i \geq j \) then the assertion follows immediately. The number of cubes \( N_{j,i} \) is smaller than the number of cubes \( Q_{j,m} \) contained in a ball of radius \( 4 \sqrt{n} \cdot 2^{-j+i} + 1 \) and larger as the number of cubes \( Q_{j,m} \) contained in an annulus with radii \( \sqrt{n} \cdot 2^{-j+i} + 1 \) and \( 4 \sqrt{n} \cdot 2^{-j+i} - 1 \). Since \( 2^{-j+i} \geq 1 \) by volume argument this gives

\[
N_{j,i} \sim 2^{i} \quad i \geq j.
\]

Step 2. We use the Whitney decomposition of \( \mathbb{R}^n \setminus U \). Let \( Q_{k,m,k,r} : k \in \mathbb{Z}, m_k,r \in \mathbb{Z}^n, r = 1, 2, \ldots, r(k) \) be (open) cubes in \( \mathbb{R}^n \) with sides parallel to the axes of coordinates, centered in \( 2^{-k} m_k,r \) with side length \( 2^{-k} \) such that for suitable disjoint cubes \( Q_{k,m,k,r} \)

\[
\mathbb{R}^n \setminus U = \bigcup_{k,r} \overline{Q}_{k,m,k,r} \quad \text{and} \quad \text{dist}(Q_{k,m,k,r}, U) \sim 2^{-k},
\]

more precise

\[
\sqrt{n} \cdot 2^{-k} \leq \text{dist}(Q_{k,m,k,r}, U) \leq 4 \sqrt{n} \cdot 2^{-k},
\]

cf. the proof of Theorem 1 in [St, page 167].

Let \( \{ B(x_i, 2^{-k}) \}_{i=1,\ldots,m} \) be a covering of \( U \) with the centers \( x_i \) belonging to \( U \), and such that the balls \( B(x_i, 2^{-k-1}) \) are pairwise disjoint. Then as we mention earlier \( m \sim 2^{dk} \) with the constants independent of \( k \). The family \( \{ B(x_i, 6 \sqrt{n} \cdot 2^{-k}) \}_{i=1,\ldots,m} \) covers an “annulus” \( \{ x \in \mathbb{R}^n : \sqrt{n} \cdot 2^{-k} \leq \text{dist}(x, U) \leq 5 \sqrt{n} \cdot 2^{-k} \} \) therefore by disjointness of cubes in the Whitney decomposition we get

\[
2^{-kn} r(k) \leq c 2^{dk} (6 \sqrt{n} \cdot 2^{-k})^n.
\]

On the other hand, if we blow up any cube \( Q_{k,m,k,r} \) by the factor \( \lambda = 9 \sqrt{n} \), then any of the balls \( B(x_i, 2^{-k-1}) \), \( i = 1, \ldots, m \), is contained in at least one enhance cube, so

\[
c 2^{-(k-1)n} 2^{kd} \leq r(k) (9 \sqrt{n} \cdot 2^{-k})^n.
\]

In consequence

\[
r(k) \sim 2^{kd}.
\]

Step 3. We estimate now the number \( N_{j,i} \) in case \( i < j \).

Estimate from below: By (6) we have at least \( c 2^{(2-j) \cdot 2^{(j-i)d}} \) cubes of side length \( 2^{-j+i} \) with

\[
\sqrt{n} \cdot 2^{-j+i} < \text{dist}(Q_{j-i,m}, U) \leq 4 \sqrt{n} \cdot 2^{-j+i}.
\]

Each of these cubes can be divided into \( 2^{ni} \) cubes of side length \( 2^{-j} \) and so we have at least \( c 2^{(2-j+i) \cdot 2^{(j-i)d}} \) cubes with (5).
Estimate from above: Let \( Q_{j,m} \) be a cube with side length \( 2^{-j} \) that satisfies (5). Then it is contained in some, maybe larger, cube of the Whitney decomposition - \( Q_{j-l,m'} \), \( 0 \leq l \leq i \). Because the last cube belongs to the Whitney decomposition we have

\[
\sqrt{n} \, 2^{-j+l} < \text{dist} \left( Q_{j-l,m'}, U \right) \leq 4 \sqrt{n} \, 2^{-j+l}.
\]

There exist at most \( 2^{(j-l)d} \) such cubes, cf. (6). But each cube \( Q_{j-l,m'} \) from the Whitney decomposition can be appointed by at most \( 2^{ln} \) original cubes \( Q_{j,m} \). For this reason we have

\[
N_{j,i} \leq \max_{l=0, \ldots, i} 2^{(j-l)d} \leq 2^{jd(n-d)}.
\]

**Step 4.** If the \( d \)-set \( U \) is contained in a larger ball \( \{ x : |x| \leq R \} \) then by dilation with \( 2^{-J} \leq R^{-1} \) for a suitable fixed \( J \) we can reduce the considerations to the previous case. Afterwards blowing up the cubes by \( 2^{J} \) we have the same estimates, but now with new constants where \( 2^{-Jn} \) and \( 2^{Jn} \) come in.

**Theorem 2** Let \( U \) be a \( d \)-set and \( w_{j,m} = w_{j,2^{-j}m} = (1 + 2^{j} \text{dist}(2^{-j}m,U))^{s'} \).

Then the embedding

\[
\ell_{q_{1}}(2^{j\delta} \ell_{p_{1}}(w)) \hookrightarrow \ell_{q_{2}}(\ell_{p_{2}}).
\]

is compact if and only if

\[
s' > n/p * \quad \text{and} \quad \delta > d/p * .
\]

**Proof** We introduce a decomposition of the identity which will be of great use for us also in the proof of the next theorem. Let

\[
\Lambda := \{ \lambda = (\lambda_{j,m})_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} : \lambda_{j,m} \in C, \ j \in \mathbb{N}_0, m \in \mathbb{Z}^n \}.
\]

Let \( I_{j,i} \subset \mathbb{N}_0 \times \mathbb{Z}^n \) s.t.

\[
I_{j,i} := \{ (j, m) : \sqrt{n} \, 2^{-j+i-1} < \text{dist}(2^{-j}m, U) \leq \sqrt{n} \, 2^{-j+i} \}, \ i \in \mathbb{N}, \ j \in \mathbb{N}_0
\]

\[
I_{j,0} := \{ (j, m) : \text{dist}(2^{-j}m, U) \leq \sqrt{n} \, 2^{-j} \}, \quad j \in \mathbb{N}_0.
\]

Further, let \( P_{j,i} : \Lambda \rightarrow \Lambda \) be the canonical projection with respect to \( I_{j,i} \), i.e., for \( \lambda \in \Lambda \) we put

\[
(P_{j,i} \lambda)_{u,v} := \begin{cases} 
\lambda_{u,v} & (u, v) \in I_{j,i}, \\
0 & \text{otherwise}.
\end{cases}, \quad u \in \mathbb{N}_0, \ v \in \mathbb{Z}^n.
\]

Observe

\[
\text{id}_\Lambda = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} P_{j,i}.
\]
The benefit of the decomposition is, that we have for all \((j, m)\) in the same index set \(I_{j,i}\) an uniform estimate for the weight

\[
    w_j(2^{-j}m) \sim (1 + 2^j 2^{-(j-i)})^s' \sim 2^{is'} \quad \text{if} \quad (j, m) \in I_{j,i}. \quad (7)
\]

The cardinality of \(I_{j,i}\) is denoted by \(M_{j,i}\). In case \(0 < i < j\) this cardinality depends on the structure of \(U\) and is fundamental in the following. If \(U\) is a \(d\)-set, then an easy observation and the result of Lemma 1 gives

\[
    M_{j,i} \leq N_{j,i+2} + N_{j,i+3} \sim \left\{ \begin{array}{ll}
        2^{in} 2^{(j-i)d} & 0 \leq i < j, \\
        2^{in} & j \leq i
    \end{array} \right. \quad (8)
\]

and

\[
    M_{j,i+1} + M_{j,i+2} \geq N_{j,i} \sim \left\{ \begin{array}{ll}
        2^{in} 2^{(j-i)d} & 0 \leq i < j, \\
        2^{in} & j \leq i
    \end{array} \right. \quad (9)
\]

Using the second part of Theorem 1 we can prove now necessary and sufficient conditions for the compactness of the embedding. If we take \(\beta_j = 2^{j^d}\) then by the estimation (7) we have

\[
    ||\beta_j^{-1}(w_{j,m})^{-1}|l_q(l_p)||| \sim \left( \sum_{j=0}^{\infty} 2^{-j d^q} \left( \sum_{i=0}^{\infty} M_{j,i} 2^{-is'^p} \right) \right)^{\frac{q^*_p}{q^*_p}} \sim \left( \sum_{j=0}^{\infty} 2^{-j d^q} \left( \sum_{i=0}^{j} 2^{in+j(j-i)d} 2^{-is'^p} + \sum_{i=j}^{\infty} 2^{in} 2^{-is'^p} \right) \right)^{\frac{q^*_p}{q^*_p}} \quad (\frac{1}{q^*_p})^s\).
\]

We can estimate this norm from below by the same series

\[
    \left( \sum_{j=0}^{\infty} 2^{-j d^q} \left( \sum_{i=0}^{\infty} M_{j,i} 2^{-is'^p} \right) \right)^{\frac{1}{q^*_p}} \sim \left( \sum_{j=0}^{\infty} 2^{-j d^q} \left( \sum_{i=0}^{j} 2^{in+j(j-i)d} 2^{-is'^p} + \sum_{i=j}^{\infty} 2^{in} 2^{-is'^p} \right) \right)^{\frac{1}{q^*_p}} \quad (\frac{1}{q^*_p})^s.
\]

In case \(p^*_p < \infty\) the sums over \(i\) converges if and only if \(n < s' p^*_p\). So in that case

\[
    \left( \sum_{j=0}^{\infty} 2^{-j d^q} \left( \sum_{i=0}^{j-1} 2^{in+j(j-i)d} 2^{-is'^p} + \sum_{i=j}^{\infty} 2^{in} 2^{-is'^p} \right) \right)^{\frac{1}{q^*_p}} \sim \left( \sum_{j=0}^{\infty} 2^{-j d^q} (2^{jd} + 2^{jn} 2^{-js'^p}) \right)^{\frac{1}{q^*_p}} \sim \left( \sum_{j=0}^{\infty} 2^{-j d^q} 2^{jd q*/p^*_p} \right)^{\frac{1}{q^*_p}}.
\]
Thus in case $p^* < \infty$ and $q^* < \infty$ the norm is finite if and only if $s' > n/p^*$ and
$
\delta > d/p^*.
$
If either $p^* = \infty$ or $q^* = \infty$ then one should consider in addition (3) or (4).

## 4 Entropy numbers of embeddings of special weighted sequence spaces

We are interested in measuring the compactness of the embedding of
$
\ell_{q_1}(2^{j(s_1-n/p_1)} \ell_{p_1}(w_1)) \hookrightarrow \ell_{q_2}(2^{j(s_2-n/p_2)} \ell_{p_2}(w_2)).
$

### 4.1 Preliminaries

Let us recall briefly the definition of entropy numbers.

**Definition 4** Let $X, Y$ be two complex Banach spaces and let $P$ be a linear and continuous operator from $X$ into $Y$. Let $k \in \mathbb{N}$. The $k$-th entropy number $e_k(P : X \to Y)$ is the infimum of all numbers $\varepsilon > 0$ such that there exist $2^{k-1}$ balls in $Y$ of radius $\varepsilon$ which cover the image of the unit ball $U := \{ x \in X : \|x\|_X \leq 1 \}$ under the mapping $P$.

In particular, $P$ is compact if and only if $\lim_{k \to \infty} e_k(P : X \to Y) = 0$. For details and basic properties we refer for example to the monographs [CS, ET].

Let $w_j(x) = \frac{w_{1,j}(x)}{w_{2,j}(x)}$ and consequently $\ell_{q_1}(2^{j\delta^*} \ell_{p_1}(w)) = \ell_{q_1}(2^{j\delta^*} \ell_{p_1}(w_1/w_2))$.

Then the mapping $I$ defined by
$
\lambda_{j,m} \underset{\text{I}}{\mapsto} \lambda_{j,m} 2^{j(s_2-n/p_2)} w_2(2^{-j}m), \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n,
$
yields an isometry of $\ell_{q_1}(2^{j(s_1-n/p_1)} \ell_{p_1}(w_1))$ onto $\ell_{q_1}(2^{j\delta} \ell_{p_1}(w))$. Furthermore, $I^{-1}$ yields an isometry of $\ell_{q_1}(\ell_{p_2})$ onto $\ell_{q_2}(2^{j(s_2-n/p_2)} \ell_{p_2}(w_2))$. As a consequence of the definition of the entropy numbers and the properties of $I, I^{-1}$ we obtain the identity $k = 1, 2, \ldots$

\begin{align*}
\ e_k(\text{id} : \ell_{q_1}(2^{j(s_1-n/p_1)} \ell_{p_1}(w_1))) & \to \ell_{q_2}(2^{j(s_2-n/p_2)} \ell_{p_2}(w_2))) = \\
&= e_k(\text{id} : \ell_{q_1}(2^{j\delta^*} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})).
\end{align*}

Hence, we may concentrate on $e_k(\text{id} : \ell_{q_1}(2^{j\delta^*} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}))$.

The abstract concept of operator ideal, see [Pi 1, Pi 2], has been proved to be an useful tool in various situation. Here it simplifies again the estimates of the entropy numbers. We refer to [KLSS 2] were this concept was used extensively.
Definition 5 Let $X,Y$ be quasi-Banach spaces, $E$ a quasi-normed symmetric sequence space and let $T \in \mathcal{L}(X,Y)$. Then we put
\[ L^e_E(T) := \| e_k(T) \|_E \]
and
\[ \mathcal{L}^e_E(X,Y) := \left\{ T \in \mathcal{L}(X,Y) : L^e_E(T) < \infty \right\}. \]

Remark 3 Let $E$ be a quasi-normed symmetric sequence space which is maximal, too. Then $\mathcal{L}^e_E(X,Y)$ is a complete quasi-normed space, i.e. a quasi-Banach space and there exist an equivalent $\varrho$-norm on it.

For details see once again [KLSS 2]. Moreover we will specify again the operator ideal. We choose $E = \ell_{r,\infty}$ as the Lorentz sequence space.

4.2 Estimates for the entropy numbers

We come to the main result of our paper. One of the tools in proving it will be the characterization of the asymptotic behaviour of the entropy numbers of the embeddings $\ell^N_{p_1} \hookrightarrow \ell^N_{p_2}$. For $0 < p_1 \leq p_2 \leq \infty$ and for all $k \in \mathbb{N}$ we have
\[ e_k(\text{id} : \ell^N_{p_1} \rightarrow \ell^N_{p_2}) \sim \begin{cases} 1 & \text{if } 1 \leq k \leq \log 2N, \\ \left(\frac{\log(1+\frac{k}{N})}{k}\right)^{\frac{1}{p_1} - \frac{1}{p_2}} & \text{if } \log 2N \leq k \leq 2N, \\ 2^{-k N^{\frac{1}{p_1} - \frac{1}{p_2}}} & \text{if } 2N \leq k. \end{cases} \] (10)

and in case $0 < p_2 < p_1 \leq \infty$ it holds
\[ e_k(\text{id} : \ell^N_{p_1} \rightarrow \ell^N_{p_2}) \sim 2^{-k N^{\frac{1}{p_2}} \cdot \frac{1}{p_1}} \quad \text{for all } k \in \mathbb{N}. \] (11)

If $1 \leq p_1 \leq p_2 \leq \infty$ this has been proved by Schütt [Sch]. For $p_1 < 1$ and/or $p_2 < 1$, and $p_2 < p_1$ we refer to Edmunds and Triebel [ET, 3.2.2] and Triebel [Tr 1, 7.2, 7.3] (with the supplement in [Kü]). The abbreviations $\delta$, $p^*$ and $q^*$ have the same meaning as before.

We specify again the operator ideal. We choose $E = \ell_{r,\infty}$ as the Lorentz sequence space and write instead of $L^e_E(T)$ and $\mathcal{L}^e_E(X,Y)$ simply $L^e_{r,\infty}(T)$ and $\mathcal{L}^e_{r,\infty}(X,Y)$, respectively. Then we have
\[ \mathcal{L}^e_{r,\infty}(T) := \sup_k \{ e_k(T) k^{1/r} \} < c \quad \text{if and only if } e_k(T) \leq c k^{-\frac{1}{r}}. \] (12)

The characterization of the asymptotic behaviour of the entropy numbers, cf. (10) and (11), implies
\[ L^e_{r,\infty}(\text{id} : \ell^N_{p_1} \rightarrow \ell^N_{p_2}) \sim N^{\frac{1}{r} - \left(\frac{1}{p_1} \frac{1}{p_2}\right)} \quad \text{if } \frac{1}{r} > \max \left(0, \frac{1}{p_1} - \frac{1}{p_2}\right). \] (13)
Theorem 3  Let $U$ be a $d$-set , $0 \leq d \leq n$, $w_{j,m} = w_j(2^{-j}m) = (1 + 2^j \text{dist}(2^{-j}m, U))$ a sequence of weights. Let

$$
\delta = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{d}{p^*} = d\left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \quad \text{and} \quad s' = s'_1 - s'_2 > \frac{n}{p^*} .
$$

Then

$$
e_k\left(\text{id} : \ell_q(2^j \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})\right) \sim k^{-\frac{1}{p_1} + \frac{1}{p_2} - \min\left(\frac{1}{q}, \frac{1}{q'}\right)} .
$$

Proof  Step 1. Preparations. We use the same decomposition of identity as in the proof of Theorem 2 and employ the same notations as before. Recall that

$$w_j(2^{-j}m) \sim 2^{is'} \quad \text{if} \quad (j, m) \in I_{j,i} .$$

Monotonicity arguments and elementary properties of the entropy numbers yield

$$e_k\left(P_{j,k} : \ell_q(2^j \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})\right) \leq c 2^{-j\delta} 2^{-is'} e_k\left(\text{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}\right) , 
$$

with a constant $c$ independent of $k, j$ and $i$.

Step 2. Now the operator ideal comes into play. Using (12) and (14) we find

$$L_{r,\infty}^{(e)}(P_{j,i}) \leq c 2^{-j\delta} 2^{-is'} L_{r,\infty}^{(e)}\left(\text{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}\right) .$$

To shorten notations let $1/p = 1/p_1 - 1/p_2$. Under the assumption $1/r > \max(0, 1/p)$ we conclude from Lemma 1, (8) and (13) that

$$L_{r,\infty}^{(e)}\left(\text{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}\right) \leq \begin{cases} 2^{(in + d(j-i))(\frac{1}{q} - \frac{1}{p})}, & 0 \leq i < j \\ 2^{in(\frac{1}{q} - \frac{1}{p})}, & 0 < j \leq i \end{cases}$$

and consequently

$$L_{r,\infty}^{(e)}(P_{j,i}) \leq c 2^{-j\delta} 2^{-is'} \begin{cases} 2^{(in + d(j-i))(\frac{1}{q} - \frac{1}{p})}, & 0 \leq i < j \\ 2^{in(\frac{1}{q} - \frac{1}{p})}, & 0 < j \leq i \end{cases} .$$

Now, for given $M \in \mathbb{N}_0$ let

$$P^1 := \sum_{j=0}^{M} \sum_{i=0}^{j-1} P_{j,i} , \quad P^2 := \sum_{j=M+1}^{\infty} \sum_{i=0}^{j-1} P_{j,i} ,$$

$$Q^1 := \sum_{j=0}^{M} \sum_{i=j}^{M} P_{j,i} , \quad Q^2 := \sum_{j=0}^{\infty} \sum_{i=M}^{\infty} P_{j,i} , \quad Q^3 := \sum_{j=M+1}^{\infty} \sum_{i=j}^{\infty} P_{j,i} .$$
Substep 2.1. First we estimate $L_{r,∞}^{(e)}(P^1)$. Recall, for any $r > 0$ there exists an equivalent $\varrho$-norm on $L_{r,∞}^{(e)}$ with $0 < \varrho \leq 1$. Hence we have

$$L_{r,∞}^{(e)}(P^1)^{\varrho} \leq \sum_{j=0}^{M} \sum_{i=0}^{j-1} L_{r,∞}^{(e)}(P_{j,i})^{\varrho} \leq c_1 \sum_{j=0}^{M} \sum_{i=0}^{j-1} 2^{-j\varrho\delta} 2^{-is'\varrho} 2^{(m+d(j-i))\varrho(\frac{1}{r} - \frac{1}{p})}$$

$$\leq c_2 \sum_{j=0}^{M} 2^{-j\varrho\delta} 2^{dj(\frac{1}{r} - \frac{1}{p})} \sum_{i=0}^{j-1} 2^{-is'\varrho} 2^{i(n-d)\varrho(\frac{1}{r} - \frac{1}{p})}$$

$$\leq c_3 \sum_{j=0}^{M} 2^{-j\varrho\delta} 2^{dj(\frac{1}{r} - \frac{1}{p})} 2^{-js'\varrho} 2^{j(n-d)\varrho(\frac{1}{r} - \frac{1}{p})} \leq c_4 2^{-M\varrho\delta} 2^{-M's'\varrho} 2^{Mn\varrho(\frac{1}{r} - \frac{1}{p})}$$

if $r$ is chosen such that

$$\left(\frac{1}{r} - \frac{1}{p}\right)(n - d) > s' \quad \text{and} \quad d\left(\frac{1}{r} - \frac{1}{p}\right) > \delta. \quad (15)$$

These imply

$$n\left(\frac{1}{r} - \frac{1}{p}\right) > s' + \delta$$

in the last summation. If $0 < d < n$ then one can choose $r$ according to (15) and in view of (12) this gives

$$e_{2Mn}(P^1 : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})) \leq c_4 2^{nM(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{d}{n} + s')}.$$}

On the other hand, in case $d = 0$, we have to chose $r$ in such a way that again

$$n\left(\frac{1}{r} - \frac{1}{p}\right) > s' + \delta$$

holds and we obtain the same result.

If we consider the case $d = n$ we obtain in a similar way

$$e_{2Mn}(P^1 : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})) \leq c_5 2^{nM(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{d}{n})}.$$}

Now we estimate $L_{r,∞}^{(e)}(Q^1)$.

$$L_{r,∞}^{(e)}(Q^1)^{\varrho} \leq \sum_{j=0}^{M} \sum_{i=0}^{j} L_{r,∞}^{(e)}(P_{j,i})^{\varrho} \leq c_1 \sum_{j=0}^{M} \sum_{i=0}^{j} 2^{-j\varrho\delta} 2^{-is'\varrho} 2^{i(m+j-i)\varrho(\frac{1}{r} - \frac{1}{p})}$$

$$\leq c_2 \sum_{j=0}^{M} 2^{-j\varrho\delta} \sum_{i=0}^{j} 2^{-is'\varrho} 2^{i(n-j)\varrho(\frac{1}{r} - \frac{1}{p})} \leq c_3 \sum_{j=0}^{M} 2^{-j\varrho\delta} 2^{-M's'\varrho} 2^{Mn\varrho(\frac{1}{r} - \frac{1}{p})}$$

$$\leq c_4 2^{-M's'\varrho} 2^{Mn\varrho(\frac{1}{r} - \frac{1}{p})}$$

if $r$ is chosen such that

$$n\left(\frac{1}{r} - \frac{1}{p}\right) > s'.$
This implies
\[ e_{2Mn} (Q^1 : \ell_{q_1} (2j^{\delta} \ell_{p_1} (w)) \to \ell_{q_2} (\ell_{p_2})) \leq c_4 2^{nM (\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n})}. \]

**Substep 2.2.** To estimate \( L_{r, \infty}^{(e)} (P^2) \), \( L_{r, \infty}^{(e)} (Q^2) \) and \( L_{r, \infty}^{(e)} (P^3) \) we assume that
\[
\begin{align*}
&n \left( \frac{1}{r} - \frac{1}{p} \right) < s' \quad \text{and} \quad d \left( \frac{1}{r} - \frac{1}{p} \right) < \delta.
\end{align*}
\] (16)

Because of
\[
\begin{align*}
s' > n/p^* = n \max(0, -1/p) \quad \text{and} \quad \delta > d/p^* = d \max(0, -1/p)
\end{align*}
\]
we have
\[
\begin{align*}
n \max \left( 0, \frac{1}{p} \right) < s' + \frac{n}{p} \quad \text{and} \quad d \max \left( 0, \frac{1}{p} \right) < \delta + \frac{d}{p}.
\end{align*}
\]

Hence, there exists for \( 0 \leq d \leq n \) an appropriate \( r \) with
\[
\max \left( 0, \frac{1}{p} \right) < \frac{1}{r}
\]
and
\[
\begin{align*}
n \left( \frac{1}{r} - \frac{1}{p} \right) < s' \quad \text{and} \quad d \left( \frac{1}{r} - \frac{1}{p} \right) < \delta.
\end{align*}
\]

We proceed now as before and obtain for \( 0 \leq d \leq n \) and with condition (16)
\[
\begin{align*}
L_{r, \infty}^{(e)} (P^2) & \leq c_2 \sum_{j=0}^{M} 2^{-j \delta} \sum_{i=M+1}^{\infty} 2^{-i s' \varphi} 2^{i n \varphi (\frac{1}{r} - \frac{1}{p})} \\
& \leq c_3 \sum_{j=0}^{M} 2^{-j \delta} 2^{-(M+1) s' \varphi} 2^{(M+1) n \varphi (\frac{1}{r} - \frac{1}{p})} \\
& \leq c_4 2^{-M s' \varphi} 2^{M n \varphi (\frac{1}{r} - \frac{1}{p})}.
\end{align*}
\]

This gives
\[
\begin{align*}
e_{2Mn} (P^2 : \ell_{q_1} (2j^{\delta} \ell_{p_1} (w)) \to \ell_{q_2} (\ell_{p_2})) \leq c_4 2^{nM (\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n})}.
\end{align*}
\]

Now we estimate \( L_{r, \infty}^{(e)} (Q^2) \). We assume for \( 0 < d \leq n \) again condition (16)
\[
\begin{align*}
L_{r, \infty}^{(e)} (Q^2) & \leq c_2 \sum_{j=M+1}^{\infty} 2^{-j \delta} \sum_{i=0}^{j} 2^{-i s' \varphi} 2^{j (n-d) \varphi (\frac{1}{r} - \frac{1}{p})} \\
& \leq c_3 \sum_{j=M+1}^{\infty} 2^{-j \delta} 2^{j (n-d) \varphi (\frac{1}{r} - \frac{1}{p})} \\
& \leq c_4 2^{-M s' \varphi} 2^{M d \varphi (\frac{1}{r} - \frac{1}{p})}.
\end{align*}
\]
This gives
\[ e_{2d} \left( Q^2 : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_4 2^{dM\left(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{\delta}{n} - \frac{s'}{n}\right)}, \quad d \neq 0. \]

In case \( d = 0 \) we chose instead of condition (16) the parameter \( r \) such that
\[ s' + \delta > n \left( \frac{1}{r} - \frac{1}{p} \right) > s' \]
holds and obtain in this case
\[ e_{2Mn} \left( Q^2 : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_4 2^{nM\left(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n} - \frac{\delta}{n}\right)}. \]

To estimate \( L_{r,\infty}^{(e)}(P^3) \) we assume (16) for \( 0 \leq d \leq n \)
\[
L_{r,\infty}^{(e)}(P^3)^e \leq c_2 \sum_{j=M+1}^{\infty} 2^{-j\delta} \sum_{i=j}^{\infty} 2^{-is'\theta} 2^{i n\theta\left(\frac{1}{r} - \frac{1}{p}\right)} \\
\leq c_3 \sum_{j=M+1}^{\infty} 2^{-j\delta} 2^{-js'\theta} 2^{jn\theta\left(\frac{1}{r} - \frac{1}{p}\right)} \\
\leq c_4 2^{-(M+1)\delta} 2^{-(M+1)s'\theta} 2^{(M+1)n\theta\left(\frac{1}{r} - \frac{1}{p}\right)} .
\]

This gives
\[ e_{2Mn} \left( P^3 : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_4 2^{nM\left(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n} - \frac{\delta}{n}\right)}. \]

Summarizing all the estimates we get by the additivity and monotonicity of the entropy numbers
\[ e_k \left( id : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_k 2^{-\frac{1}{p_1} + \frac{1}{p_2} - \min\left(\frac{s'}{n}, \frac{\delta}{n}\right)} \leq c_k 2^{-\frac{1}{p_1} + \frac{1}{p_2} - \min\left(\frac{s'}{n}, \frac{\delta}{n}\right)}, \]
where we use the convention \( \frac{\delta}{0} = \infty. \)

**Step 3.** For the estimate from below we consider the commutative diagram
\[
\begin{array}{ccc}
\ell_{p_1}^{M_{j,i}} & \xrightarrow{S_{j,i}} & \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \\
\downarrow \text{id}_1 & & \downarrow \text{id} \\
\ell_{p_2}^{M_{j,i}} & \xleftarrow{T_{j,i}} & \ell_{q_2}(\ell_{p_2})
\end{array}
\]

Here
\[
(T_{j,i} \lambda)_{\varphi(u,v)} := \lambda_{u,v}, \quad (u,v) \in I_{j,i}, \quad \text{if } (u,v) \in I_{j,i},
\]
\[
(S_{j,i} \eta)_{u,v} := \begin{cases} 
\eta_{\varphi(u,v)} & \text{if } (u,v) \in I_{j,i}, \\
0 & \text{otherwise},
\end{cases}
\]

\( 17 \)
where ϕ denotes a bijection of $I_{j,i}$ onto $\{1, 2, \ldots M_{j,i}\}$. Observe
\[
\left\| T_{j,i} \right\|_{\ell_{p_1}(\ell_{2}) \to \ell_{p_2}^{M_{j,i}}} = 1
\]
and
\[
\left\| S_{j,i} \right\|_{\ell_{p_1}^{M_{j,i}} \to \ell_{q_1}(2^{j^d} \ell_{p_1}(w))} \sim 2^{j^d 2^{i^*}}.
\]
Hence we obtain
\[
e_k(\text{id}_1 : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}) \leq \| S_{j,i} \| \| T_{j,i} \| e_k(\text{id} : \ell_{q_1}(2^{j^d} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})) \leq c 2^{j^d 2^{i^*}} e_k(\text{id} : \ell_{q_1}(2^{j^d} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})).
\]
(17)
Using once more the characterization from (10) and (11) and now the estimate of the $M_{j,i}$ from below (9). In case $i = 0$ either $M_{j,1}$ or $M_{j,2}$ can be estimated from below by $N_{j,0}/2$, respectively because of Lemma 1 by $c2^j$ with a constant $c > 0$. Therefore without lost of generality we can assume $2^{jd} \sim M_{j,2}$ where the estimate from above is justified again by (8) and Lemma 1. Now we find from (17) with $k = 2^{jd}$
\[
k^{-\frac{1}{\alpha} - \frac{1}{m} + \frac{1}{p_1}} \leq c e_k(\text{id} : \ell_{q_1}(2^{j^d} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})) \quad \text{if} \quad i = 2, d \neq 0.
\]
In the same way, using $M_{0,i+1}$ or $M_{0,i+2}$ we get from (17) with $k = 2^{in}$
\[
k^{-\frac{1}{\alpha} - \frac{1}{m} + \frac{1}{p_2}} \leq c e_k(\text{id} : \ell_{q_1}(2^{j^d} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})) \quad \text{if} \quad j = 0.
\]
The estimate for the remaining $k$ follows by monotonicity of the entropy numbers. This finishes the proof. ■

5 Entropy numbers of embeddings of $2$-microlocal Besov spaces

The next theorem, which is the main theorem of the paper, follows now immediately from Proposition 2, Theorem 2 and Theorem 3.

**Theorem 4** Let $U$ be a $d$-set, $0 \leq d \leq n$ and $w_{i,j}(x) = (1 + 2^j \text{dist}(x, U))^{s_i}$ \quad $i = 1, 2$. Let $\frac{1}{p^*} := (\frac{1}{p_2} - \frac{1}{p_1})_+$ and
\[
\delta = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \quad \text{and} \quad s' = s_1' - s_2'.
\]
The embedding
\[
B_{p_1,q_1}^{s_1,s_1'}(\mathbb{R}^n, U) \hookrightarrow B_{p_2,q_2}^{s_2,s_2'}(\mathbb{R}^n, U)
\]
is compact if and only if $\delta > d/p^*$ and $s' > n/p^*$.
Moreover, for the entropy numbers of the embedding we have
\[
e_k(\text{id} : B_{p_1,q_1}^{s_1,s_1'}(\mathbb{R}^n, U) \to B_{p_2,q_2}^{s_2,s_2'}(\mathbb{R}^n, U)) \sim k^{-\frac{1}{\alpha} - \frac{1}{m} - \min(\frac{\delta}{\alpha}, s')}.
\]
Corollary 2 If $U = \{x_0\}$ then the embedding (18) is compact if and only if $\delta > 0$ and $s' > n/p^*$. Moreover

$$e_k(id : B^{s_1, s'_1}_{p_1, q_1}(\mathbb{R}^n, \{x_0\}) \to B^{s_2, s'_2}_{p_2, q_2}(\mathbb{R}^n, \{x_0\})) \sim k^{-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n}}.$$ 

In particular the embedding $H^{s_1, s'}_{x_0}(\mathbb{R}^n) \hookrightarrow C^{s_2, s'}_{x_0}(\mathbb{R}^n)$ is compact if and only if $s_1 - s_2 > n/2$ and $s'_1 > s'_2$. Furthermore

$$e_k(id : H^{s_1, s'}_{x_0}(\mathbb{R}^n) \to C^{s_2, s'}_{x_0}(\mathbb{R}^n)) \sim k^{-\frac{1}{2} - \frac{s'}{n}}.$$ 

Remark 4 At last we want to compare our main theorem with the known results about entropy numbers of embeddings of Besov spaces with polynomial weights. We will consider the case $U = \{0\}$. For the weighted spaces we have

$$e_k(id : B^{s_1}_{p_1, q_1}(\mathbb{R}^n, <x>^\alpha) \hookrightarrow B^{s_2}_{p_2, q_2}(\mathbb{R}^n)) \sim k^{-\frac{1}{p_1} - \frac{1}{p_2} - \min(\frac{s}{n}, \alpha)}.$$ 

if $\min(\delta, \alpha) > n/p^*$ and $\delta \neq \alpha$ where again $\delta = s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2})$ - see for example [KLSS 2].

In the case $U = \{0\}$ the weights of the admissible weight sequence become $w_j(x) = (1 + 2^j |x|)^{s'}$, that is for each fixed $j$ they are equivalent to the polynomial weight

$$w(x) = (1 + |x|^2)^{s'/2} = <x>^{s'}.$$ 

The spaces $B^{t}_{p,q}(\mathbb{R}^n, <x>^\alpha)$ have a wavelet characterization with the same wavelet basis as described in Section 2.2. Therefore we can reduce the comparison of entropy numbers of the embeddings to the comparison of corresponding weights. Moreover we can shift all considerations to the case were the target space is $B^{n/p_2}_{p_2, q_2}(\mathbb{R}^n)$ since

$$e_k(id : B^{s_1, s'_1}_{p_1, q_1}(\mathbb{R}^n, \{0\}) \to B^{s_2, s'_2}_{p_2, q_2}(\mathbb{R}^n, \{0\})) \sim e_k(id : B^{\delta + n/p_1}_{p_1, q_1}(\mathbb{R}^n, \{0\}) \to B^{n/p_2}_{p_2, q_2}(\mathbb{R}^n))$$ 

and

$$e_k(id : B^{s_1}_{p_1, q_1}(\mathbb{R}^n, <x>^\alpha) \to B^{s_2}_{p_2, q_2}(\mathbb{R}^n))$$

$$\sim e_k(id : B^{\delta + n/p_1}_{p_1, q_1}(\mathbb{R}^n, <x>^\alpha) \to B^{n/p_2}_{p_2, q_2}(\mathbb{R}^n)).$$

(19)

If $s' > 0$ then by easy calculations we have

$$2^{-j s'} w_j(2^{-j} m)^{s'} \leq <2^{-j} m>^{s'} \leq w_j(2^{-j} m)^{s'} \leq 2^{j s'} <2^{-j} m>^{s'}.$$ 

These inequalities gives for $\alpha = s'$ the following embeddings

$$B^{t + s'}_{p,q}(\mathbb{R}^n, <x>^s) \hookrightarrow B^{t,s'}_{p,q}(\mathbb{R}^n, \{0\}) \hookrightarrow B^{t}_{p,q}(\mathbb{R}^n, <x>^s) \hookrightarrow B^{t-s'}_{p,q}(\mathbb{R}^n, \{0\}).$$

(20)

If $\delta > s'$ then

$$B^{\delta + n/p_1-s', s'}_{p_1, q_1}(\mathbb{R}^n, \{0\}) \hookrightarrow B^{n/p_2}_{p_2, q_2}(\mathbb{R}^n).$$

Hence the estimates for the entropy numbers of embeddings of the spaces with polynomial weight follows from (20) and Corollary 2 with $\{x_0\} = \{0\}$. Vice versa the result for the spaces with polynomial weight gives in this particular case the result of Corollary 2.
References


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