On Coloring Resilient Graphs

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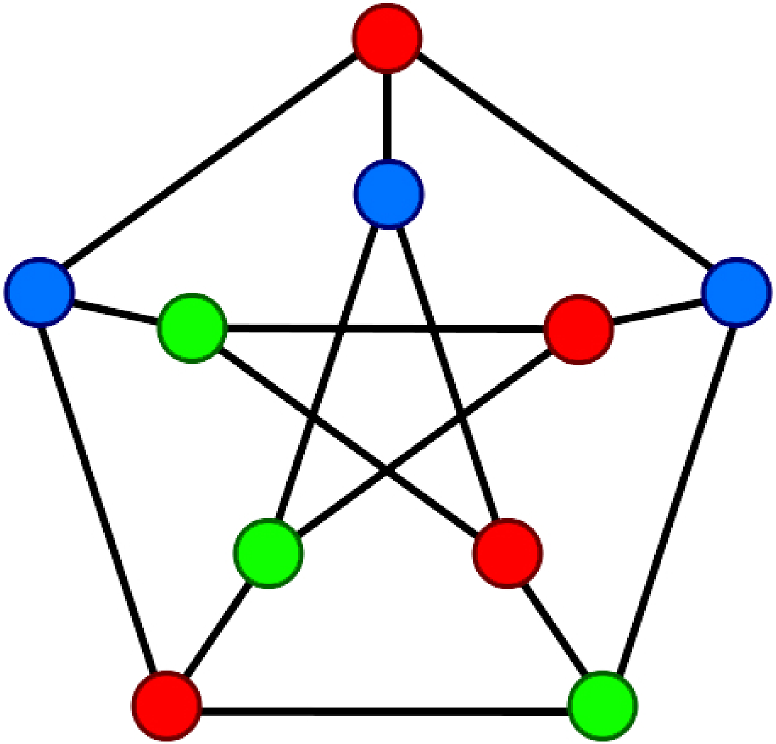
Let's talk about graph coloring



Let's talk about graph coloring

Can we color k-colorable graphs if we allow more colors?

- 4-coloring 3-colorable graphs is NP-hard.
  [Khanna-Linial-Safra '00]
- $2^{\sqrt[3]{k}}$-coloring k-colorable graphs is NP-hard for $k \gg 0$.
  [Huang '13]
- $\exists c > 0 : \log^c(n)$-coloring 4-colorable graphs is UGC*-hard.
  [Dinur-Shinkar '13]
- $O(n^{0.19996})$-coloring 3-colorable graphs is in P.
  [Kawarabayashi-Thorup '14]

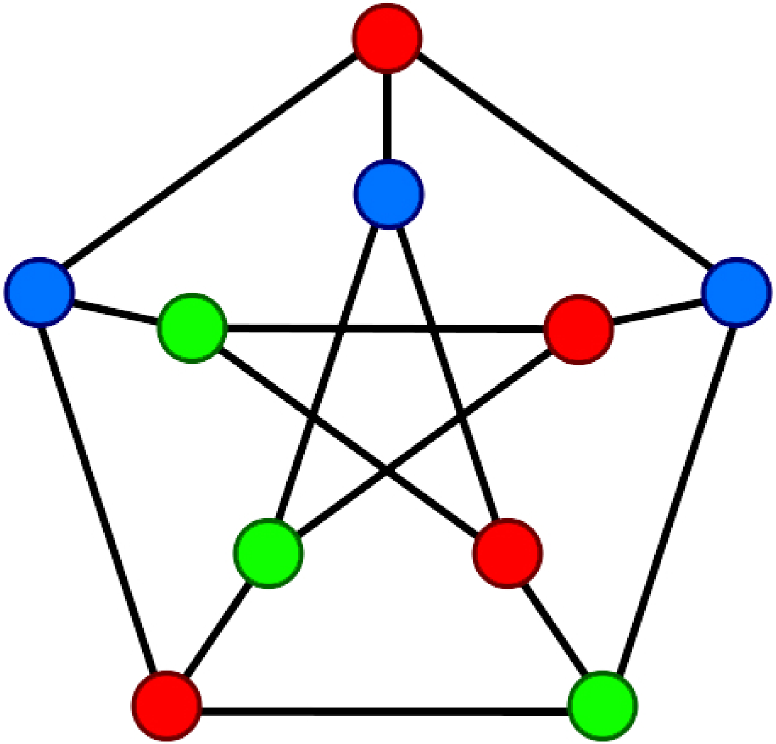
Resiliently colorable graphs

Can we k-color k-colorable graphs that are really k-colorable?

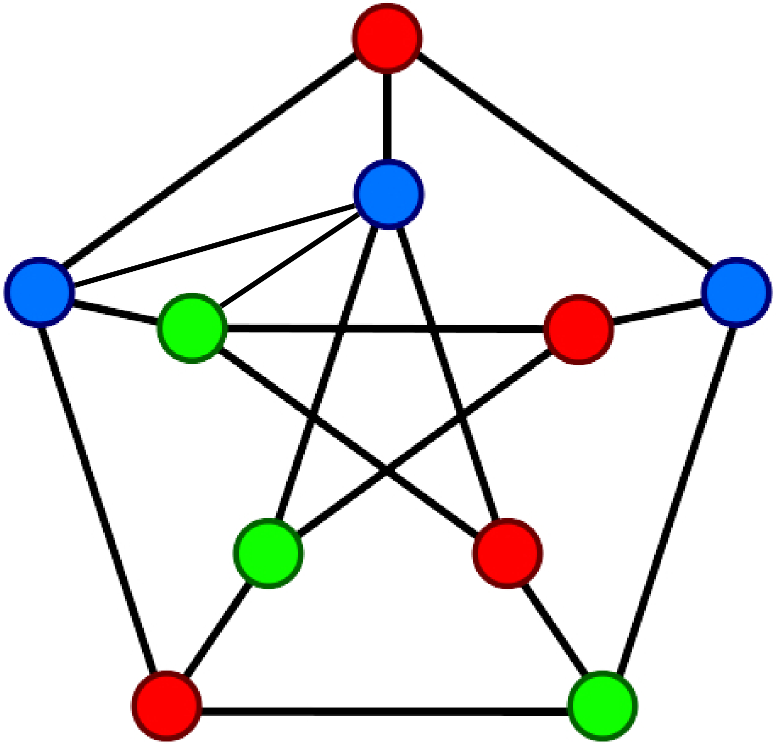
Call G 1-resiliently k-colorable if it's k-colorable, and remains k-colorable no matter which edge is added.

More generally, G is r-resiliently k-colorable if it remains k-colorable no matter which set of r new edges is added.

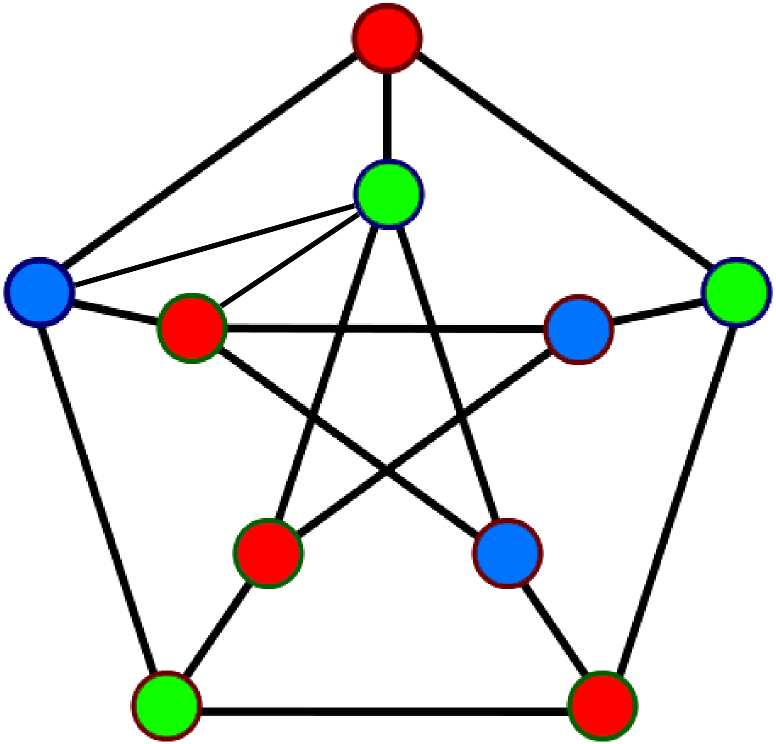
Resiliently colorable graphs



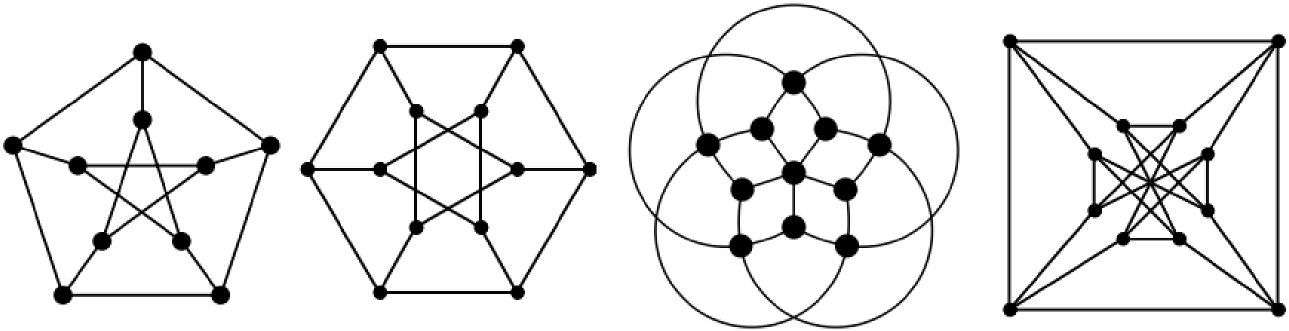
Resiliently colorable graphs



Resiliently colorable graphs



Resiliently colorable graphs



From left to right: the Petersen graph, 2-resiliently 3-colorable; the Dürer graph, 4-resiliently 4-colorable; the Grötzsch graph, 4-resiliently 4-colorable; and the Chvátal graph, 3-resiliently 4-colorable. These are all maximally resilient (no graph is more resilient than stated) and chromatic (no graph is colorable with fewer colors).

Resilient combinatorial problems

Can turn any combinatorial problem into a resilient problem.

1. r-resilient k-SAT is resilient up to fixing variables.
2. r-resilient Hamiltonian cycle is resilient up to removal of edges.
3. r-resilient 3D matching is resilient up to the addition of triples.

Let's talk about SAT

Fact: 1-resilient 6-SAT is NP-hard

Proof: Reduce from 3-SAT.

$$\varphi \mapsto \varphi \vee \varphi'$$

Let's talk about SAT

Theorem [K-Reyzin '14] r-resilient k-SAT is NP-hard for all $k \geq 3$, $r < k$.

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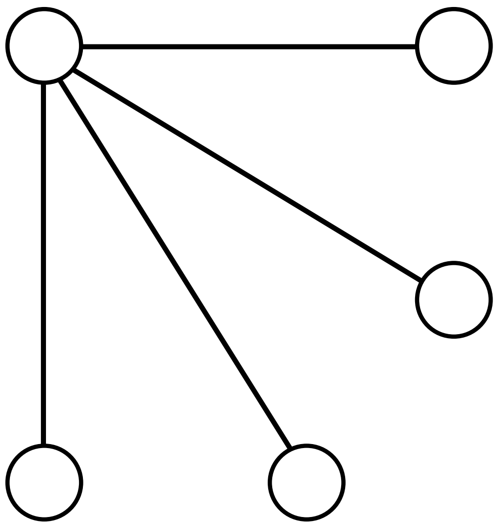
Lemma (blowing up): For all $r \geq 0, s \geq 1, k \geq 3$, r-resilient k-SAT reduces to $[(r+1)s - 1]$-resilient (sk)-SAT in poly time.

Lemma (shrinking down): Let $r \geq 1, k \geq 2$, $q = \min(r, \lfloor k/2 \rfloor)$. Then r-resilient k-SAT reduces to q-resilient $(\lceil k/2 \rceil + 1)$-SAT in poly time.

Back to coloring

Coloring has a gradient from easy to hard: $\binom{k}{2}$-resilient k-coloring is in P, but not vacuously trivial. e.g. 6-resilient 4-coloring is easy



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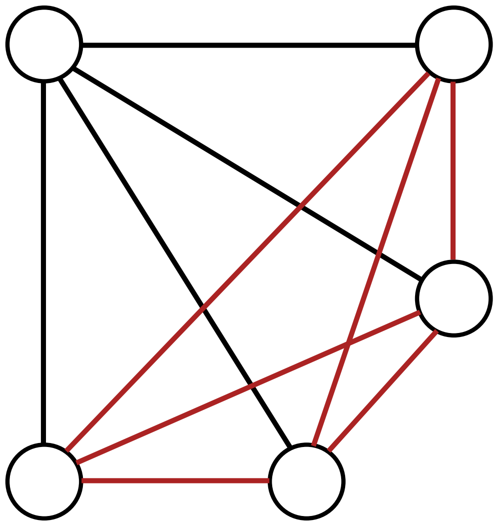


Back to coloring

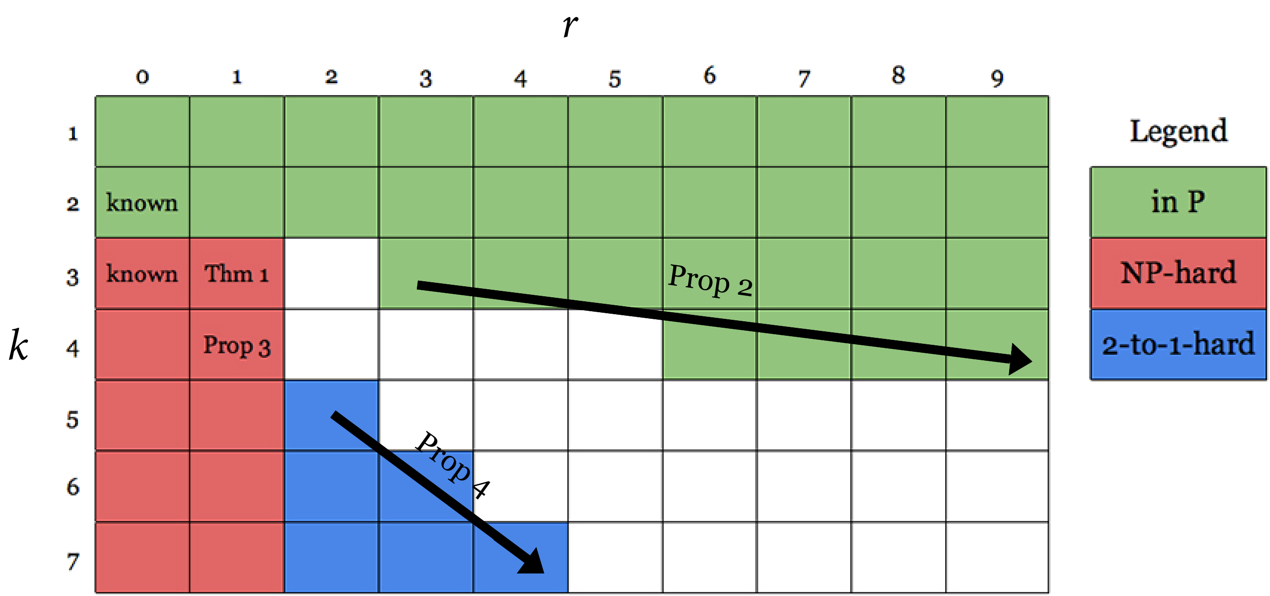
1-resilient 4-coloring is NP-hard.

Every 3-colorable graph is 1-resiliently 4-colorable, and it's hard to 4-color a 3-colorable graph.

Observation: If it is NP-hard to $f(k)$-color a k-colorable graph, then it is NP-hard to $f(k)$-color an $(f(k) - k)$-resiliently $f(k)$-colorable graph.

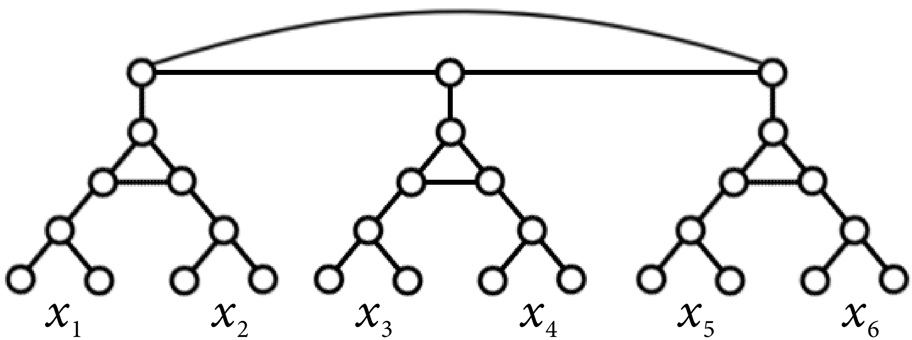
Classifying resilient coloring

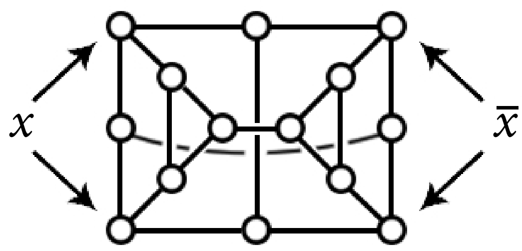
Complexity of r-resilient k-coloring



1-resilient 3-coloring is NP-hard

Theorem [K-Reyzin '14] It is NP-hard to 3-color a 1-resiliently 3-colorable graph.





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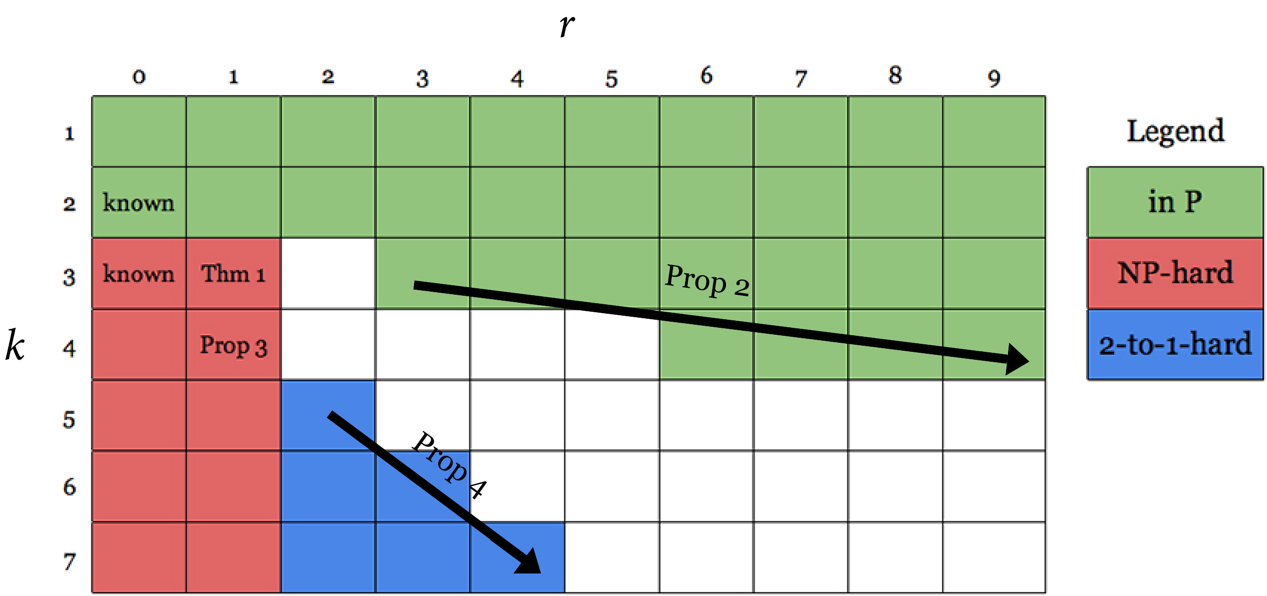

Open problems:

1. Find hardness boundary for resilient coloring (upper bounds!).
2. Analyze resilience of your favorite combinatorial problem.
3. Find a suitable notion of resilience for general CSPs.

Questions?