On the spectrum of $r$-self-orthogonal Latin squares

Yunqing Xu$^{a,b}$, Yanxun Chang$^a$

$^a$Department of Mathematics, Northern Jiaotong University, Beijing 100044, China
$^b$Department of Mathematics, Xinyang Teachers College, Xinyang, Henan 464000, China

Received 7 October 2002; received in revised form 12 March 2003; accepted 9 June 2003

Abstract

Two Latin squares of order $n$ are $r$-orthogonal if their superposition produces exactly $r$ distinct ordered pairs. If the second square is the transpose of the first one, we say that the first square is $r$-self-orthogonal, denoted by $r$-SOLS($n$). It has been proved that the necessary condition for the existence of an $r$-SOLS($n$) is $n \leq r \leq n^2$ and $r \not\in \{n + 1, n^2 - 1\}$. Zhu and Zhang conjectured that there is an integer $n_0$ such that for any $n \geq n_0$, there exists an $r$-SOLS($n$) for any $r \in [n, n^2] - \{n + 1, n^2 - 1\}$. In this paper, we show that $n_0 \leq 28$.© 2003 Elsevier B.V. All rights reserved.

Keywords: Latin square; $r$-Orthogonal; $r$-Self-orthogonal

1. Introduction

Two Latin squares of order $n$, $L = (l_{ij})$ and $M = (m_{ij})$, are said to be $r$-orthogonal if their superposition produces exactly $r$ distinct pairs, that is

$$| \{(l_{ij}, m_{ij}): 0 \leq i, j \leq n - 1\}| = r.$$  

Belyavskaya (see [2–4]) first systematically treated the following question: For which integers $n$ and $r$ does a pair of $r$-orthogonal Latin squares of order $n$ exist? Evidently, $n \leq r \leq n^2$, and an easy argument establishes that $r \not\in \{n + 1, n^2 - 1\}$. In papers by Colbourn and Zhu [5] and Zhu and Zhang [7,8], this question has been completely answered, and the final result is in the following theorem.

$^{\star}$ Research supported by NSFC 10071002.

E-mail addresses: yqxu@mail.edu.cn (Y. Xu), yxchang@center.njtu.edu.cn (Y. Chang).

0012-365X/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0012-365X(03)00288-7
Table 1

<table>
<thead>
<tr>
<th>Order $n$</th>
<th>Genuine exceptions of $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>5, 6, 7</td>
</tr>
<tr>
<td>4</td>
<td>7, 10, 11, 13, 14</td>
</tr>
<tr>
<td>5</td>
<td>8, 9, 20, 22, 23</td>
</tr>
<tr>
<td>6</td>
<td>36, 33</td>
</tr>
</tbody>
</table>

Theorem 1.1 (Zhu and Zhang [8, Theorem 2.1]). For any integer $n \geq 2$, there exists a pair of $r$-orthogonal Latin squares of order $n$ if and only if $n \leq r \leq n^2$ and $r \not\in \{n + 1, n^2 - 1\}$ with the exceptions of $n$ and $r$ shown in Table 1.

In a pair of $r$-orthogonal Latin squares of order $n$, if the second square is the transpose of the first one, we say that the first square is $r$-self-orthogonal, denoted by $r$-SOLS($n$). When an $r$-SOLS($n$) exists, we can simply list only one square for a pair of $r$-orthogonal Latin squares of order $n$.

For the existence of an $r$-SOLS($n$), we have the necessary condition in Theorem 1.1, i.e., $n \leq r \leq n^2$ and $r \not\in \{n + 1, n^2 - 1\}$. It is well-known that an SOLS($n$) exists if and only if $n \not\equiv 2 \pmod{3}$ (see, for example, [6]). This solves the case of $r = n^2$. For the case of $r = n$, we take the square

$L = (a_{ij})$, $a_{ij} = j - i$, $i, j \in \mathbb{Z}_n$.

It is easily seen that $L$ is an $n$-SOLS($n$). So, we can focus on the cases $n + 1 < r < n^2 - 1$ for the existence of an $r$-SOLS($n$).

For small orders, we have the following results.

Theorem 1.2 (Zhu and Zhang [8]). For order $n = 4$, there is only one $r \in [n + 1, n^2 - 1]$, namely $r = 9$, such that an $r$-SOLS($n$) exists.

For $n = 5$ and $n + 1 < r < n^2 - 1$, there is an $r$-SOLS($5$) for $r \in \{7, 10, 11, 13, 14, 15, 17, 19, 21\}$ only.

For $n = 6$ and $n + 1 < r < n^2 - 1$, there is an $r$-SOLS($6$) for $r \in [8, 31]$ only.

For $n = 7$ and $n + 1 < r < n^2 - 1$, there is an $r$-SOLS($7$) for all $r \in [9, 47] - \{46\}$ only.

For $n = 8$, there exists a 61-SOLS($8$).

Zhu and Zhang [8, Conjecture 3.1] conjectured that there is an integer $n_0$ such that for any $n \geq n_0$, there exists an $r$-SOLS($n$) for any $r \in [n, n^2] - \{n + 1, n^2 - 1\}$. In this paper, we show that $n_0 \leq 28$.

Some constructions in this paper rely on information regarding the location of transversals in certain Latin squares. Suppose $L$ is an Latin squares on symbol set $S$. A transversal is a set $T$ of $|S|$ cells in $L$ such that every symbol of $S$ occurs in exactly one cell of $T$ and the $|S|$ cells in $T$ intersect each row and each column in exactly once. A transversal $T$ is symmetric if $(i, j) \in T$ implies $(j, i) \in T$. Two transversals $T_1$ and $T_2$ are called a symmetric pair of transversals if $(i, j) \in T_1$ if and
only if \((j, i) \in T_2\). A set of transversals are said to be *disjoint* if they have no cell in common.

2. Existence of \(r\)-SOLS\((n)\) for \(r \in \{n^2 - 3, n^2 - 5\}\)

A *self-orthogonal* Latin square of order \(n\), or SOLS\((n)\), is a Latin square of order \(n\) which is orthogonal to its transpose.

Let \(S\) be a set and \(\mathcal{H} = \{S_1, S_2, \ldots, S_k\}\) be a set of nonempty sub-sets of \(S\). An *incomplete Latin square* having *hole set* \(\mathcal{H}\) is a \(|S| \times |S|\) array, \(L\), indexed by \(S\), which satisfies the following properties:

1. every cell of \(L\) is either empty or contains a symbol of \(S\),
2. every symbol of \(S\) occurs at most once in any row or column of \(L\),
3. the sub-arrays \(S_i \times S_i\) are empty for \(1 \leq i \leq k\) (these sub-arrays are referred to as *holes*),
4. symbol \(x \in S\) occurs in row or column \(y\) if and only if \((x, y) \in (S \times S) \setminus \bigcup_{i=1}^{k} (S_i \times S_i)\).

The *order* of \(L\) is \(|S|\). \(L\) is said to be *self-orthogonal* if the superposition with its transpose yields every ordered pair in \((S \times S) \setminus \bigcup_{i=1}^{k} (S_i \times S_i)\).

Theorem 2.1 (Abel et al. [1, Theorem 2.10]). There exists an ISOLS\((n; u)\) for all values of \(n\) and \(u\) satisfying \(n \geq 3u + 1\), except for \((n, u) = (6, 1), (8, 2)\) and possibly for \(n = 3u + 2, \ u \in \{6, 8, 10\}\).

Lemma 2.2. If there exists an ISOLS\((n; u)\) and an s-SOLS\((u)\), then there exists an \(r\)-SOLS\((n)\) for \(r = n^2 - u^2 + s\).

Proof. Fill in the empty sub-array of the ISOLS\((n; u)\) with the s-SOLS\((u)\). \(\Box\)

Lemma 2.3. (1) There exists an \((n^2 - 3)\)-SOLS\((n)\) for \(n \geq 25\) and \(\neq 26\); (2) there exists an \((n^2 - 5)\)-SOLS\((n)\) for \(n \geq 19\) and \(\neq 20\).
Proof. (1) From Theorems 2.1 and 1.2, we know that there exist an ISOLS($n, 8$) for $n \geq 25$ and $n \neq 26$ and a 61-SOLS(8). Applying Lemma 2.2 with $u = 8$ and $s = 61$ we then obtain a $(n^2 - 3)$-SOLS($n$).

(2) Applying Theorem 2.1 and Lemma 2.2 with $u = 6$ and $s = 31$ we get a $(n^2 - 5)$-SOLS($n$) for $n \geq 19$ and $n \neq 20$. Where the input design 31-SOLS(6) is from Theorem 1.2.

3. Existence of $r$-SOLS($n$) for $r \in \{n^2 - 2, n^2 - 7\}$

In this section and the next two sections, we give input designs used in the main constructions in Section 6.

Lemma 3.1. Suppose that $n$ is an integer and $n \geq 7$. There exists an $(n^2 - 2)$-SOLS($n$).

Proof. From Theorem 2.1, we know that there is an ISOLS($n, 2$), $L = (a_{ij})$, for $n \geq 7$ and $n \neq 8$. The Latin square $L$ is based on the set $Z_n$ with hole set $\{0, 1\}$. Then the hole is in the upper left corner of $L$. By filling the hole with a sub-array

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

we get a $(n^2 - 2)$-SOLS($n$), where the set of distinct ordered pairs is $P = (Z_n \times Z_n) \setminus \{(0, 1), (1, 0)\}$.

A 62-SOLS(8) with 7 on the main diagonal can be obtained from Zhu and Zhang’s web site (http://www.cs.uiowa.edu/~hzhang/sr/).

Lemma 3.2. Suppose that $m$ is an integer and $m \geq 6$. There exists an $((m+1)^2 - 3)$-ISOLS($m+1, 1$), $L$, on symbol set $Z_m \cup \{x\}$, in which the pair $(x, x)$ does not appear in the superposition of $L$ with its transpose.

Proof. For $m \geq 6$ and $m \neq 7$, from the proof of Lemma 3.1 we know that there exists an $((m+1)^2 - 2)$-SOLS($m+1$), $L$, with element $m$ on the main diagonal and the set of distinct ordered pairs is $(Z_{m+1} \times Z_{m+1}) \setminus \{(0, 1), (1, 0)\}$, where $(0, 0)$ and $(1, 1)$ are the repeated pairs in the superposition of $L$ with its transpose.

For $m = 7$, we have a 62-SOLS(8), $L$, from http://www.cs.uiowa.edu/~hzhang/sr/. $L$ has the properties that 7 is on the main diagonal and the pair $(7, 7)$ appears only once in the superposition of $L$ with its transpose.

Replace all elements $m$ in $L$ with $x$, and then delete $x$ from the main diagonal we get the desired incomplete Latin square.

Lemma 3.3. Suppose that $n$ is an integer, $n \geq 4$ and $n \neq 5$. Then there exists an $(n^2 - 7)$-SOLS($n$).

Proof. From Theorem 1.2, we know there exists a 9-SOLS(4) on $Z_4$. From Theorem 2.1, we know that there exists an ISOLS($n, 4$) for $n \geq 13$. We assume that the hole set of $L$ is $\{0, 1, 2, 3\}$, and then the hole is in the upper left corner of $L$. By filling the
hole with the 9-SOLS(4), we get an \((n^2 - 7)\)-SOLS\((n)\) for \(n \geq 13\). For \(6 \leq n \leq 12\), \((n^2 - 7)\)-SOLS\((n)\) are shown in Figs. 1 and 2, where the first two are from http://www.cs.uiowa.edu/~hzhang/sr/, \(a, b\) denote 10, 11, respectively. □

**Lemma 3.4.** Suppose that \(m\) is an integer and \(m \geq 5\). There exists an \(((m + 1)^2 - 8)\)-ISOLS\((m + 1, 1)\), \(L\), on symbol set \(Z_m \cup \{x\}\), in which the pair \((x, x)\) does not appear in the superposition of \(L\) with its transpose.

**Proof.** If \(m \geq 12\), from the proof of Lemma 3.3 we know that there exists an \(((m + 1)^2 - 7)\)-SOLS\((m + 1)\), \(L\), with \(m\) on the main diagonal and \((m, m)\) appears only once in the superposition of \(L\) with its transpose. We replace the all elements \(m\) in \(L\) with \(x\), and then delete \(x\) from the main diagonal to get the desired Latin square.

For \(m \in [5, 11]\), from Figs. 1 and 2 we know that there is an \(((m+1)^2-7)\)-SOLS \((m+1),L\), with \(m\) in the bottom right corner of \(L\) and \((m, m)\) appears only once in the superposition of \(L\) with its transpose. Replace all elements \(m\) in \(L\) with \(x\), and then delete \(x\) from the bottom right corner where we get the desired Latin square. □
Lemma 3.5. Suppose that \( m \) is a positive integer and \( m \neq 1, 2, 5 \). There exists a \( ((m + 1)^2 - 1)\)-ISOLS\((m + 1, 1)\), \( L \), on symbol set \( \mathbb{Z}_m \cup \{x\} \), in which the pair \((x, x)\) does not appear in the superposition of \( L \) with its transpose.

Proof. Replace all the elements \( m \) in an SOLS\((m + 1)\) on \( \mathbb{Z}_m \) with \( x \), and then delete \( x \) from the main diagonal we get the desired incomplete Latin square.

4. Existence of \( r \)-SOLS\((n)\) for \( r \in \{n + 2, n + 3\} \)

Lemma 4.1. Suppose that \( n \geq 5 \) is an integer. There exists an \( (n+2)\)-SOLS\((n)\) on \( \mathbb{Z}_n \), where the set of \( n+2 \) distinct ordered pairs is \( \{(i, i) : 0 \leq i \leq (n-1)\} \cup \{(0,1),(1,0)\} \), and element \( n - 1 \) in the bottom right corner.

Proof. First, we suppose that \( n \geq 5 \) is odd. We construct a symmetric Latin square \( L(n-2) = (l_{ij}) \) of order \( n-2 \) with symbol set \( \mathbb{Z}_n \setminus \{0,1\} \) as follows: \( l_{ij} = [i + j + 1 \mod (n-2)] + 2 \), where \( 0 \leq i, j \leq n-3 \). It’s easy to see that \( S = \{(i, i+1) : 0 \leq i \leq n-4\} \cup \{(n-3,0)\} \) is a transversal. So is \( T = \{(i+1, i) : 0 \leq i \leq n-5\} \cup \{(0, n-3)\} \).

Replace the elements of \( L(n-2) \) in cells in \( S \) with \( 0 \); replace the elements in cells in \( T \) with \( 1 \). Then we get an array denoted by \( A(n-2) \).

Let \( M(2) = (m_{ij}) \) be a \( 2 \times (n-2) \) array with symbol set \( \mathbb{Z}_n \setminus \{0,1\} \), row set \( Z_2 \) and column set \( Z_{n-2} \), where \( m_{ij} = [2j - 2i + 2 \mod (n-2)] + 2 \).

Let
\[
N(2) = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]
and \( M(2)^T \) denote the transpose of \( M(2) \). We construct an array \( L \) as follows:
\[
L = \begin{pmatrix}
N(2) & M(2) \\
M(2)^T & A(n-2)
\end{pmatrix}.
\]

Fig. 3 is an example of \( L \) for \( n = 5 \).

It is obvious that \( L \) is an \( (n+2)\)-SOLS\((n)\) on \( \mathbb{Z}_n \), where the set of \( n+2 \) distinct ordered pairs is \( \{(i, i) : 0 \leq i \leq (n-1)\} \cup \{(0,1),(1,0)\} \), and element \( n - 1 \) in the bottom right corner.

Second, we suppose that \( n \geq 6 \) is even. We construct an idempotent, symmetric Latin square \( L(n-3) = (l_{ij}) \) of order \( n-3 \) with symbol set \( \mathbb{Z}_{n-3} \) as follows: \( l_{ij} = i + j \mod (n-3)(0 \leq i, j \leq n-4) \). It is easy to see that \( S = \{(i, i+1) : 0 \leq i \leq n-5\} \cup \{(n-4, 0)\} \) is a transversal. So is \( T = \{(i+1, i) : 0 \leq i \leq n-5\} \cup \{(0, n-4)\} \).
Replace the elements of \( L(n-3) \) on the main diagonal with \( n-1 \); replace the elements in cells in \( S \) with \( n-2 \); replace the elements in cells in \( T \) with \( n-3 \). Then we get an array denoted by \( A(n-3) \).

Let \( M(3) = (m_{ij}) \) be a \( 3 \times (n-3) \) array with symbol set \( Z_{n-3} \), row set \( Z_3 \) and column set \( Z_{n-3} \), where

\[
m_{ij} = \begin{cases} 
  i + 2j \mod (n-3) & \text{when } i \in \{0, 1\} \\
  i + 2j - 3 \mod (n-3) & \text{when } i = 2 
\end{cases} \quad (0 \leq j \leq n-4).
\]

Let

\[
N(3) = \begin{pmatrix} 
  n-3 & n-1 & n-2 \\
  n-1 & n-2 & n-3 \\
  n-2 & n-3 & n-1 
\end{pmatrix}
\]

and \( M(3)^T \) denote the transpose of \( M(3) \). We construct an array \( L \) as follows:

\[
L = \begin{pmatrix} 
  A(n-3) & M(3)^T \\
  M(3) & N(3) 
\end{pmatrix}.
\]

Fig. 4 is an example of \( L \) for \( n = 8 \).

It is obvious that \( L \) is an \( (n+2) \)-SOLS\((n)\) on \( Z_n \), where the set of \( n+2 \) distinct ordered pairs is \( \{(i,i) : 0 \leq i \leq (n-1)\} \cup \{(n-3,n-2),(n-2,n-3)\} \), and element \( n-1 \) in the bottom right corner. Give a permutation \((0,n-3)(1,n-2)\) to the elements of \( L \) we then obtain the desired square. \( \square \)

**Lemma 4.2.** If \( n \geq 6 \) is an integer, then there exists an \( (n+2+1) \)-ISOLS\((n+1,1)\) on symbol set \( Z_n \cup \{x\} \), hole set \( \{x\} \), and with \( n+3 \) distinct ordered pairs form the set \( \{(i,i) : 0 \leq i \leq n-1\} \cup \{(0,1),(1,0),(x,x)\} \).

**Proof.** From Lemma 4.1 we know that there exists an \( (n+3) \)-SOLS\((n+1)\), \( L \), on \( Z_{n+1} \), where the set of \( n+3 \) distinct ordered pairs is \( \{(i,i) : 0 \leq i \leq n\} \cup \{(0,1),(1,0)\}, \).
and element \( n \) in the bottom right corner. We replace all the elements \( n \) in \( L \) by \( x \) and then delete \( x \) from the bottom right corner to get the desired square. \( \square \)

Let \( A(2t) \) be a \( 2t \times 2t \) empty array with row and column sets based on \( Z_{2t} \). Its cells form a set \( S = Z_{2t} \times Z_{2t} \). We divide \( S \) into \( 2t \) sub-sets \( T_0, T_1, \ldots, T_{2t-1} \), where \( T_j = \{(i, i + j)|0 \leq i \leq 2t - 1\} \) \((0 \leq j \leq 2t - 1)\), the column indices of the cells are calculated modulo \( 2t \). If we treat every \( T_j \) \((0 \leq j \leq 2t - 1)\) as a transversal, then \( T_0 \) and \( T_i \) are symmetric transversals, \( T_j \) and \( T_{2t-j} \) \((1 \leq j \leq t - 1)\) form a symmetric pair of transversals. If we fill each cell of \( A(2t) \) with an element, then we get an array of side \( 2t \).

Let \( B(2t) = (b_{ij}) \) and \( C(2t) = (c_{ij}) \) be two Latin squares of order \( 2t \) with row and column sets based on \( Z_{2t} \) and symbol sets based on \{\( y_i \)|\( 0 \leq i \leq 2t - 1\}\}, and \( b_{ij} = y_{i+j}, c_{ij} = y_{i+j} \) \((0 \leq i, j \leq 2t - 1)\), where the indices of \( y_i \) are calculated modulo \( 2t \). It is obvious that \( B(2t) \) and \( C(2t) \) are symmetric Latin squares and they form a pair of \( 2t \)-orthogonal Latin squares of order \( 2t \) where their superposition yields distinct ordered pairs in \{\((y_i, y_{i+j}),(y_{i+j}, y_i)\): \(0 \leq i \leq t - 1\)\}.

Deleting the first column in \( B(2t) \), we get an array denoted by \( B_1(2t) \); we delete the first and the second columns in \( B(2t) \) to get an array denoted by \( B_2(2t) \); we delete the first column, second column, and the column indexed by \( t \) in \( B(2t) \) to get an array denoted by \( B_3(2t) \).

Deleting the first row in \( C(2t) \), we get an array denoted by \( C_1(2t) \); we delete the first and the second rows in \( C(2t) \) to get an array denoted by \( C_2(2t) \); we delete the first row, second row, and the row indexed by \( t \) in \( C(2t) \) to get an array denoted by \( C_3(2t) \).

The above arrays, \( A(2t) \), \( B(2t) \), \( C(2t) \), \( B_i(2t) \) and \( C_i(2t) \) \((1 \leq i \leq 3)\) will be used in the proofs of Lemmas 4.3 and 4.4, and \( t \) will be \( k \), or \( k + 1 \), or \( k + 2 \) there.

**Lemma 4.3.** (1) If \( n \geq 7 \) is odd, then there exists an \((n + 3)\)-SOLS\((n)\) on \( Z_n \), where the set of \( n + 3 \) distinct ordered pairs is \{\((0, 0),(1, 1),(0, 2),(2, 0)\)\} \(\cup\) \{(\(2i - 1\), \(2i\)): \(1 \leq i \leq (n - 1)/2\)\}.

(2) If \( n \geq 8 \) is even, then there exists an \((n + 3)\)-SOLS\((n)\) on \( Z_n \), where the set of \( n + 3 \) distinct ordered pairs is \{\((0, 0),(1, 1),(0, 2),(2, 0)\)\} \(\cup\) \{(\(2i - 1\), \(2i\)): \(1 \leq i \leq (n - 2)/2\)\} \(\cup\) \{(\(n - 1\), \(n - 1\))\} and element \( n - 1 \) in the bottom right corner.

**Proof.** (1) Suppose that \( n \) is odd. For \( n \in \{7, 9, 11, 13\}\), the \((n + 3)\)-SOLS\((n)\) with symbol set \( Z_n \), where the set of \( n + 3 \) distinct ordered pairs is \{\((0, 0),(1, 1),(0, 2),(2, 0)\)\} \(\cup\) \{(\(2i - 1\), \(2i\)), \((2i, 2i - 1)\): \(1 \leq i \leq (n - 1)/2\}\}, are shown in Fig. 5, where \( a, b, c \) denote \(10, 11, 12\), respectively.

For \( n \geq 15 \), we construct \((n + 3)\)-SOLS\((n)\) recursively.

Suppose that \( L_{2k-1} \) is an \((m + 3)\)-SOLS\((m)\), where the set of \( m + 3 \) distinct ordered pairs is \{\((0, 0),(1, 1),(0, 2),(2, 0)\)\} \(\cup\) \{(\(2i - 1\), \(2i\)), \((2i, 2i - 1)\): \(1 \leq i \leq (m - 1)/2\}\} for \( m = 2k - 1 \).

If \( n = 4k - 1 \) \((k \geq 4)\), we take an empty array \( A(2k) \), as mentioned above in the statement of this lemma, the set of all its cells is partitioned into \( 2k \) sub-sets \( T_j \)
Fig. 5. \((n + 3)\)-SOLS\((n)\) for \(n \in \{7, 9, 11, 13\} \).

\[
\begin{array}{cccccccc}
0 & 1 & 3 & 5 & y_0 & 6 & 4 & 2 \\
2 & 0 & 1 & 3 & 5 & y_1 & 6 & 4 \\
4 & 2 & 0 & 1 & 3 & 5 & y_2 & 6 \\
6 & 4 & 2 & 0 & 1 & 3 & 5 & y_3 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
y_4 & 6 & 4 & 2 & 0 & 1 & 3 & 5 \\
y_5 & 6 & 4 & 2 & 0 & 1 & 3 & 5 \\
y_6 & 6 & 4 & 2 & 0 & 1 & 3 & 5 \\
y_7 & 6 & 4 & 2 & 0 & 1 & 3 & 5 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\
y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\
y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\
y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\
\end{array}
\]

Fig. 6. \((4k + 2)\)-SOLS\((4k - 1)\) for \(k = 4\).

\((0 \leq j \leq 2k - 1)\). We fill the cells of \(A(2k)\) as follows, where the row and column indices of \(A(2k)\) are calculated modulo \(2k\):

(a) fill all the cells in \(T_0\) with 0;
(b) fill all the cells in \(T_j\) with \(2j - 1\) and all the cells in \(T_{2k-j}\) with \(2j\) for \(1 \leq j \leq k - 1\);
(c) fill all the cells in \(T_k\) with the first column of \(B(2k)\).

We construct an array \(L_{4k-1}\) of order \(n = 4k - 1\) as follows:

\[
L_{4k-1} = \begin{pmatrix}
A(2k) & B_1(2k) \\
C_1(2k) & L_{2k-1}
\end{pmatrix}.
\]

Fig. 6 is an example of \(L_{4k-1}\) for \(k = 4\).
We now prove (2) of the lemma. Suppose that

We construct an array \( L_{4k+1} \) of order \( n = 4k + 1 \) as follows:

\[
L_{4k+1} = \begin{pmatrix}
    A(2k+2) & B_3(2k+2) \\
    C_3(2k+2) & L_{2k-1}
\end{pmatrix}
\]

Replace the elements \( y_i \) with \( 2i + (2k - 1) \) and \( y_{i+k} \) with \( 2i + 2k \) for \( 0 \leq i \leq k - 1 \) in \( L_{4k+1} \). We then get an \((n+3)\)-SOLS(n) with symbol set \( Z_n \), where the set of \( n + 3 \) distinct ordered pairs is \( \{(0,0),(1,1),(0,2),(2,0)\} \cup \{(2i-1,2i),(2i,2i-1): 1 \leq i \leq (n-1)/2\} \) for \( n = 4k - 1 \).

If \( n = 4k + 1 (k \geq 4) \), we fill the cells of \( A(2k+2) \) mentioned above in the statement of this lemma as follows, where the row and column indices of \( A(2k+2) \), the indices of \( y_i \) are calculated modulo \( 2k + 2 \):

(a) fill all the cells in \( T_0 \) with 0;
(b) fill all the cells in \( T_j \) with \( 2j - 1 \) and all the cells in \( T_{2k-2-j} \) with \( 2j \) for \( 1 \leq j \leq k - 1 \);
(c) fill all the cells in \( T_k \) with the first column of \( B(2k+2) \);
(d) fill all the cells in \( T_{k+2} \) with the second column of \( B(2k+2) \);
(e) fill all the cells in \( T_{k+1} \) with the column indexed by \( k + 1 \) of \( B(2k + 2) \).

For \( n \equiv 16 \), we construct \((n+3)\)-SOLS(n) recursively.

Suppose that \( L_{2k} \) is an \((m+3)\)-SOLS(m) with \( m+3 \) distinct ordered pairs form the set \( \{(0,0),(1,1),(0,2),(2,0)\} \cup \{(2i-1,2i),(2i,2i-1): 1 \leq i \leq (m-2)/2\} \cup \{(m-1,m-1)\} \) and element \( m - 1 \) in the bottom right corner for \( m = 2k \).
Fig. 8. \((n + 3)\)-SOLS\((n)\) for \(n \in \{12, 14\}\).

If \(n = 4k\) \((k \geq 4)\), we construct an array \(L_{4k}\) of order \(n\) as follows.

\[
L_{4k} = \begin{pmatrix}
L_{2k} & B(2k) \\
C(2k) & L_{2k}
\end{pmatrix}.
\]

Replace the elements \(2k - 1\) with \(4k - 1\), \(y_i\) with \(2i + (2k - 1)\) and \(y_{i+k}\) with \(2i + 2k\) for \(0 \leq i \leq k - 1\) in \(L_{4k+1}\). We then get an \((n + 3)\)-SOLS\((n)\) with symbol set \(Z_n\) and \(n + 3\) distinct ordered pairs form the set \(\{(0, 0), (1, 1), (0, 2), (2, 0)\} \cup \{(2i - 1, 2i), (2i, 2i - 1): 1 \leq i \leq (n - 2)/2\} \cup \{(n - 1, n - 1)\}\) and element \(n - 1\) in the bottom right corner for \(n = 4k\).

If \(n = 4k + 2\) \((k \geq 4)\), we fill the cells of \(A(2k + 2)\) as follows, where the row and column indices of \(A(2k + 2)\), the indices of \(y_i\) are calculated modulo \(2k + 2\):

(a) fill all the cells in \(T_0\) with \(2k - 1\) and all the cells in \(T_{k+1}\) with \(0\);
(b) fill all the cells in \(T_j\) with \(2j - 1\) and all the cells in \(T_{2k+2-j}\) with \(2j\) for \(1 \leq j \leq k - 1\);
(c) fill all the cells in \(T_k\) with the first column of \(B(2k + 2)\);
(d) fill all the cells in \(T_{k+2}\) with the second column of \(B(2k + 2)\).

We construct a square \(L_{4k+2}\) of order \(n = 4k + 2\) as follows:

\[
L_{4k+2} = \begin{pmatrix}
A(2k + 2) & B_2(2k + 2) \\
C_2(2k + 2) & L_{2k}
\end{pmatrix}.
\]

Replace the elements \(2k - 1\) with \(4k + 1\), \(y_i\) with \(2i + (2k - 1)\) and \(y_{i+k+1}\) with \(2i + 2k\) for \(0 \leq i \leq k\) in \(L_{4k+2}\). We then get an \((n + 3)\)-SOLS\((n)\) with symbol set \(Z_n\) and \(n + 3\) distinct ordered pairs form the set \(\{(0, 0), (1, 1), (0, 2), (2, 0)\} \cup \{(2i - 1, 2i), (2i, 2i - 1): 1 \leq i \leq (n - 2)/2\} \cup \{(n - 1, n - 1)\}\) and element \(n - 1\) in the bottom right corner for \(n = 4k + 2\).

This completes the proof. □
Lemma 4.4. (1) If \( n \geq 7 \) is odd, then there exists an \((n + 3 + 1)\)-ISOLS\((n + 1,1)\) on \(Z_n \cup \{x\}\) with \(n + 4\) distinct ordered pairs form the set \(\{(0,0),(1,1),(0,2),(2,0)\}\) \(\cup\) \(\{(2i - 1,2i),(2i,2i - 1): 1 \leq i \leq (n - 1)/2\}\) \(\cup\) \(\{(x,x)\}\).

(2) If \( n \geq 6 \) is even, then there exists an \((n + 3 + 1)\)-ISOLS\((n + 1,1)\) on \(Z_n \cup \{x\}\) with \(n + 4\) distinct ordered pairs form the set \(\{(0,0),(1,1),(0,2),(2,0)\}\) \(\cup\) \(\{(2i - 1,2i),(2i,2i - 1): 1 \leq i \leq (n - 2)/2\}\) \(\cup\) \(\{(n - 1,n - 1),(x,x)\}\).

Proof. (1) If \( n \geq 7 \) is odd, from Lemma 4.3(2) we know that there exists an \((n + 4)\)-ISOLS\((n + 1,1)\), \(L\), on symbol set \(Z_{n+1}\), where the set of \(n + 4\) distinct ordered pairs is \(\{(0,0),(1,1),(0,2),(2,0)\}\) \(\cup\) \(\{(2i - 1,2i),(2i,2i - 1): 1 \leq i \leq (n - 1)/2\}\) \(\cup\) \(\{(x,x)\}\) and element \(n\) in the bottom right corner. Replace all the elements \(n\) in \(L\) with \(x\), and then delete the element \(x\) of \(L\) in the bottom right corner we get the desired \((n + 4)\)-ISOLS\((n + 1,1)\). This completes the proof of (1) of the lemma.

We now prove (2) of the lemma. Suppose that \( n \geq 6 \) is even. For \( n \in \{6,8,10,12\}\), \((n+4)\)-ISOLS\((n+1,1)\) on \(Z_n \cup \{x\}\) with \(n+4\) distinct ordered pairs form the set \(\{(0,0),(1,1),(0,2),(2,0)\}\) \(\cup\) \(\{(2i - 1,2i),(2i,2i - 1): 1 \leq i \leq (n - 2)/2\}\) \(\cup\) \(\{(n - 1,n - 1),(x,x)\}\) are shown in Fig. 9, where \(a, b\) denote 10, 11, respectively.

For \( n \geq 14 \), we construct \((n+4)\)-ISOLS\((n+1,1)\) recursively.

Suppose that \(L_{2k+1}\) is an \((m+4)\)-ISOLS\((m+1,1)\) on symbol set \(Z_m \cup \{x\}\), where the set of \(m+4\) distinct ordered pairs is \(\{(0,0),(1,1),(0,2),(2,0)\}\) \(\cup\) \(\{(2i - 1,2i),(2i,2i - 1): 1 \leq i \leq (m - 2)/2\}\) \(\cup\) \(\{(m - 1,m - 1),(x,x)\}\) for \(m = 2k\).

If \( n = 4k + 2\) \((k \geq 3)\), we fill the cells of \(A(2k + 2)\) as follows, where the row and column indices of \(A(2k + 2)\) are calculated modulo \(2k + 2\):

(a) fill all the cells in \(T_0\) with \(0\);
(b) fill all the cells in \(T_1\) with \(2k - 1\) and \(x\) alternately, fill all the cells in \(T_{2k+1}\) with \(x\) and \(2k - 1\) alternately, that is \(a(2i,2i + 1) = 2k - 1\), \(a(2i + 1,2i) = 2k - 1\), \(a(2i + 1,2i + 2) = x\), \(a(2i + 2,2i + 1) = x\) for \(0 \leq i \leq k\);
(c) fill all the cells in \(T_j\) with \(2j - 4\) and all the cells in \(T_{2k+2-j}\) with \(2j - 3\) for \(2 \leq j \leq k\);
(d) fill all the cells in \(T_{k+1}\) with the first column of \(B(2k)\).
We construct an array \( L_{4k+3} \) of order \( n + 1 = 4k + 3 \) as follows:
\[
L_{4k+3} = \begin{pmatrix}
A(2k + 2) & B_1(2k + 2) \\
C_1(2k + 2) & L_{2k+1}
\end{pmatrix}.
\]

Replace the elements \( 2k - 1 \) with \( 4k + 1 \), \( y_i \) with \( 2i + (2k - 1) \) and \( y_{i+k+1} \) with \( 2i+2k \) for \( 0 \leq i \leq k \) in \( L_{4k+3} \). We then get an \((n+4)\)-ISOLS\((n+1,1)\) with symbol set \( Z_n \cup \{ x \} \), where the set of \( n + 4 \) distinct ordered pairs is \{\((0,0),(1,1),(0,2),(2,0)\)\} \( \cup \{ (2i - 1,2i),(2i,2i-1) : 1 \leq i \leq (n-2)/2 \} \cup \{ (n-1,n-1),(x,x) \} \) for \( n = 4k + 2 \).

If \( n = 4k + 4 \ (k \geq 3) \), we fill the cells of \( A(2k+4) \) as follows, where the row and column indices of \( A(2k+4) \) are calculated modulo \( 2k + 4 \):

(a) fill all the cells in \( T_0 \) with 0;
(b) fill all the cells in \( T_1 \) with \( 2k - 1 \) and \( x \) alternately, fill all the cells in \( T_{2k+3} \) with \( x \) and \( 2k - 1 \) alternately, that is \( a(2i,2i+1) = 2k - 1 \), \( a(2i+1,2i) = 2k - 1 \), \( a(2i+1,2i+2) = x \), \( a(2i,2i+1) = x \) for \( 0 \leq i \leq k + 1 \);
(c) fill all the cells in \( T_j \) with \( 2j - 4 \) and all the cells in \( T_{2k+4-j} \) with \( 2j - 3 \) for \( 2 \leq j \leq k \);
(d) fill all the cells in \( T_{k+1} \) with the first column of \( B(2k+4) \);
(e) fill all the cells in \( T_{k+3} \) with the second column of \( B(2k+4) \);
(f) fill all the cells in \( T_{k+2} \) with the column indexed by \( k + 2 \) of \( B(2k+4) \).

We construct an array \( L_{4k+5} \) of order \( n + 1 = 4k + 5 \) as follows:
\[
L_{4k+5} = \begin{pmatrix}
A(2k + 4) & B_3(2k + 4) \\
C_3(2k + 4) & L_{2k+1}
\end{pmatrix}.
\]

Replace the elements \( 2k - 1 \) with \( 4k + 3 \), \( y_i \) with \( 2i + (2k - 1) \) and \( y_{i+k+2} \) with \( 2i+2k \) for \( 0 \leq i \leq k+1 \) in \( L_{4k+5} \). We then get an \((n+4)\)-ISOLS\((n+1,1)\) with symbol set \( Z_n \cup \{ x \} \), where the set of \( n + 4 \) distinct ordered pairs is \{\((0,0),(1,1),(0,2),(2,0)\)\} \( \cup \{ (2i - 1,2i),(2i,2i-1) : 1 \leq i \leq (n-2)/2 \} \cup \{ (n-1,n-1),(x,x) \} \) for \( n = 4k + 4 \).

This completes the proof. \( \square \)

Combining Lemmas 4.1–4.4, we have the following lemma.

**Lemma 4.5.** Suppose that \( n \) is an integer and \( n \geq 7 \). For \( r \in \{ n + 2, n + 3 \} \), there exists an \( r \)-SOLS\((n)\) on \( Z_n \), where the set of distinct ordered pairs form a set denoted by \( P \); and there exists an \((r+1)\)-ISOLS\((n+1,1)\) on \( Z_n \cup \{ x \} \), where the set of distinct ordered pairs is \( P \cup \{ (x,x) \} \), where \( |P| = r \).

5. Existence of \( n \)-SOLS\((n)\) with associated Latin squares

**Lemma 5.1.** For each positive integer \( n \), there exists an \( n \)-SOLS\((n)\) on \( Z_n \), where the set of distinct ordered pairs is \{\((i,i) : 0 \leq i \leq n-1\)\}; there exists an \((n+1)\)-ISOLS\((n+1,1)\) on \( Z_n \cup \{ x \} \), where the set of distinct ordered pairs is \{\((i,i) : 0 \leq i \leq n-1\)\} \( \cup \{ (x,x) \} \).
Proof. Suppose that $A=(a_{ij})$ is an $n$ by $n$ array on symbol set $Z_n$ and $a_{ij}=i+j+1 \mod n$. It is obvious that $A$ is an $n$-SOLS($n$) on $Z_n$ with distinct ordered pairs form the set \{(i,i): 0 \leq i \leq n-1\}.

Suppose that $B=(b_{ij})$ is an $(n+1) \times (n+1)$ array on symbol set $Z_{n+1}$ and $b_{ij}=i+j+1 \mod (n+1)$. It is obvious that $B$ is an $(n+1)$-SOLS($n+1$) on $Z_{n+1}$ with distinct ordered pairs form the set \{(i,i): 0 \leq i \leq n\} and element $n$ in the bottom right corner. Replace all the elements $n$ in $B$ by $x$ and then delete the element $x$ from the bottom right corner of $B$, we get an $(n+1)$-ISOLS($n+1$; 1) on $Z_n \cup \{x\}$, where the set of distinct ordered pairs is \{(i,i): 0 \leq i \leq n-1\} \cup \{(x,x)\}. □

Lemma 5.2 (Colbourn and Zhu [5, Lemma 2.4]). If there exists a pair of $r$-orthogonal Latin squares of order $n$, then there exists a pair of $r$-orthogonal Latin squares of order $n$ in which the superposition contains every ordered pair in \{(i,i): 0 \leq i \leq n-1\}.

Combining Theorem 1.1 and Lemma 5.2, we have the following lemma.

Lemma 5.3. Suppose that $n \geq 7$ and $r \in [n,n^2] \setminus \{n+1,n^2-1\}$. There exists a pair of $r$-orthogonal Latin squares on symbol set $Z_n$ in which the superposition contains every ordered pair in \{(i,i): 0 \leq i \leq n-1\}.

6. Main result

In Sections 3–5, we have completed the construction of necessary input designs. In this section, we establish the existence of $r$-SOLS($n$) for general $r$ and $n \geq 28$.

The key construction in this section is referred to as Inflation Construction. It essentially “blows up” every cell of an $r$-SOLS into one of a pair of $s$-orthogonal Latin squares indexed by the element in that cell, and if one cell is filled with the first square, then its symmetric cell is filled with the transpose of the second one.

Lemma 6.1. Suppose $n \geq 28$ is an integer and $m = \lfloor n/4 \rfloor$. Then there exists an $r$-SOLS($n$) for $r \in [n,m+3m^2] \setminus \{n+1\}$.

Proof. Fig. 10 is a 4-SOLS(4) such that the superposition with its transpose yields every ordered pairs in \{(0,0),(3,3),(1,2),(2,1)\}. It has four disjoint transversals, $T_0 = \{(0,0),(1,1),(2,3),(3,2)\}$, $T_1 = \{(0,1),(1,0),(2,2),(3,3)\}$, $T_2 = \{(0,2),(1,3),(2,1),(3,0)\}$, $T_3 = \{(0,3),(1,2),(2,0),(3,1)\}$. $T_0$ and $T_1$ are symmetric ones, $T_2$ and $T_3$ form a symmetric pair. We construct an $r$-ISOLS($4m+k,k$) for $0 \leq k \leq 3$ as follows.
Note that each of the four distinct ordered pairs in \{(0,0), (3,3), (1,2), (2,1)\} repeats four times in the superposition of the 4-SOLS(4) with its transpose. For pairs (0,0) and (3,3), we fill every pair with a \(t\)-SOLS\((m)\) or a \(t\)-ISOLS\((m+1,1)\) with distinct ordered pairs form a set \(Q\), and other three repeated pairs with a \(t_1\)-SOLS\((m)\) or a \(t_1\)-ISOLS\((m+1,1)\) with distinct ordered pairs form a set \(P \subseteq Q\), these input designs are from Lemmas 3.5, 4.5 and 5.1. For pairs (1,2) and (2,1), we fill every pair with a pair of \(r_1\)-orthogonal Latin square of order \(m\) from Lemma 5.3, in which the superposition contains every ordered pair in \(P = \{(i, i) : 0 \leq i \leq n - 1\}\), and other three repeated pairs with an \(m\)-SOLS\((m)\) with distinct ordered pair form the set \(P\) or an \((m+1)\)-ISOLS\((m+1,1)\) with distinct ordered pairs form the set \(P \cup \{(x, x)\}\) from Lemma 5.1.

The cells in the same transversal are all filled with complete Latin squares or all filled with incomplete Latin squares.

We give the following explicit construction.

Suppose that \(k_1, k_2, s_0, s_{10}, s_{11}, s_{12}, r_1\) are integers satisfying the following conditions:

(a) \(k_1, k_2 \in \{0, 1\}\);
(b) \(s_0 \in \{0, 2, 3\}\);
(c) \(s_{10} \in \{0, 2, 3, m^2 - m\}\);
(d) \(s_{11} = s_{12} = s_{10}\) when \(s_{10} \in \{0, 2, 3\}\);
(e) \(s_{11} = (m + k_1)^2 - (m + k_1) - k_1\) and \(s_{12} = (m + k_2)^2 - (m + k_2) - k_2\) when \(s_{10} = m^2 - m\);
(f) \(r_1 \in \{n, n^2\} \cup \{n + 1, n^2 - 1\}\).

**Step 1:** In \(T_0\): fill the cell occupied by 0 with an \((m+s_0)\)-SOLS\((m)\) from Lemmas 5.1, 4.5; fill the cell occupied by 3 with an \((m+s_{10})\)-SOLS\((m)\) from Lemmas 5.1, 4.5, 3.5 or an SOLS\((m)\); fill the cell occupied by 1 with the first square of a pair of \(r_1\)-orthogonal Latin squares of order \(m\) that their superposition yields every ordered pair in \(\{(i, i) : 0 \leq i \leq m - 1\}\), and the cell occupied by 2 with the transpose of the second square.

**Step 2:** In \(T_1\): fill the cell occupied by 0 with an \((m+k_1+s_0)\)-ISOLS\((m+k_1,k_1)\) from Lemmas 5.1, 4.5; fill the cell occupied by 3 with an \((m+k_1+s_{11})\)-ISOLS\((m+k_1,k_1)\) from Lemmas 5.1, 4.5, 3.5 or an SOLS\((m)\); fill the two cells occupied by 1 and 2 with an \((m+k_1)\)-ISOLS\((m+k_1,k_1)\) from Lemma 5.1.

**Step 3:** In \(T_2\) and \(T_3\): fill the two cells occupied by 0 with an \((m+k_2+s_0)\)-ISOLS\((m+k_2,k_2)\) from Lemmas 5.1, 4.5; fill the two cells occupied by 3 with an \((m+k_2+s_{12})\)-ISOLS\((m+k_2,k_2)\) from Lemmas 5.1, 4.5, 3.5 or an SOLS\((m)\); fill the four cells occupied by 1 and 2 with an \((m+k_2)\)-ISOLS\((m+k_2,k_2)\) from Lemma 5.1.

Now we get an \(r\)-ISOLS\((4m+k,k)\) with \(r = 2r_1 + 2m + s_0 + s_1 + k\), where \(s_1 = \max\{s_{11} + k_1, s_{12} + k_2\}, k = k_1 + 2k_2\).

**Case 1:** \(k_1 = k_2 = 0\). In this case, the resulting \(r\)-ISOLS\((4m+k,k)\) is an \(r\)-SOLS\((4m)\) for \(r = 2r_1 + 2m + s_0 + s_1\). Simple counting shows that \(r\) can be any integer in \([4m, m + 3m^2] \cup \{4m + 1\}\).

**Case 2:** \(k_1 = 1, k_2 = 0\). We assume that the hole set of the input designs used in Step 2 is \(\{x\}\). Fill in the hole of the resulting \(r\)-ISOLS\((4m+1,1)\) with \(x\), we obtain an \(r\)-SOLS\((4m+1)\) for \(r = 2r_1 + 2m + s_0 + s_1 + 1\). Simple counting shows that \(r\) can be any integer in \([4m+1, m + 3m^2 + 1] \cup \{4m + 2\}\).
Fig. 11. Input designs for the construction of a 27-SOLS(17).

Case 3: \( k_1 = 0, k_2 = 1 \). We assume that the hole set of the input designs used in Step 3 to fill the cells occupied by 0 and 3 is \{y\}, the hole set of the input designs used in Step 3 to fill the cells occupied by 1 and 2 is \{z\}. By filling the hole of the resulting \( r \)-ISOLS(\( 4m + 2 \), 2) with a sub-square

\[
\begin{pmatrix}
  y & z \\
  z & y
\end{pmatrix}
\]

we obtain an \( r \)-SOLS(\( 4m + 2 \)) for \( r = 2r_1 + 2m + s_0 + s_1 + 2 \).

Simple counting shows that \( r \) can be any integer in \([4m + 3; m + 3m^2 + 2]\) \{4m + 3\}.

Case 4: \( k_1 = k_2 = 1 \). By filling in the hole of the resulting \( r \)-ISOLS(\( 4m + 3 \), 3) with a sub-square

\[
\begin{pmatrix}
  x & z & y \\
  z & y & x \\
  y & x & z
\end{pmatrix}
\]

we obtain an \( r \)-SOLS(\( 4m + 3 \)) for \( r = 2r_1 + 2m + s_0 + s_1 + 3 \).

Simple counting shows that \( r \) can be any integer in \([4m + 3; m + 3m^2 + 3]\) \{4m + 4\}.

This completes the proof. \( \square \)

We give an example of the construction in Lemma 6.1 in Figs. 11 and 12. For convenience, we take a smaller \( m \), \( m = 4 \), \( r_1 = 9 \), \( s_0 = 0 \), \( s_{10} = s_{11} = s_{12} = 0 \), \( k_1 = 1 \), \( k_2 = 0 \). The input designs are shown in Fig. 11, where (a) and (b) form a pair of 9-orthogonal Latin square of order 4, in which the superposition contains every ordered pair in \{(i,i): 0 \leq i \leq 3\}, (c) is a 4-SOLS(4) with four distinct pairs form the set \{(i,i): 0 \leq i \leq 3\}, (d) is a 5-ISOLS(5,1) on \( \mathbb{Z}_4 \cup \{x\} \) where the set of distinct ordered pairs is \{(i,i): 0 \leq i \leq 3\} \cup \{(x,x)\}. The obtained 27-SOLS(17) is shown in Fig. 12.

Lemma 6.2. Suppose \( n \geq 28 \) is an integer and \( m = \lfloor n/4 \rfloor \). Then there exists an \( r \)-SOLS(n) for \( r \in [16m + 5, n^2 - 3, n^2 - 1] \).

Proof. Fig. 13 is an SOLS(4) (16-SOLS(4)). Each of the 16 distinct ordered pairs appears only one time in the superposition of the SOLS(4) with its transpose, and it has four disjoint symmetric transversals, \( T_0 = \{(0,0), (1,1), (2,2), (3,3)\} \), \( T_1 = \{(0,1), (1,0), (2,3), (3,2)\} \), \( T_2 = \{(0,2), (1,3), (2,0), (3,1)\} \), \( T_3 = \{(0,3), (1,2), (2,1), (3,0)\} \). Start with the SOLS(4), we construct an \( r \)-ISOLS(\( 4m + k \), \( k \)) as follows.
In the first three transversals, $T_0$, $T_1$ and $T_2$, we fill an $r'$-ISOLS($m+k_i$) in each cell and the symmetric cell, where $k_i \in \{0,1\}$. In the fourth transversal, $T_3$, we fill a pair of $r'$-orthogonal Latin squares in each cell and the symmetric cell. We give the following explicit construction.

Suppose that $k_0, k_1, k_2, s_i (0 \leq i \leq 7), r_1, r_2$ are integers satisfying the following conditions:

(a) $k_i \in \{0,1\}$ for $0 \leq i \leq 2$;
(b) $s_i \in \{0,2,3,(m+k_0)^2-(m+k_0)-7-k_0,(m+k_0)^2-(m+k_0)-2-k_0,(m+k_0)^2-(m+k_0)-k_0\}$ for $0 \leq i \leq 3$;
(c) $s_i \in \{0,(m+k_1)^2-(m+k_1)-k_1\}$ for $4 \leq i \leq 5$;
(d) $s_i \in \{0,(m+k_2)^2-(m+k_2)-k_2\}$ for $6 \leq i \leq 7$;
(e) $r_i \in [m,m^2]\{m+1,m^2-1\}$ for $1 \leq i \leq 2$.

**Step 1:** In $T_3$: fill the cell occupied by $i$ with an $(m+k_0+s_i)$-ISOLS($m+k_0$) for $0 \leq i \leq 3$, these input designs are from Lemmas 3.1–3.5, 4.5 and 5.1.
Step 2: In $T_1$: fill the two cells occupied by 0 and 1 with an $(m+k_1+s_4)$-ISOLS$(m+k_1,k_1)$; fill the two cells occupied by 2 and 3 with an $(m+k_1+s_5)$-ISOLS$(m+k_1,k_1)$.

Step 3: In $T_2$: fill the two cells occupied by 0 and 2 with an $(m+k_2+s_6)$-ISOLS$(m+k_2,k_2)$; fill the two cells occupied by 1 and 3 with an $(m+k_2+s_7)$-ISOLS$(m+k_2,k_2)$.

Step 4: In $T_3$: fill the cell occupied by 0 with the first square of a pair of $r_1$-orthogonal Latin squares of order $m$ from Theorem 1.1, and the cell occupied by 3 with the transpose of the second square; fill the cell occupied by 1 with the first square of a pair of $r_2$-orthogonal Latin squares of order $m$, and the cell occupied by 2 with the transpose of the second one.

In Steps 1–3, the incomplete Latin squares filled in the same transversal have the same hole set, and those filled in different transversals have different hole sets.

For an input design $(m+1+s_i)$-ISOLS$(m+1,1)$, denote the set of the distinct ordered pairs by $Q$. From Lemmas 3.2, 3.4, 3.5, 4.5 and 5.1 we know that if the hole set is $\{x\}$, then the pair $(x,x) \in Q$ when $s_i \in \{0,2,3\}$, and $(x,x) \notin Q$ when $s_i \in \{(m+1)^2-(m+1)-8,(m+1)^2-(m+1)-3,(m+1)^2-(m+1)-1\}$. So, from Steps 1–4, we get a $t$-ISOLS$(4m+k,k)$ with $k=k_0+k_1+k_2$ and $t=2(r_1+r_2)+12m+\sum_{i=0}^3 (s_i+k_0 \times \text{sig}(s_i))+2 \sum_{i=4}^5 (s_i+k_1 \times \text{sig}(s_i))+2 \sum_{i=6}^7 (s_i+k_2 \times \text{sig}(s_i))+\sum_{j=0}^2 k_j \times \text{sig}(T_j),$

where

\[
\text{sig}(s_i) = \begin{cases} 
0 & \text{when } s_i \in \{0,2,3\}, \\
1 & \text{otherwise}, 
\end{cases} \quad 0 \leq i \leq 7,
\]

\[
\text{sig}(T_0) = \begin{cases} 
0 & \text{if } \sum_{i=0}^3 \text{sig}(s_i) = 4, \\
1 & \text{otherwise}, 
\end{cases}
\]

\[
\text{sig}(T_1) = \begin{cases} 
0 & \text{if } \sum_{i=4}^5 \text{sig}(s_i) = 2, \\
1 & \text{otherwise}, 
\end{cases}
\]

\[
\text{sig}(T_2) = \begin{cases} 
0 & \text{if } \sum_{i=6}^7 \text{sig}(s_i) = 2, \\
1 & \text{otherwise}.
\end{cases}
\]

Case 1: $k_0=k_1=k_2=0$. In this case, the resulting $t$-ISOLS$(4m+k,k)$ is an $r$-SOLS$(4m)$ for $r=2(r_1+r_2)+12m+\sum_{i=0}^3 s_i+2 \sum_{i=4}^7 s_i$. Simple counting shows that $r$ can be any integer in $[16m+2,(4m+1)^2]\{(4m)^2-5,(4m)^2-3,(4m)^2-1\}$.

Case 2: $k_0=1$, $k_1=k_2=0$. We assume that the hole set of the input $(m+1+s_i)$-ISOLS$(m+1,1)$ used in $T_0$ is $\{x\}$. By filling the hole of the resulting $t$-ISOLS$(4m+1,1)$ with $x$, we obtain an $r$-SOLS$(4m+1)$ for $r=2(r_1+r_2)+12m+\sum_{i=0}^3 (s_i+\text{sig}(s_i))+2 \sum_{i=4}^7 s_i+1$.

Simple counting shows that $r$ can be any integer in $[16m+3,(4m+1)^2]\{(4m+1)^2-5,(4m+1)^2-3,(4m+1)^2-1\}$.
Case 3: \( k_0 = k_1 = 1, k_2 = 0 \). We assume that the hole set of the input \((m+1+s_j)\text{-ISOLS}(m+1,1)\) used in \(T_j\) is \(\{x_j\}\) (\(0 \leq j \leq 1\)). By filling the hole of the resulting \(t\text{-ISOLS}(4m+2,2)\) with a sub-square

\[
\begin{pmatrix}
  x_0 & x_1 \\
  x_1 & x_0
\end{pmatrix},
\]

we obtain an \(r\text{-SOLS}(4m+2)\) for \(r = 2(r_1 + r_2) + 12m + \sum_{i=0}^{3}(s_i + \text{sig}(s_i)) + 2 \sum_{i=4}^{7}(s_i + \text{sig}(s_i)) + 2 \sum_{i=6}^{7} s_i + 2 \).

Simple counting shows that \(r\) can be any integer in \([16m+4, (4m+2)^2]\) \(\backslash S\), where

\[ S = \bigcup_{i=1}^{3}\{(m+2)^2 - (2i - 1)\} \cup \bigcup_{j=1}^{2}\{(4m+3)^2 - 2j\} \].

\((4m+2)^2 - 7)\text{-SOLS}(4m+2)\) can be obtained from Lemma 3.3.

Case 4: \( k_0 = k_1 = k_2 = 1 \). We assume that the hole set of the input \((m+1+s_j)\text{-ISOLS}(m+1,1)\) used in \(T_j\) is \(\{x_j\}\) (\(0 \leq j \leq 2\)). By filling in the hole of the resulting \(t\text{-ISOLS}(4m+3,3)\) with a sub-square

\[
\begin{pmatrix}
  x_0 & x_1 & x_2 \\
  x_1 & x_2 & x_0 \\
  x_2 & x_0 & x_1
\end{pmatrix},
\]

we obtain an \(r - \text{SOLS}(4m+3)\) for \(r = 2(r_1 + r_2) + 12m + \sum_{i=0}^{3}(s_i + \text{sig}(s_i)) + 2 \sum_{i=4}^{7}(s_i + \text{sig}(s_i)) + 3 \).

Simple counting shows that \(r\) can be any integer in \([16m+5, (4m+3)^2]\) \(\backslash S\), where

\[ S = \bigcup_{i=0}^{3}\{(4m+3)^2 - (2i - 1)\} \cup \bigcup_{j=1}^{2}\{(4m+3)^2 - 2j\} \].

By filling the hole of an ISOLS(4m+3,6) from Theorem 2.1 with a 25-SOLS(6) or a 27-SOLS(6) from Theorem 1.2, we get a \(((4m+3)^2 - 11)\text{-SOLS}(4m+3)\) or a \(((4m+3)^2 - 9)\text{-SOLS}(4m+3)\). \(((4m+3)^2 - 7)\text{-SOLS}(4m+3)\) can be obtained from Lemma 3.3. By filling the hole of an ISOLS(4m+3,5) with a 21-SOLS(5) from Theorem 1.2, we get a \(((4m+3)^2 - 4)\text{-SOLS}(4m+3)\). \(((4m+3)^2 - 2)\text{-SOLS}(4m+3)\) can be obtained from Lemma 3.1.

Since \(m + 3m^2 \geq 16m + 5\) when \(m \geq 7\), combining Lemmas 6.1, 6.2 and 2.3 we have the following lemma.

**Lemma 6.3.** If \(n \geq 28\) is an integer, then there exists an \(r\text{-SOLS}(n)\) for \(r \in [n,n^2]\) \(\backslash \{n+1,n^2-1\}\).

We are now in the position to give the main result of this paper.

**Theorem 6.4.** For any integer \(n \geq 28\), there exists an \(r\text{-SOLS}(n)\) if and only if \(n \leq r \leq n^2\) and \(r \not\in \{n+1,n^2-1\}\).

**Proof.** The necessity comes from Theorem 1.1. The sufficiency comes from Lemma 6.3. \(\square\)
Acknowledgements

The authors are thankful to Professor L. Zhu for supplying them Ref. [8].

References