A Measure of Pareto Superiority?

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Abstract

We first confirm the notions of Pareto optimality, superiority, and inefficiency. Then, we discuss a definition of the measure of Pareto superiority.

keyword Pareto superiority/inferiority, Nash equilibrium, Wardrop equilibrium

1 Introduction

The notions of Pareto optimality, superiority, and inefficiency have already been established. We first confirm the notions and their definitions. Then, we discuss a definition of the measure of Pareto superiority.

2 Pareto Optimality, and Superiority

We consider a system consisting of a number of users. For each state of the system, each user has its own utility. Denote a combination of utilities of all users in a system $S$ by $U(S) = (U_1(S), U_2(S), \ldots, U_n(S))$. We consider only the cases where $U_i(S) > 0$ has a positive real value, for all $i$. Denote by $\mathcal{R}$ the set of positive real numbers. Thus, $U(S) \in \mathcal{R}^n$.

[Achievable set of utilities]: Naturally, the set of achievable $U(S)$ does not cover all the elements of $\mathcal{R}^n$.

[Pareto optimality and efficiency]: There may exist a state of the system where we cannot improve the utility of each user without decreasing the utility of some other user. This is called a Pareto optimum or efficient state. In general, there are infinitely many Pareto optimum states for a system. The set of Pareto optimum points forms the border (Pareto border) separating the set of achievable $U(S)$ from the set of unachievable $U(S)$.

[Pareto superiority and inferiority]: Consider an arbitrary pair of two (achievable) states of the system, $S^a$ and $S^b$: If $U_i(S^a) \leq U_i(S^b)$ for all $i$ and $U_i(S^a) < U_i(S^b)$, then $S^a$ is Pareto inferior to $S^b$ and $S^b$ is Pareto superior to $S^b$. Define $k_i = U_i(S^b)/U_i(S^a)$. Then, $S^b$ is Pareto superior to $S^a$ if and only if $k_i > 1$ for some $i$ and $k_j \geq 1$ for all other $j$. $S^b$ is Pareto inferior to $S^a$ if and only if $k_i < 1$ for some $i$ and $k_j \leq 1$ for all other $j$.

We define strong Pareto superiority and inferiority. That is, $S^b$ is strongly Pareto superior to $S^a$ iff $k_i > 1$ for all $i$. $S^b$ is strongly Pareto inferior to $S^a$ iff $k_i < 1$ for all $i$. A state to which some other state is (strongly) Pareto superior is (strongly) Pareto inefficient.

[A measure of Pareto superiority/inferiority]: As we see in the above the definition of Pareto superiority/inferiority has already given and well accepted. It seems, however, that the measure of the degree of Pareto superiority/inferiority has not been generally accepted. The measure is necessary, e.g., for defining the degree of the inefficiency of Nash equilibria.

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The Pareto superiority depends on the vector \( (k_1, k_2, \ldots, k_n) \). It would, however, be convenient to express the degree of Pareto superiority by a single scalar measure. The primary concern must be the requirement that the value of the measure clearly distinguish cases of Pareto inferiority and, thus, paradoxes, from other cases, simply and almost clearly. Define \( k_{\min} = \min_i k_i \) and \( k_{\max} = \max_i k_i \). If \( k_{\min} > 1 \), the state \( S^a \) is (strongly) Pareto superior to \( S^b \), and if \( k_{\min} < 1 \), the state \( S^a \) is Pareto indifferent or inferior to \( S^b \). If \( k_{\max} < 1 \), the state \( S^a \) is (strongly) Pareto inferior to \( S^b \), and if \( k_{\max} < 1 \), the state \( S^a \) is Pareto indifferent or superior to \( S^b \). Thus, the measures \( k_{\min} \) and \( k_{\max} \) may be used as primary measures of the degree of Pareto superiority and inferiority, respectively. We note that if \( k_{\min} < 1 \) and \( k_{\max} > 1 \), states \( S^a \) and \( S^b \) are mutually Pareto indifferent to each other. On the other hand, for example, a measure \( X \) based on a certain average of all of \( k_i \) should be rejected, since it can hold that \( X > 1 \) even if some \( k_i < 1 \) for some \( i \) but if \( k_j \gg 1 \) for all other \( j \)'s. Such a measure may be used as a secondary measure. (In many practical situations, the variables may have continuous values and truly exact equalities occur rarely. Or, the tie-breaking of the case that \( k_{\min} = 1 \) may depend on such a secondary measure.)

We propose that \( k_{\min} \) and \( k_{\max} > 1 \) are used as primary measures showing the degrees of Pareto superiority and inferiority, respectively. The tie-breaking of the case that \( k_{\min} = 1 \) may depend on some other secondary measure.

**Individual and Class Optima** Some systems have infinitely many infinitesimal users. Examples are road traffic systems and communications networks. In such systems we can think of the scheme wherein each such infinitesimal user has its own decision making for optimizing its utility. This is regarded as a noncooperative game with infinitely many infinitesimal decision makers. The Nash equilibria for such a game is called ‘individual optima,’ ‘Wardrop equilibria,’ or ‘Nash equilibria with non-atomic users.’ We can also consider such situations where infinitely many infinitesimal users are partitioned into a finite number of groups, and where one decision maker corresponds to each such group. The decision maker of the group strives to optimize the total utility of only the group members. The Nash equilibria for such a game is called ‘class optima,’ simply ‘Nash equilibria,’ or ‘Nash equilibria with atomic users.’

Note that Nash equilibria may not be Pareto efficient, however, since the conditions of Pareto efficiency and noncooperative equilibria are not mutually identical (Smale, 1973; Dubey, 1986), except some cases (see, e.g., Cohen (1998), Altman et al. (2002)) whereas all overall optima are Pareto efficient.

### 3 Application of the Measure

In this section, we present some example where the properties of systems are presented in terms of the measures as defined above.

**An Example: Globalizing separated monopolies to a Nash-Cournot oligopoly** The system considered here consists of \( n \) producers and \( n \) corresponding markets (Kameda and Ui, 2002). Producers and markets are numbered \( 1, 2, \ldots, n \). In each market, the commodity of the same and single kind is demanded. There is one producer at each market. Consider the two cases as to the market.

1) Markets are separated, and each producer serves only the demand of the corresponding market.

2) Markets are united into a globalized market. Call it market \( w \).

Let \( q_i \) denote the quantity that producer \( i \) produces. \( q_i \) is the variable determined by producer \( i \). Assume the demand function of each separated market as follows: Denote by \( p_i \) the price of the commodity in the market \( i \). Then, \( p_i = a(1 - q_i/b_i) \). If \( p_w \) is the price in the globalized market, then \( p_w = a[1 - (\sum_i q_i)/(\sum_i b_i)] \). Assume that the cost that producer \( i \) produces the amount \( q_i \) of the commodity is \( c_i q_i + d_i \). \( d_i \) is the fixed cost of producer \( i \).
If the markets are united by bringing the zero-cost connection, such a situation may occur that all the producers suffer the degradation of their profits or producer surpluses. The surplus of a producer is the sum of the profit for the producer and the fixed cost of producer \((d_i)\). \(k^R_i\) denote the ratio of the surplus for producer \(i\) before globalization to that after globalization. \(k_{\text{min}}\) is defined as follows:

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k_{\text{min}} = \min_i \{ k^R_1, k^R_2, \ldots, k^R_n \}.
\]

\(k_{\text{min}}\) is considered the measure of coincident producer surplus degradation due to globalization (that of Pareto inferiority of producer surpluses of the united market over those of the separated markets). It has been shown that the worst-case ratio, \(k_{\text{min}}\), of coincident surplus degradation for all producers due to globalization is reached in complete symmetry, i.e., \(b_i = B, c_i = C\), for all \(i\) (Kameda and Ui, 2002). That is, the measure of such surplus degradation for all producers due to globalization is worst when the system is in complete symmetry.

### 4 Concluding Remarks

We have first confirmed the notions of Pareto optimality, superiority, and inefficiency. Then, we have proposed a definition of the measure of Pareto superiority. We have also mentioned some example.

### References


