Planarity in ROMDD's of Multiple-Valued Symmetric Functions*

Jon T. Butler  Jeffrey L. Nowlin  Tsutomu Sasao

Department of Electrical and Computer Eng.  Naval Postgraduate School, Code EC/Bu  Monterey, CA 93943-5121 U.S.A.

Dept. of Computer Science and Electronics  Kyushu Institute of Technology  Iizuka 820, Japan

Abstract
We show that a multiple-valued symmetric function has a planar ROMDD (reduced ordered multiple-valued decision diagram) if and only if it is a pseudo-voting function. We show that a multiple-valued symmetric function has a planar ROMDD (reduced ordered multiple-valued decision diagram) if and only if it is a pseudo-voting function. We show that a multiple-valued symmetric function has a planar ROMDD (reduced ordered multiple-valued decision diagram) if and only if it is a pseudo-voting function. We show that a multiple-valued symmetric function has a planar ROMDD (reduced ordered multiple-valued decision diagram) if and only if it is a pseudo-voting function. We show that a multiple-valued symmetric function has a planar ROMDD (reduced ordered multiple-valued decision diagram) if and only if it is a pseudo-voting function.

The number of such functions is \( b \cdot \frac{n!}{(r-1)！} \), where \( r \) is the number of logic values and \( n \) is the number of variables. It follows from this that the fraction of symmetric multiple-valued functions that have planar ROMDD's approaches 0 as \( n \) approaches infinity. Further, we show that the worst case and average number of nodes in planar ROMDD's of symmetric functions is \( n^r \frac{1}{2} \) and \( n^r \frac{1}{(r+1)} \) respectively, when \( n \) is large.

Index Terms: multiple-valued function, decision diagrams, ROBDD's, ROMDD's, symmetric functions, voting functions.

1 Introduction
For more than a decade, binary decision diagrams (BDD's) have been used to efficiently represent switching functions. Introduced by Akers[1] and others, it was not until 1986 with a paper by Bryant[2], that BDD's have become a predominant data structure for switching function representation.

Multiple-valued logic functions can be represented by multiple-valued decision diagrams (MDD's) which are a natural extension of BDD's. MDD's have been treated by Miller[5] and Sasao[6]. In this paper, we extend our results on planar MDD's as described in Sasao and Butler [7].

We consider two types of functions. In the first type, a multiple-valued function, \( f: \mathbb{R}^r \rightarrow \mathbb{R} \), where \( \mathbb{R} = \{0,1,\ldots,r-1\} \), both the function and the variables take on values from \( \mathbb{R} \). We denote a function with \( r = 2 \) as a switching function. In the second type, \( f: \mathbb{R}^r \rightarrow \{0,1\} \), the function is two-valued, but the variables are \( r \)-valued.

An MDD of a function \( f(x_1, x_2, \ldots, x_r) \) is a directed graph that has a root node (i.e. no incoming edges) which represents \( f \). From this node, there are \( r \) outgoing edges labeled 0,1,...,r-1 directed to nodes that represent \( f(0, x_2, \ldots, x_r), f(1, x_2, \ldots, x_r), \ldots, f(r-1, x_2, \ldots, x_r) \) respectively. For each of these nodes, there are \( r \) outgoing edges, that go to nodes that have \( r \) outgoing edges, etc. A terminal node is a node with no outgoing edges. It is labeled by 0,1,..., or \( r-1 \), and corresponds to a logic value of the function. An MDD is a data structure. To reduce storage requirements, the following rules will be applied.

- **merging rule** - if nodes 0 and 1 represent the same function, they are combined into one, as are descendant nodes and edges.
- **elimination rule** - if all children of node 0, are the same node 0, then 0 is eliminated and all incoming edges to 0 go to 0.

**Definition 1** An ordered multiple-valued decision diagram (OMDD) is an MDD in which the relative order of any pair of variables is the same for all paths from the root node to any terminal node.

**Definition 2** A reduced OMDD or ROMDD is an OMDD in which the merging and elimination rules have been applied to the greatest extent possible.

Bryant[2] has shown that, for any given ordering of variables, the ROBDD is unique. Therefore, regardless of what order the merging and elimination rules are applied, the final ROBDD is the same. The same argument applies to ROMDD's. A special type of OMDD is examined in this paper. In VLSI, a significant source of delay is interconnect, and a significant component of interconnect delay occurs at crossings. For example, in FPGA's, a significant source of delay occurs in crossings among interconnections between cells. Via resistance, and thus delay, increases as feature size is decreased. For a discussion of circuit implementations based on MDD's and the role of crossings in such realizations, the reader is referred to [7,8]. We adopt the restrictions in [7,8] restated as follows.

**Restriction 1**

\( a \): All edges are directed down throughout their length.

\( b \): All edges emerging from a node are labeled 0,1,...,r-1 from left to right, and

\( c \): The terminal nodes (representing constant functions) are labeled 0,1,...,r-1 from left to right.

Restriction 1a) precludes, for example, arcs that extend around the root node or terminal nodes (e.g. Fig. 13 of [7]). It is a simplifying assumption that makes uniform the interconnections between levels. Restriction 1b) and 1c) are also simplifying assumptions. However, they can be removed, enlarging the set of functions for which our results apply. For our purposes, these restrictions allow a tractable analysis.

**Definition 3** An OMDD is planar if it can be drawn without crossings.
Because of their importance in logic design, we conside r symmetric functions. Symmetric functions are indispensable in arithmetic circuits; indeed, such circuits represent one of the most important applications of multiple-valued logic [4].

**Definition 4** A symmetric function is a function that is unchanged by any permutation of variables.

In this paper, we consider multiple-valued functions and their representation using decision diagrams. We show necessary and sufficient conditions for planarity in the ROMDD of symmetric functions.

### 2 Background

In this section we consider conditions that cause non-planarity in ROMDD's.

**Lemma 1** If the ROMDD of a multiple-valued variable, two-valued function has at least two nodes associated with the lowest variable, then it is non-planar.

**Proof** Assume \( x_i \) labels the variable just above the termina l nodes. Consider a node \( 0 \) at the \( x_i \) level. Because of the elimination rule, not all of its edges go to 0 and not all go to 1. With no crossings among edges from 0 to 0 and 1, the \( x_i = 0 \) edge must go to the terminal node 0 and the \( x_i = r - 1 \) edge must go to the terminal node 1. If there are two nodes at the \( x_n \) level, each must satisfy this requirement, and there is at least one crossing. Q.E.D.

This result allows us to make the following observation.

**Definition 5** \( f \) is a voting function if \( f = g \) if \( T_j \leq \sum_{i=1}^{n} x_i < T_{j+1} \), where 0 \# \( T_0 \# T_1 \# \ldots \# T_{r-1} \# T_r = n(r - 1) + 1 \). \( g \) is a binary voting function on multiple-valued variables if it is a voting function with \( T_0 = T_1 = \ldots = T_r = n(r - 1) + 1 \). Associated with \( g \) is a weight-threshold vector \((1,1,\ldots,1;T)\), where \( T = T_y \).

**Lemma 2** For any \( n > 1 \) and \( r > 2 \), there exists a function \( f \) with an ROMDD that is non-planar for any ordering of the \( r \) variables.

**Proof** Consider a binary voting function \( f \) on multiple-value d variables, with weight-threshold old vector \((1,1,\ldots,1;2)\). There are two nodes associated with the last variable in the ordering, one that can be reached with a cumulative weight(CW) of 0 and the other with a CW = 1. There are \( r - 2 \) unavoidable crossings. Since \( f \) is totally symmetric, altering the variable order will not change the ROMDD. Q.E.D.

We now consider symmetric multiple-valued logic functions. A necessary and sufficient condition for planarity of ROBDD's of binary voting functions exists[8]. We extend this result to functions with \( r \)-valued variables for \( r > 2 \).

**Lemma 3** Let \( f(x_1, x_2, \ldots, x_n) \) be a binary voting function with \( n \)-valued variables, where \( n > 1 \) and \( r > 2 \). \( f \) has a non-planar ROMDD if \( f \) has a weight-threshold vector \((1,1,\ldots,1;T)\), where \( 1 < T < n(r - 1) \).

**Proof (if)** Let \( f \) be a symmetric threshold function with weight-threshold vector \((1,1,\ldots,1;T)\), where \( 1 < T < n(r - 1) \). We show that it has a non-planar ROMDD as follows.

Assume without loss of generality, that the order of the \( n \) variables from top to bottom is \( x_1, x_2, \ldots, x_n \). Consider two assignments \( A \) and \( B \) of values to the upper \( n-1 \) variables \( x_1, x_2, \ldots, x_{n-1} \), such that \( \sum_{i=1}^{n-1} x_i = \max(0,T-(r-1)) \) and \( \sum_{i=1}^{n-1} x_i = \min((n-1)(r-1),T-1) \), respectively. Since all weights in the weight-threshold vector are 1, \( \sum_{i=1}^{n-1} x_i \) is the number of variables equal to 1 in the assignments \( A \) and \( B \).

Consider two assignments \( A_{k=0} \) and \( A_{k=r-1} \) to all variables \( x_1, x_2, \ldots, x_n \) such that \( x_k = 0 \) and \( r - 1 \), respectively, while the values assigned to \( x_1, x_2, \ldots, x_{n-1} \) are made according to \( A \). Since \( T > 1 \) and \( r > 2 \), assignment \( A_{k=0} \) results in \( \sum_{i=1}^{n} x_i < T \). Therefore, \( f = 0 \) for \( A_{k=0} \). However, \( A_{k=r-1} \) results in \( \sum_{i=1}^{n} x_i > T \). That is, if \( \sum_{i=1}^{n-1} x_i = T - (r - 1) \), we have \( \sum_{i=1}^{n} x_i = T \), and if \( \sum_{i=1}^{n} x_i = 0 \), then \( T \# (r - 1) \), since \( \sum_{i=1}^{n} x_i = \max((T - (r - 1)) \). It follows that \( \sum_{i=1}^{n} x_i \geq T \), since \( x_n = r - 1 \). Therefore, \( f = 1 \) for \( A_{k=r-1} \). Because the value of \( x_n \) determines whether \( f = 0 \) or 1 with assignment \( A \), it follows that \( A \) corresponds to a path to a node \( O_2 \) at the \( y \) level. Further, there is an edge from \( O_2 \) to 0 labeled 0 and an edge from \( O_2 \) to 1 labeled \( r - 1 \).

By a similar argument, it can be shown that assignment \( B \) corresponds to a path to a node \( O_2 \) with an edge labeled 0 going to 0 and an edge labeled \( r - 1 \) going to 1.

We now show that \( O_2 \) and \( O_2 \) are distinct nodes, by showing that the weight accumulated across \( x_1, x_2, \ldots, x_n \) is different for these two nodes. For \( 1 < T < r \), \( O_2 \) is associated with a weight of \( \max(0,T - (r - 1)) \), while \( O_2 \) is associated with a weight of \( \min((n - 1)(r - 1),r - 1) \). For \( r \# T \# (n - 1)(r - 1) \), \( O_2 \) is associated with a weight of \( \max(0,T - (r - 1)) \), while \( O_2 \) is associated with a weight of \( \min((n - 1)(r - 1),T - 1) \). \( O_2 \) is associated with a weight of \( \min((n - 1)(r - 1),T - 1) \). \( O_2 \) is associated with a weight of \( \min((n - 1)(r - 1),T - 1) \). This means, since \( r > 2 \), \( O_2 \) is associated with a weight of \( \min((n - 1)(r - 1),T - 1) \). \( O_2 \) is associated with a weight of \( \min((n - 1)(r - 1),T - 1) \). This means, since \( (n - 1)(r - 1) > T - (r - 1) \) for this range. Thus \( O_2 \) and \( O_2 \) are distinct nodes for all \( T \) bounded by \( 1 < T < n(r - 1) \). Since there are two distinct nodes at the \( x_n \) level, Lemma 1 applies, and we conclude that the ROMDD for \( f \) is non-planar.

**(only if)** Assume that \( f \) has a non-planar ROMDD, and assume, on the contrary, that either \( T \# 1 \) or \( n(r - 1) \# T \). If \( T = 1 \), then \( f \) has an ROMDD as shown in Fig. 1(a), which has no crossings, contradicting the assumption that \( f \) has a non-planar ROMDD. That is, the ROMDD for \( f \) is unique; no reordering of variables produces a different structure, specifically one with crossings.

If \( T < 1 \), then \( f = 1 \) and is represented by a single terminal 1 node labeled 1 which is planar, contradicting the assumption.
If \( T = n(r - 1) \), \( f \) has the ROMDD shown in Fig. 1(b) which is planar, again contradicting the assumption. If \( n(r - 1) < T \), then \( f = 0 \) and is represented by a single terminal node labeled 0, which is again planar. Thus, it must be that \( 1 < T < n(r - 1) \).

Q.E.D.

It is interesting that Lemma 3 cannot be stated for \( n \) and \( r \) outside the range \( n > 1 \) and \( r > 2 \). That is, if \( n = 1 \), then all ROMDDs for \( f \) are represented by the structure shown in Fig. 2, which is planar.

Consider \( r = 2 \). We find that the ROMDD for the function \( f \) associated with weight-threshold vector \((1,1,1;2)\) as shown in Fig. 3 is planar. For this case, we have a weight-threshold vector \((1,1,1;T)\) with \( 1 < T < n(r - 1) \) that corresponds to an ROMDD which is planar. Therefore, Lemma 3 does not apply when \( r = 2 \).

3 Planar ROMDD's of Symmetric Functions

In this section, we show a necessary and sufficient condition for an \( r \)-valued symmetric function to have a planar ROMDD. Such a condition has already been established for \( r = 2 \). Specifically,

**Lemma 4** [8] A symmetric switching function \( f \) has a planar ROBDD iff \( f \) is a voting function.

It is tempting to believe that this extends to multiple-valued functions. However, a counterexample exists for the same statement when the radix \( r \) exceeds 2. The function whose symmetric ROMDD is shown in Fig. 4 is symmetric and has a planar ROMDD. However, it is not a voting function. For example, \( x_1x_2 = 11 \), yields \( f = 0 \), while \( x_1x_2 = 02 \) yields \( f = 1 \). That is, two assignments of values to the variables with the same sum yield a different value of \( f \).

**Definition 6** Let \( M = \{a+1, a+2, \ldots, r-1\} \) be a proper subset of logic values, where \( 0 \neq a \neq r-2 \). Given an assignment \( A \) of values to variables \( x_1, x_2, \ldots, x_n \), let \( n_f(A) \) be the number of variables whose value is in \( M \). A multiple-valued function \( f \) is a pseudo-voting function if there exists a value \( a \) such that \( f(A) \) depends only on \( n_f(A) \) and \( f(A) \leq f(A') \) iff \( n_f(A) \leq n_f(A') \).

**Theorem 1** A multiple-valued symmetric function \( f \) with \( n > 1 \) variables has a planar ROMDD iff \( f \) is a pseudo-voting function.

In a multiple-valued pseudo-voting function, the variables are partitioned into two parts. For some assignment \( A \) of values to these variables, a count, \( n_f(A) \), is made of the number of variables that fall in the upper part of the variable logic value partition, and this determines the function value. A further restriction exists that the function value for some assignment \( A \) is never greater than for another assignment \( A' \), if \( n_f(A) < n_f(A') \).

**Example 1** Consider the 3-valued function shown in Fig. 5. This function is a pseudo-voting function with \( a = 1 \). Hatched regions show variable values in \( M \).

Note that, when \( r = 2 \), a pseudo-voting function is a conventional voting function. The function in Fig. 5 has a non-OMDD of the form shown in Fig. 4 above, which is planar.

Consider \( f \) a pseudo-voting function in which the variable values are divided into two contiguous parts, the upper part being \( M \). Then, \( f \) is realized by the planar OMDD shown in Fig. 6 below. Here, all nodes are shown, even nodes that can be eliminated by the merging and elimination rules. Such an OMDD is called a complete symmetric decision diagram [7]. A terminal node 0 is labeled by the number of variables that belong to \( M \) in the assignment \( A \) of values to variables that corresponds to the path from the root node to 0.
Proof (if) Since \( f \) is a pseudo-voting function, functional logic values labeling the terminal nodes are in ascending order left to right. We can apply the merging and elimination rules to produce an ROMDD of \( f \). For example, two adjacent nodes labeled by the same logic value and their parent node can be replaced by a single node. Both rules preserve planarity. Since the original OMDD, as given in Fig. 6, is planar, the resulting ROMDD is also planar.

(only if) Consider a multiple-valued symmetric function \( f \) that has a planar ROMDD. First, we show that every node \( e \) with children has exactly two children. Then, we show that the distribution of edges to children is the same for every node. This allows us to show that \( f \) is realized by a complete symmetric decision diagram, as in Fig. 6 with the termina nodes labeled by logic values in ascending order left to right. We can then conclude that \( f \) is a pseudo-voting function.

Consider a node \( 0 \) in the ROMDD of \( f \), as shown in Fig 7.

![Figure 7: A node 0 of the ROMDD of f.](image)

Assume that \( 0 \) is associated with \( x_1 \) and there is at least one child of \( 0 \) that is associated with \( x_{i+1} \). That is, there are at least two variables between \( 0 \) and a terminal node. Such a node \( e \) exists because \( n > 1 \). Let \( 0_0, 0_+ \), and \( 0_- \) be the children nodes of \( 0 \) associated with edges labeled by \( 0, a +, \) and \( a - \), respectively where \( 0 < a < r - 1 \).

First, \( 0_0 \) and \( 0_{+1} \) are distinct. Indeed, if \( 0_0 = 0_{+1} \), then edges labeled \( 1, 2, \ldots, r - 2 \) from \( 0 \) must also go to \( 0_0 = 0_{+1} \). Otherwise, there are crossings. However, by the elimination rule, \( 0 \) would be eliminated. Second, \( 0_0 \) and \( 0_{-1} \) are not both terminal nodes. Indeed if they were both terminal nodes, they would have to be the same node, since by symmetry of \( f \), \( x_{i+1} = 0 \) and \( r - 1 \) 0 must lead to the same node. However, as discussed earlier, \( 0_0 \) and \( 0_{-1} \) must be distinct.

Consider the paths originating from \( 0_+ \). Not all go to \( 0_{0+1} \). Otherwise, \( 0_+ \) does not exist by the elimination rule. But, for the edges from \( 0_+ \) to go to nodes outside the diamond shown in Fig. 7 above, crossings are required. It follows, therefore, that either \( 0_0 = 0_+ \) or \( 0_0 = 0_{+1} \), regardless of the value of \( a \). Since the planarity of the ROMDD of \( f \) excludes crossings among edges from \( 0 \) to its children, there exists an \( a \) such that \( 0_0 \) = \( 0_+ \) or \( 0_0 \) = \( 0_{+1} \).

We now show that \( a \) is the same for every node. Consider, for example, the root node and the children nodes, as shown in Fig. 8 below. We claim \( a' = a \). On the contrary, suppose \( a' \neq a \). First, suppose that \( a' < a \). Then, \( a' \) which is reached if \( x_{i+1} = a' \), must be the same as \( 0_+ \), which is reached if \( x_{i+1} = a \). Since \( f \) is symmetric. Since \( 0_0 = 0_+ \), all children nodes of \( 0_0 \) are the same and, by the elimination rule, \( 0_1 \) does not exist. Next, suppose that \( a' > a \). Since \( f \) is symmetric, the node corresponding to \( x_1 x_2 = a \) must be the same as the node corresponding to \( x_1 x_2 = (r-1) a \). Thus, it follows that \( a' \neq a \). Since \( a' > a \) and \( a'' \neq a \), the node corresponding to \( x_1 x_2 = a' \) is \( 0_0 \). Since \( x_1 = a \) corresponds to \( 0_0 \) and \( f \) is symmetric, \( 0_0 = 0_+ \), and all children nodes of \( 0_0 \) are the same. By the elimination rule, \( 0_1 \) does not exist. From this, we can conclude that \( a'' = a \). Similarly, we can show that \( a'' = a \), and that all left-going edges of all such nodes are labeled by \( \{0, 1, \ldots, a \} \). From this, it follows that all right-going edges are labeled by \( \{a +, a +2, \ldots, r - 1 \} \). Therefore, the function realized by the RO MDD depends only on whether the value of \( x \) is in \( \{0, 1, \ldots, a\} \) or \( \{a +, a +2, \ldots, r - 1 \} \).

Edges from a node 0 can go only to the next level down (if the final value of the function is, up to this point, undetermined) or to a terminal node (if the final value of the function is, up to this point, completely determined). That is, certain variables cannot be skipped, and others not skip the causal function to be dependent on some variables and not on others, since the function is symmetric.

The OMDD (not neccessarily reduced) that realizes \( f \) has the structure shown in Fig. 6. The characteristic diamond shape, as shown in Fig. 9, occurs because the function realized when \( x_{i+1} = \$ \), where \( x \in \{0, 1, \ldots, a\} \) or \( 0 \{a +, a +2, \ldots, r - 1 \} \) is the same as the function realized when \( x_{i+1} = \$ \).

From Restriction 1, the terminal nodes of a planar ROMDD, are labeled in ascending order left to right. The function realized by this ROMDD is a pseudo-voting function. Q.E.D.

![Figure 8: Root node 0 and its children nodes.](image)

![Figure 9: Characteristic diamond shape in an ROMDD of a symmetric function.](image)

By comparing Lemma 3 with Theorem 1, we can state,

**Corollary 1** A two-valued function with multiple-value \( d \) variables associated with weight-threshold vector \( (1, 1, \ldots, 1; T) \) is a pseudo-voting function iff \( T = 1 \) or \( n(r-1) \).

We end this section by counting pseudo-voting functions.

**Lemma 5** The number of pseudo-voting functions with \( n \) variables and \( r \) values is

\[
M_{\text{pseudo-voting}} = \frac{1}{n+1} \cdot \frac{f^{r+n}}{f^{r+1}}
\]

**Proof** We count the ways to configure a complete symmetric decision diagram of a pseudo-voting function. There are \( r-1 \) ways to partition the \( r \) labels of outgoing edges from each node. The number of ways \( s \) to assign logic values in ascending order left to right of the terminal nodes is the number of ways to choose \( n+1 \) objects (the terminal nodes in a complete symmetric decision diagram of the function) from the \( r \) labels \( \{0, 1, \ldots, r \} \) with repetition, which is

\[
\binom{r+n+1}{n+1} = \frac{f^{r+n}}{f^{r+1}}
\]

Q.E.D.
It is interesting to further compare the number of pseudo-voting functions with the number of symmetric functions on $n$ variables and $r$ logic values. The latter is $\sum_{i=0}^{r} \binom{r}{i}(\text{viz.} [3]).$ When $r = 2$, this expression yields for the number of symmetric switching functions $2^{n+1}$. We have, therefore, from Theorem 1, the following Lemma 6:

**Lemma 6** The fraction of $r$-valued symmetric functions that have planar ROMDD’s approaches 0 as $n$ approaches infinity, where $n$ is the number of variables.

## 4 Number of Nodes in ROMDD’s

Consider the worst case number of nodes $WC_r(n)$ in ROMDD’s of pseudo-voting functions. In a complete symmetric decision diagram, there are $1 + 2 + \ldots + (n+1) = (n+2)(n+1)/2$ nodes. However, sequences of identical logic values yield nodes with identical children nodes, that can be eliminated by the elimination rule. For example, Fig. 10 shows how a group of three 1’s and of three identical children nodes, that can be eliminated by the elimination rule. Specifically, the group of three 1’s has three nodes, which can be eliminated by the elimination rule.

For $m_1$ and $m_2$ held constant for a given $i$ and $j$, any logic value can occur on terminal nodes. From Lemma 5, we have the total number of nodes in complete symmetric decision diagrams is

$$N_r = (1 + 2 + \ldots + m_i - 1) = (m_i + 1)/2 - 1.$$ 

When $r = 2$, we achieve the minimum total reduction over all $m_i$ and $m_j$ with the most uniform distribution of terminal nodes. This occurs when

$$m_i = n+1,$$ $\text{as } n \to \infty,$ $\text{for } 0 \leq i \leq r - 1.$

The total reduction for this worst case is

$$R_{\min} = \frac{n^2}{4}.$$ 

For the total number of nodes before reduction, we have

$$N_r = (1 + 2 + \ldots + n) = (n+2)(n+1)/2.$$ 

When $r = 2$, $WC_r(n) = n^2/4$ for large $n$. We now derive the average number of nodes $A_r(n)$ as follows,

$$A_r(n) = \frac{N_{\text{complete}} - N_{\text{reduction}}}{M_{\text{pseudo-voting}}}.$$ 

where $N_{\text{complete}}$ is the total number of nodes in complete symmetric decision diagrams of pseudo-voting functions, $N_{\text{reduction}}$ is the reduction of nodes that occurs because of consecutive logic values on terminal nodes. From Lemma 5, we have the total number of pseudo-voting functions, $M_{\text{pseudo-voting}}$. Therefore, the total number of nodes in complete symmetric decision diagrams of pseudo-voting functions is

$$N_{\text{complete}} = (r-1) \frac{n^2}{2} + 1.$$ 

We calculate $N_{\text{reduction}}$ as follows. Any logic value can occur $m$ times at the terminal nodes of a complete symmetric decision diagram, where $0 \leq m \leq n+1$. As shown previously, $m(m+1)/2$ nodes are replaced by a single node, yielding a reduction of $m(m+1)/2 - 1$ nodes.

There are

$$\binom{n - 1 + m - 1}{n - 1} = \binom{n + m - 1}{n - 1}$$ 

ways to choose a distribution of $r - 1$ remaining logic values to the $n+1-m$ remaining terminal nodes. Specifically, these are chosen by selecting $n+1-m$ objects (terminal nodes) from $r - 1$ objects.
objects (remaining logic values) with repetition. Since this is true for any of the $r$ logic values and for any of the $r - 1$ ways to partition $r$ logic values into two parts corresponding to labels on outgoing edges of each node, we have for $N_{\text{reduction}}$

$$N_{\text{reduction}} = \sum_{i=0}^{r-1} \binom{r}{i}(r-1)^i b x g$$

We solve this sum using generating functions. First, it is convenient to substitute $i = m - 1$. Doing this and rearranging yields

$$N_{\text{reduction}} = \sum_{i=0}^{r-1} \binom{r}{i}(r-1)^i b x g$$

A generating function $G(x)$ in which the coefficient of $x^n$ in the above sum is

$$G(x) = A(x) B(x),$$

where,

$$A(x) = r(r-1)^2 x^2 b x g$$

and,

$$B(x) = \frac{1}{b x g}.$$ 

Therefore,

$$G(x) = r(r-1)^2 x^2 b x g.$$ 

The coefficient of $x^n$ in this expression is

$$N_{\text{reduction}} = r(r-1)(n-1)^n b x g.$$ 

Therefore, we have,

**Lemma 9** The average number of nodes in ROMDD's of $r$-valued $n$-variable pseudo-voting functions is

$$A_r(n) = \frac{(r+1)(n+1)-2}{2}.$$ 

Consider the expression for $A_r(n)$ when $n$ is large. $N_{\text{reduction}}$ can be written as

$$r(r-1)^2 x^2 b x g.$$ 

When $n$ is large compared to $r$, each term in the numerator is approximately $n$, and so, for large $n$, this expression is

$$r(r-1)^2 x^2 b x g.$$ 

Since $r(r-1)^2 x^2 b x g = n^{r-2}$ when $n$ is large, we have,

**Lemma 10** The average number of nodes in ROMDD's of $r$-valued $n$-variable pseudo-voting functions for $n \to \infty$ is

$$A_r(n) \sim n^{r-2}.$$ 

where $f(n) - g(n)$ means $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$. When $r = 2$, the average number of nodes is $n^{r-2}$. It is interesting to compare this result with the average number of nodes in the ROMDD's of $r$-valued $n$-variable symmetric functions. It is shown in [3] that this number is $n^{r-1}$, when $n$ is large. That is, the average number of nodes in both cases is polynomial in $n$. However, the average for general symmetric multiple-valued functions grows at a greater rate than the average for planar symmetric multiple-valued functions, suggesting that planarity restricts the number of nodes possible. It follows that the latter require less storage in computer representations.

**5 Concluding Remarks**

Our results can be extended in a number of ways. Restriction 1 has allowed us to make specific statements about the planarity of a class of functions. Allowing other permutations of node assignments and/or terminal node assignments enlarges the class of functions with planar ROMDD's considerably. This class can be enlarged further by allowing unary functions along the edges. That is, two nodes can be combined if their function differs by a mapping among function (output) values. In binary, such mappings are described as complemented edges.

**References**


