Weak Completeness Theorem for Propositional Linear Time Temporal Logic

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Summary. We prove weak (finite set of premises) completeness theorem for extended propositional linear time temporal logic with irreflexive version of until-operator. We base it on the proof of completeness for basic propositional linear time temporal logic given in [20] which roughly follows the idea of the Henkin-Hasenjaeger method for classical logic. We show that a temporal model exists for every formula which negation is not derivable (Satisfiability Theorem). The contrapositive of that theorem leads to derivability of every valid formula. We build a tree of consistent and complete PNPs which is used to construct the model.

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The papers [25], [14], [28], [21], [4], [1], [30], [11], [26], [31], [13], [24], [2], [3], [5], [6], [7], [12], [15], [9], [23], [8], [10], [19], [27], [29], [22], [16], [17], and [18] provide the notation and terminology for this paper.

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1. Preliminaries

For simplicity, we use the following convention: $A, B, p, q$ denote elements of the LTLB-WFF, $M$ denotes a LTL Model, $j, k, n$ denote elements of $\mathbb{N}$, $i$ denotes a natural number, $X$ denotes a subset of the LTLB-WFF, $F$ denotes a finite subset of the LTLB-WFF, $f$ denotes a finite sequence of elements of the LTLB-WFF, and $P, Q, R$ denote positive-negative pairs.

Let $X$ be a finite set. We see that the enumeration of $X$ is a one-to-one finite sequence of elements of $X$.

Let $E$ be a set and let $F$ be a finite subset of $E$. We see that the enumeration of $F$ is a one-to-one finite sequence of elements of $E$.

Let $D$ be a set. One can verify that there exists a set of finite sequences of $D$ which is non empty and finite.

We now state the proposition

(1) Let $X$ be a set and $G$ be a non empty finite set of finite sequences of $X$.
Then there exists a finite sequence $A$ such that $A \in G$ and for every finite sequence $B$ such that $B \in G$ holds $\text{len } B \leq \text{len } A$.

Let $T$ be a decorated tree, let us consider $n$, and let $t$ be a node of $T$. Then $t|n$ is a node of $T$.

We now state the proposition

(2) $p$ is a finite sequence of elements of $\mathbb{N}$.
Let us consider $A$. We introduce $A$ is s-until as a synonym of $A$ is conjunctive.
Let us consider $A$. Let us assume that $A$ is s-until. The right argument of $A$ yields an element of the LTLB-WFF and is defined by:

(Def. 1) There exists $p$ such that $pU$ the right argument of $A = A$.

Let us consider $A$. We say that $A$ is satisfiable if and only if:

(Def. 2) There exist $M, n$ such that $\text{SAT}_M(\langle n, A \rangle) = 1$.

We now state four propositions:

(3) $\emptyset_{\text{LTLB-WFF}} \models A$ iff $\neg A$ is not satisfiable.
(4) If $\top \land \land A$ is satisfiable, then $A$ is satisfiable.
(5) Let $i$ be an element of $\mathbb{N}$. Then $\text{SAT}_M(\langle i, pUq \rangle) = 1$ if and only if there exists $j$ such that $j > i$ and $\text{SAT}_M(\langle j, q \rangle) = 1$ and for every $k$ such that $i < k < j$ holds $\text{SAT}_M(\langle k, p \rangle) = 1$.
(6) $\text{SAT}_M(\langle n, \text{conjunction } f |_{\text{len } \text{conjunction } f} \rangle) = 1$ iff for every $i$ such that $i \in \text{dom } f$ holds $\text{SAT}_M(\langle n, f_i \rangle) = 1$.

One can prove the following three propositions:

(7) $\hat{W} \models \top \land \land \neg A$, where $W = \langle \epsilon (\text{the LTLB-WFF}), \langle A \rangle \rangle$.
(8) For every complete positive-negative pair $P$ such that $\text{UN}(A, B) \in \text{rng } P$ holds $A, B, A \cup B \in \text{rng } P$.
(9) $\text{rng } P \subseteq \cup \sigma(\text{rng } P)$. 
2. Set of PNP-formulas. Completions of Formulas and PNPs

In the sequel $P$ is an element of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$.

Let $F$ be a subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$. The functor $\hat{F}$ yields a subset of the LTLB-WFF and is defined by:

\begin{equation}
\hat{F} = \{ \hat{P} : P \in F \}.
\end{equation}

Let $F$ be a non empty subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$. Note that $\hat{F}$ is non empty.

Let $F$ be a finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$. Observe that $\hat{F}$ is finite.

We now state the proposition

(10) For all subsets $F, G$ of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ holds $\hat{F} \cup \hat{G} = \hat{F \cup G}$.

One can prove the following proposition

(11) $\hat{W} = \{ \top_t \land \top_t \}$, where $W = \{ \langle \varepsilon_{\text{the LTLB-WFF}}, \varepsilon_{\text{the LTLB-WFF}} \rangle \}$.

In the sequel $Q$ denotes a positive-negative pair.

Let $F$ be a finite subset of the LTLB-WFF. The functor comp $F$ yielding a non empty finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ is defined as follows:

\begin{equation}
\text{comp}(F) = \{ Q : \text{rng} Q = \tau(F) \land \text{rng}(Q_1) \text{ misses } \text{rng}(Q_2) \}.
\end{equation}

Let $F$ be a finite subset of the LTLB-WFF. Note that every element of comp $F$ is complete.

One can prove the following proposition

(12) $\text{comp}(\emptyset_{\text{the LTLB-WFF}}) = \{ \langle \varepsilon_{\text{the LTLB-WFF}}, \varepsilon_{\text{the LTLB-WFF}} \rangle \}$.

Let us consider $P, Q$. We say that $Q$ is completion of $P$ if and only if:

(Def. 5) $\text{rng}(P_1) \subseteq \text{rng}(Q_1)$ and $\text{rng}(P_2) \subseteq \text{rng}(Q_2)$ and $\tau(\text{rng} P) = \text{rng} Q$.

We now state the proposition

(13) If $Q$ is completion of $P$, then $Q$ is complete.

In the sequel $Q$ is a consistent positive-negative pair.

Let us consider $P$. The functor comp $P$ yields a finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ and is defined by:

\begin{equation}
\text{comp}(P) = \{ Q : Q \text{ is completion of } P \}.
\end{equation}

Let $P$ be a consistent positive-negative pair. One can check that comp $P$ is non empty. Observe that every element of comp $P$ is consistent.

In the sequel $P$ denotes an element of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$.

Let $X$ be a subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$. The functor comp $X$ yields a subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ and is defined by:
(Def. 7) \( \text{comp } X = \bigcup \{ \text{comp } P : P \in X \} \).

Let \( X \) be a finite subset of \((\text{the LTLB-WFF})_{\text{1-1}}^* \times (\text{the LTLB-WFF})_{\text{1-1}}^* \). One can check that \( \text{comp } X \) is finite.

We now state four propositions:

(14) For every non empty subset \( X \) of \((\text{the LTLB-WFF})_{\text{1-1}}^* \times (\text{the LTLB-WFF})_{\text{1-1}}^* \) such that \( Q \in X \) holds \( \text{comp } Q \subseteq \text{comp } X \).

(15) For every non empty finite subset \( F \) of the LTLB-WFF there exists \( p \) such that \( p \in \tau(F) \) and \( \tau(\tau(F) \setminus \{ p \}) = \tau(F) \setminus \{ p \} \).

(16) Let \( F \) be a finite subset of the LTLB-WFF and \( f \) be a finite sequence of elements of the LTLB-WFF. If \( \text{rng } f = \text{comp } F \), then \( \emptyset \text{ the LTLB-WFF } \vdash \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f}) \).

(17) Let \( P \) be a consistent positive-negative pair and \( f \) be a finite sequence of elements of the LTLB-WFF. If \( \text{rng } f = \text{comp } P \), then \( \emptyset \text{ the LTLB-WFF } \vdash \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f}) \).

3. Set of Possible Next-State PNPs

In the sequel \( A, B \) denote elements of the LTLB-WFF.

Let us consider \( X \). The functor \( \text{UN}(X) \) yields a subset of the LTLB-WFF and is defined as follows:

(Def. 8) \( \text{UN}(X) = \{ \text{UN}(A,B) : A \cup B \in X \} \).

Let \( X \) be a finite subset of the LTLB-WFF. One can check that \( \text{UN}(X) \) is finite.

Let us consider \( P \). The functor \( \text{UN}(P) \) yielding a non empty finite subset of \((\text{the LTLB-WFF})_{\text{1-1}}^* \times (\text{the LTLB-WFF})_{\text{1-1}}^* \) is defined by:

(Def. 9) \( \text{UN}(P) = \{ Q ; Q \text{ ranges over positive-negative pairs: } \text{rng}(Q_1) = \text{UN}(\text{rng}(P_1)) \land \text{rng}(Q_2) = \text{UN}(\text{rng}(P_2)) \} \).

One can prove the following proposition

(18) For every element \( Q \) of \( \text{UN}(P) \) holds \( \emptyset \text{ the LTLB-WFF } \vdash \P \Rightarrow X \hat{Q} \).

Let \( P \) be a consistent positive-negative pair. Note that every element of \( \text{UN}(P) \) is consistent. In the sequel \( Q \) denotes an element of \((\text{the LTLB-WFF})_{\text{1-1}}^* \times (\text{the LTLB-WFF})_{\text{1-1}}^* \).

Let us consider \( P \). The next completion of \( P \) yielding a finite subset of \((\text{the LTLB-WFF})_{\text{1-1}}^* \times (\text{the LTLB-WFF})_{\text{1-1}}^* \) is defined by:

(Def. 10) The next completion of \( P = \{ Q : Q \in \text{comp } \text{UN}(P) \} \).

Let \( P \) be a consistent positive-negative pair. One can verify that the next completion of \( P \) is non empty.
Let $P$ be a consistent positive-negative pair. One can check that every element of the next completion of $P$ is consistent.

Next we state two propositions:

(19) If $Q \in$ the next completion of $P$ and $R \in \text{UN}(P)$, then $Q$ is completion of $R$.

(20) If $Q \in$ the next completion of $P$, then $Q$ is complete.

Let $P$ be a consistent positive-negative pair. One can verify that every element of the next completion of $P$ is complete.

Next we state several propositions:

(21) If $A U B \in \text{rng}(P^2)$ and $Q \in$ the next completion of $P$, then $\text{UN}(A,B) \in \text{rng}(Q^2)$.

(22) If $A U B \in \text{rng}(P^1)$ and $Q \in$ the next completion of $P$, then $\text{UN}(A,B) \in \text{rng}(Q^1)$.

(23) If $R \in$ the next completion of $Q$ and $\text{rng} Q \subseteq \bigcup \sigma(\text{rng} P)$, then $\text{rng} R \subseteq \bigcup \sigma(\text{rng} P)$.

(24) Let $P$ be a consistent complete positive-negative pair and $Q$ be an element of the next completion of $P$. If $A U B \in \text{rng}(P^2)$, then $B \in \text{rng}(Q^2)$ but $A \in \text{rng}(Q^2)$ or $A U B \in \text{rng}(Q^2)$.

(25) Let $P$ be a consistent complete positive-negative pair and $Q$ be an element of the next completion of $P$. If $A U B \in \text{rng}(P^1)$, then $B \in \text{rng}(Q^1)$ or $A, A U B \in \text{rng}(Q^1)$.

4. A PNP-Tree and its Properties

Let us consider $P$. A finite-branching tree decorated with elements of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ is said to be a tree of positive-negative pairs of $P$ if it satisfies the conditions (Def. 11).

(Def. 11)(i) $\text{It}(\emptyset) = P$, and

(ii) for every element $t$ of dom $\text{it}$ and for every element $w$ of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ such that $w = \text{it}(t)$ holds

$succ(\text{it}(t)) = \text{the enumeration of the next completion of } w$.

In the sequel $T$ is a tree of positive-negative pairs of $P$ and $t$ is a node of $T$. Let us consider $P, T, t$. Then $T|t$ is a tree of positive-negative pairs of $T(t)$.

Next we state two propositions:

(26) For every natural number $n$ such that $t \cap \langle n \rangle \in \text{dom } T$ holds $T(t \cap \langle n \rangle) \in$ the next completion of $T(t)$.

(27) If $Q \in \text{rng } T$, then $\text{rng } Q \subseteq \bigcup \sigma(\text{rng } P)$.

Let us consider $P, T$. One can check that $\text{rng } T$ is non empty and finite.

Let $P$ be a consistent positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. One can check that every element of $\text{rng } T$ is consistent.
Let $P$ be a consistent complete positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. One can verify that every element of $\text{rng}\, T$ is complete.

Let $P$ be a consistent complete positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, and let $t$ be a node of $T$. Observe that $T(t)$ is consistent and complete as a positive-negative pair.

Let $P$ be a consistent positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, and let $t$ be an element of $\text{dom}\, T$. Observe that $\text{succ}\, t$ is non empty.

Let us consider $P$, $T$. The range of $T$ except the root node yields a finite subset of $(\text{the LTLB-WFF})^{*}_{1-1}$ and is defined as follows:

(Def. 12) The range of $T$ except the root node $= \{T(t); t \text{ ranges over nodes of } T; t \neq \emptyset\}$.

Let $P$ be a consistent positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. One can verify that the range of $T$ except the root node is non empty.

One can prove the following proposition

(28) If $R \in \text{rng}\, T$ and $Q \in \text{UN}(R)$, then $\text{comp}\, Q \subseteq \text{the range of } T$ except the root node.

One can prove the following proposition

(29) Let $P$ be a consistent complete positive-negative pair, $T$ be a tree of positive-negative pairs of $P$, and $f$ be a finite sequence of elements of the LTLB-WFF. If $\text{rng}\, f = \hat{J}$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \neg(\text{conjunction negation } f)_{\text{len conjunction negation } f} \Rightarrow \forall \neg(\text{conjunction negation } f)_{\text{len conjunction negation } f}$, where $J = \text{the range of } T$ except the root node.

5. A Path in PNP-Tree and its Properties. Existence of Temporal Model for a Consistent PNP. Weak Completeness Theorem

Let $P$ be a consistent positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. A sequence of $\text{dom}\, T$ is called a path of $T$ if:

(Def. 13) $\text{It}(0) = \emptyset$ and for every natural number $k$ holds $\text{it}(k+1) \in \text{succ}\, \text{it}(k)$.

Let $P$ be a consistent complete positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, let $t$ be a path of $T$, and let us consider $i$. Then $t(i)$ is a node of $T$.

Next we state three propositions:

(30) Let $P$ be a consistent complete positive-negative pair, $T$ be a tree of positive-negative pairs of $P$, and $t$ be a path of $T$. Suppose $A \cup B \in \text{rng}(T(t(i))_{2})$. Let given $j$. If $j > i$, then $B \in \text{rng}(T(t(j))_{2})$ or there exists $k$ such that $i < k < j$ and $A \in \text{rng}(T(t(k))_{2})$. 
Let $P$ be a consistent complete positive-negative pair and $T$ be a tree of positive-negative pairs of $P$. Suppose $A \cup B \in \text{rng}(P_1)$ and for every element $Q$ of the range of $T$ except the root node holds $B \notin \text{rng}(Q_1)$. Let $Q$ be an element of the range of $T$ except the root node. Then $B \in \text{rng}(Q_2)$ and $A \cup B \in \text{rng}(Q_1)$.

Let $P$ be a consistent complete positive-negative pair and $T$ be a tree of positive-negative pairs of $P$. Suppose $A \cup B \in \text{rng}(P_1)$. Then there exists an element $R$ of the range of $T$ except the root node such that $B \in \text{rng}(R_1)$.

Let $P$ be a consistent positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, and let $t$ be a path of $T$. We say that $t$ is complete if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let given $i$. Suppose $A \cup B \in \text{rng}(T(t(i))_1)$. Then there exists $j$ such that $j > i$ and $B \in \text{rng}(T(t(j))_1)$ and for every $k$ such that $i < k < j$ holds $A \in \text{rng}(T(t(k))_1)$.

Let $P$ be a consistent complete positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. Note that there exists a path of $T$ which is complete.

Let $P$ be a consistent positive-negative pair. Observe that $\tilde{P}$ is satisfiable.

One can prove the following proposition

(33) If $F \models A$, then $F \vdash A$.

References


$^3$Weak completeness theorem of basic propositional linear temporal logic extended with $\mathcal{U}$ operator (LTLB).
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