Solving minimal covering location problem (MinCLP) with the aid of fuzzy sets

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Abstract—The object of Minimal Covering Location Problem (MinCLP) is to find a placement of the locations which have the minimal coverage. Minimal Covering location problem is the opposite of the Maximal Covering Location Problem (MCLP) which is more studied in literature. The role of fuzzy sets in the improvement of MCLP was studied in several papers in the past and the role of fuzzy sets in the improvement of MinCLP is analyzed in this paper. Two different models for two different classes of Fuzzy MinCLP will be presented.

keywords: minimal, covering, location, distances, fuzzy

I. INTRODUCTION

The object of Minimal Covering Location Problem (MinCLP) is to find the circle or rectangle which have the minimal coverage, and it is opposite to the objective Maximum Covering Location Problem (MCLP) which is more popular than MinCLP.

Study of the Maximal Covering Location Problem (MCLP) started in the 1974 by R. Church, C. ReVelle, in the paper [4]. Objective of MCLP is to find the placement of different facilities (hospitals, schools, shops, etc.) i.e. to obtain the maximum coverage to the locations with given radius. The MCLP has evolved over the years and put in many different settings and overview of the evolution of MCLP could be find in [5]. The latest research of MCLP related to the applications of fuzzy sets in improving a MCLP models [6].

On the other hand, MinCLP is not much presented in the literature, although both problems are equally important. The main object of MinCLP is to find a placement of the locations which have the minimal coverage. MinCLP is used for finding a locations for pollution sources, like air, groundwater or soil pollutants. The main difference between MCLP and MinCLP is additional condition related to the minimal distance between facilities in MinCLP. Without this condition, the optimal solution is that all facilities will be located in the same site. MinCLP introduced by Drezner and Wesolowsky in [9], and review of literature of MinCLP is presented in [3]. There are different ideas in MinCLP (coverage with circles, rectangles, existence of forbidden zones etc.) and they presented in [3], [13], [12] and [10].

As mentioned before, fuzzyness has been used in many ways in MCLP yielding fuzzy MLCP or FMLCP. In [6] travel times are fuzzyfied in order to model their unpredictability. As we are aware this is one of the first attempts to use fuzzy sets in MinCLP.

Let us mention the degree of the coverage of a location in MCLP by a facility is has been introduced by the authors of this paper in [15]. Usually, coverage is a crisp value i.e. the location was either covered(1) or not(0), no partial coverage was allowed. In this model fuzzy travel times are fuzzy sets. Fuzzy ordering is used in order to find the coverage degree of a location by a facility. Namely, this results in a completely different model since there is no crisp cut of as it was the case in previous models and the goal function is calculated in a different manner. This makes the model more closer to the reality since the fuzziness more adequately models a vague undetermined real-time system. Moreover, the fuzzification does not increase the complexity of the algorithm, but MinCLP is belongs to the class of NP-hard problems. There are no serious papers related to methods for solving MinCLP, but all methods for solving MCLP can be applied for solving MinCLP. Many authors are adjusted plenty of metaheuristics for solving MCLP, like PSO algorithm [8], a greedy variable neighborhood search heuristic [7], a constructive genetic algorithm [1] and heuristic concentration ČiřeReVelle1. It this paper methods for solving MinCLP are not discussed, that choice depends on the concrete application.

In order to obtain the solution of the problem a method of Particle Swarm Optimization is applied. Since MinCLP have a completely different objective than MCLP the model is completely different. The main difference besides finding the minimum of the goal function is the aggregation operator...
used to aggregate coverage degrees. Its choice depends on the concrete application. The paper is structured as follows. In section 2 some basic notions from the theory of fuzzy sets are introduced and their use in these models. In section 3 the mathematical model of the problem is given and finally section 4 briefly summarises the paper.

II. BASIC NOTIONS

In this section some basic notions from the theory of fuzzy sets will be given. They were defined and developed by the father of the theory L. Zadeh in [16]. With the notion of fuzzy sets, fuzzy logic has developed. In fuzzy logic the truth value can be any number from the unit interval. The most common operations on the unit interval are described in [11]. In order to generalize conjunctions $t$-norms are used. They are symmetric, associative, non decreasing operators on the unit square with a unit element 1. Similarly, the generalization of the disjunction operators are $t$-conorms or $s$-norms. The four basic $s$-norms are:

- $S_M(x_1, ..., x_n) = \max(x_1, ..., x_n)$
- $S_P(x_1, ..., x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)$
- $S_L(x_1, ..., x_n) = \min\{\sum_{i=1}^{n} x_i, 1\}$
- $S_D(x_1, ..., x_n) = \begin{cases} x_i, & \text{if } \forall j \neq i, x_j = 0; \\ 1, & \text{otherwise.} \end{cases}$

We will use these operators later to obtain the coverage degree of the locations.

In order to describe imprecise values for example $3 \pm 2$, triangular fuzzy numbers are introduced. Actually, triangular fuzzy numbers have an increasing membership function up to a point, and then the membership function decreases. Trapezoidal fuzzy numbers fuzzify the interval $[a, b]$ i.e. make its borders not so strict. Gaussian fuzzy sets are the Gaussian probability distributions. Left and right shoulder fuzzy numbers have a strictly increasing (or decreasing ) membership function. They usually represent constraints limited from one side only (low, hi...). Fuzzy relations are actually fuzzy subsets of any universe. In order to fuzzify the equality relation similarity relations are obtained. One of the most common similarity relation is obtained in the following way:

$$Sim(A, B) = \sup_{x \in X} \min\{\mu_A(x), \mu_B(x)\}$$

where $X$ is the universe, $\mu_A$ and $\mu_B$ are the membership functions of the fuzzy sets $A$ and $B$ respectively. Actually the result of this relation has a very intuitive geometrical interpretation. An ordering relation on fuzzy sets is actually a generalization of the well known ordering on intervals. $[a_1, b_1]$ and $[a_2, b_2]$ It is given in the following way:

$$[a_1, b_1] \preceq [a_2, b_2] \text{ iff } a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

The ordering is generalised in the following way:

**Definition 2.1:** Let $X$ be a universe and $\preceq$ is an ordering on $X$. For a fuzzy set $A$ $A \in \mathcal{F}(X)$, a fuzzy superset of $A$, denoted by $LTR(A)$ is defined as:

$$\mu_{LTR(A)}(x) = \sup\{\mu_A(y) | y \preceq x\}.$$

$LTR(A)$ is the smallest fuzzy superset of $A$ with a non-decreasing membership function. Moreover, we give the definition of $RTL(A)$:

$$\mu_{RTL(A)}(x) = \sup\{\mu_A(y) | x \preceq y\}.$$ 

Similarly, $RTL(A)$ is the smallest fuzzy superset of $A$ with a non-increasing membership function.

Relying on the previous definition of $RTL$ and $LTR$ the ordering $\preceq_t$ shall be introduced.

**Definition 2.2:** If $A, B \in \mathcal{F}(X), \mathcal{F}(X)$ being the set of all fuzzy subsets of a universe $X$ then we define an ordering $\preceq_t$ on $\mathcal{F}(X)$:

$$A \preceq_t B \text{ iff } LTR(A) \supseteq LTR(B) \wedge RTL(A) \subseteq RTL(B).$$

In order to fuzzify the previous relation we introduce the compatibility relation.

**Definition 2.3:** Let $A, B \in \mathcal{F}(X)$, then the compatibility of a fuzzy set $A$ with the fuzzy set $B$ is defined in the following way:

$$C(A, B) = \frac{\text{card}(A \cap B)}{\text{card}(A)}$$

where $A \cap B$ is the intersection of $A$ and $B$, and $\text{card}(L)$ is the cardinality of a fuzzy set $L$.

Using previous definitions of $C$ and $\preceq_t$ the following similarity measure is obtained.

**Definition 2.4:** Let $A, B \in \mathcal{F}(X)$. The fuzzy similarity measure $FEL$ is defined in the following way:

$$FEL(A, B) = \left( A \preceq_t B \right) \bigcap (N(C(B, A))$$

where $C(A, B)$ is defined above and $N$ is the fuzzy equivalent of the $\neg$ operator.

Different values of $FEL(A, B)$ are given in the following table.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$FEL(A, B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>tri(2,2)</td>
<td>tri(3,2)</td>
<td>0.25</td>
</tr>
<tr>
<td>trap(2,3,2)</td>
<td>tri(3,2)</td>
<td>0.5</td>
</tr>
<tr>
<td>rs(1,2)</td>
<td>tri(3,1,4)</td>
<td>0.375</td>
</tr>
<tr>
<td>rs(1,2)</td>
<td>trap(2,3,1,4)</td>
<td>0.25</td>
</tr>
</tbody>
</table>

III. MATHEMATICAL MODELS OF MINCLP

In this section we will give two mathematical models for MinCLP. These models describe two different class of Minimal covering location problem. The main difference in these models is algorithm for calculation the degree of coverage in case if some location is covered by more that one facility.

First, it is necessary to introduce the following variables

- $P$ - number of facilities [integer]
- $D_{ij}$ - distance matrix - $(d_{ij})$ is distance between $i$ and $j$ nodes [real]
- $R$ - coverage radius [right shoulder fuzzy number].
- $A_{ij}$ - coverage matrix - $(a_{ij})$ is degree of coverage node $j$ by facility on node $i$ [real]
• $y_i$ - indicator if a facility is established in the node $i$, $y_i \in \{0, 1\}$
• $x_i$ - degree of coverage of facility $x_i \in [0, 1]$
• $m$ - minimal distance between two facilities.

As mentioned before, a location can be partially covered, if it is located in distances between $r$ and $R_f$. In that case degree of coverage of that node is in the unit interval.

First model describes the class of problems where coverage degree for some node could be added if it is partially covered with more that one facility. A typical example for this approach is the problem of determining locations for smoke pollutants. In that case, pollution is calculated as sum of degrees of all pollutants.

This approach has the mathematical model as following:

Minimize

$$\sum y_i$$

with conditions:

$$\sum (x_j \cdot A_{ij}) \geq y_i, \forall i$$

$$x_i \cdot D_{i,j} > m, \forall j$$

$$\sum x_i = P$$

$$x_i \in \{0, 1\}$$

$$y_i \in [0, 1]$$

Formula (1) represents the function which is minimized in the model, formula (2) calculates the degree of coverage of each node which reaches a sum of all coverage degrees, formula (3) insures minimal distance between two facilities and finally formula (4) insured that are exactly $p$ facilities in the solution.

The second model takes the maximum of all degrees of coverage for partially covered nodes. This model describes a class of problems where it is not allowed to sum degrees of coverage. Example for this model is the problem of determining locations for different type of pollutants, like smoke and water pollutants. In that case, total pollution is equal to maximal pollution of all pollutants.

With the same notation as in first model, mathematical model of second approach is:

Minimize

$$\sum y_i$$

with conditions:

$$\max (x_j \cdot A_{ij}) \geq y_i, \forall i$$

$$x_i \cdot D_{i,j} > m, \forall j$$

$$\sum x_i = P$$

$$x_i \in \{0, 1\}$$

$$y_i \in [0, 1]$$

The only difference in the first and the second model are formulas (2) and (8). As it is described before, the second model takes a maximum of all degrees, instead of their sum.

In order to use a CPLEX solver for problems with small dimension or some metaheuristic for big dimensions, some pre-calculations need to be performed.

Variables $a_{ij}$ are calculated in the following way: $a_{ij} = FEL(t_{ij}, R)$, where $FEL$ is the similarity measure defined in the previous section. Illustration of the computation of this variable is shown in Figure 1.

This concludes the list of all the necessary parameters to implement our model.

Illustration of differences of two proposed models are given on figures 2. and 3. Both figures contains a instance with 5 locations and two pollution sources. Figure 2. shows an optimal solution for second model - total coverage degree is 2.8 - two locations (where the pollutants are located) are fully covered and one location is assigned to the nearest pollution source (on figure marked $L_1$) and it is partially covered with degree of 0.8. In first model, location $L_2$ is also fully coverage because its coverage degree is calculated as sum of both partial coverage. Total coverage degree in that case is 3.

Figure 3. shows an optimal solution for first model on the same instance. Total coverage degree is 2.9 - two locations is fully covered (location of pollutants), degree of coverage of location $L_1$ is 0.8 and degree for $L_2$ is 0.1.

The implementation of this model into a solver is not not complicated i.e. it is necessary only to implement the calculation of the relation FLE which is a simple programming task. In the future work implementation of described models will be done and illustrated on some examples.

IV. CONCLUSION

This is one of the first contributions in the use of fuzzy sets in solving the MinCLP problems. Two different mathematical
models are presented. Moreover, the models are easily implementable and does not increase the solution complexity. Generally, the practical nature of the problem determines which of the models is used.

The testing of the model is necessary with various tools in order to obtain the optimal solution. In our previous work with other similar problems we have shown that fuzzy logic significantly improves the solution thus we hope that the same results will be obtained in MinCLP.

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