Bi-Factor Approximation Algorithms
for Hard Capacitated $k$-Median Problems

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Abstract

In the classical $k$-median problem the goal is to select a subset of at most $k$ facilities in order to minimize the total cost of opened facilities and established connections between clients and opened facilities. We consider the capacitated version of the problem, where a single facility may only serve a limited number of clients. We construct approximation algorithms slightly violating the capacities based on rounding a fractional solution to the standard LP.

It is well known that the standard LP has unbounded integrality gap if we only allow violating capacities by a factor smaller than 2, or if we only allow violating the number of facilities by a factor smaller than 2. In an earlier version of our work we showed that violating capacities by a factor of $2 + \varepsilon$ is sufficient to obtain constant factor approximation of the connection cost. In this paper we substantially extend this result in the following two directions. First, we extend the $2 + \varepsilon$ capacity violating algorithm to the more general $k$-facility location problem with uniform capacities, where opening facilities incurs a location specific opening cost. Second, we show that violating capacities by a slightly bigger factor of $3 + \varepsilon$ is sufficient to obtain constant factor approximation of the connection cost also in the case of the non-uniform hard capacitated $k$-median problem.

Our algorithms first use the clustering of Charikar et al. to partition the facilities into sets of total fractional opening at least $1 - 1/\ell$ for some fixed $\ell$. Then we exploit the technique of Levi, Shmoys, and Swamy developed for the capacitated facility location problem, which is to locally group the demand from clients to obtain a system of single node demand instances. Next, depending on the setting, we either use a dedicated routing tree on the demand nodes (for non-uniform opening cost), or we work with stars of facilities (for non-uniform capacities), to redistribute the demand that cannot be satisfied locally within the clusters.
1 Introduction

In metric location problems the input consists of a set \( C \) of clients, a set \( F \) of potential facilities (locations where a facility may potentially be built), and a metric distance function \( d \) on \( C \cup F \). The goal is to find a subset \( F' \subseteq F \) of locations for opening (building) actual facilities, and an assignment of clients to open facilities, that together minimize a certain problem-specific cost function. In the \( k \)-median setting we search for a subset \( F' \subseteq F \) of cardinality at most \( k \) and want to minimize the total cost of assigning clients in \( C \) to facilities in \( F' \), where the cost of assigning a client \( j \in C \) to a facility \( i \in F' \) equals \( d(j, i) \).

The \( k \)-median problem is a classical NP-hard problem appearing in a number of realistic optimization scenarios. Consider, for example, location of actual facilities such as voting points during elections, or power plants in an electrical grid. It also appears in the context of clustering data, where one wishes to partition objects into a fixed number of groups containing similar items.

In this paper we consider the capacitated version of the \( k \)-median problem, where we additionally require that there can be at most \( u_i \) clients assigned to facility \( i \). We focus on the version with hard capacities, where at most one facility per location may be opened, and with splittable demand where a single client may be served from more than one facility. The splittability of demands is not important in the simple case of unit demand clients and integral capacity of facilities\(^\text{1}\). The unit demand client case carries the essence of capacitated location problems with splittable demand, and hence, for the simplicity of the argument, we will only consider unit demands.

Similar to \( k \)-median is the \( k \)-center problem, where a subset of \( k \) facilities is selected but the objective is to minimize the maximal distance from a client \( j \) to the facility serving \( j \). Another related setting is the facility location problem, where instead of the strict constraint of opening at most \( k \) facilities we pay a certain cost \( f_i \) for opening a facility in location \( i \in F \). A common generalization of \( k \)-median and facility location is \( k \)-facility location, where there is both the upper bound of \( k \) on the number of open facilities, and the location specific facility opening cost. These mathematical formulations, although modeling essentially the same clustering task, behave very differently in the context of approximation.

Best understood is the \( k \)-center problem, for which Hochbaum and Shmoys\(^\text{16}\) gave a simple and best possible 2-approximation algorithm. Recently, Cygan et al.\(^\text{13}\) gave a constant-factor approximation algorithm for the capacitated version of the \( k \)-center problem. This was subsequently improved to a 9-approximation by An et al.\(^\text{6}\), which also narrows down the integrality gap of the natural LP-relaxation of the capacitated \( k \)-center problem to 7, 8, or 9.

After a long line of research, the approximability of the Uncapacitated Facility Location problem (UFL) has been nearly resolved. The 1.488-approximation algorithm of Li\(^\text{19}\) almost closed the gap with the approximability lower bound of 1.463 by Guha and Khuller\(^\text{15}\). The approximability of the capacitated variant is much less clear. We know that the soft capacitated problem admits a 2-approximation by Jain et al.\(^\text{17}\), which matches the integrality gap of the standard LP. However, the integrality gap of the standard LP for the hard capacitated facility location is unbounded and the only successful approach has been local search, which yields a 3-approximation for uniform capacities\(^\text{2}\) and a 5-approximation for general capacities\(^\text{5}\). Notably, a 5-approximation algorithm for the case with uniform facility opening costs was given by Levi et al.\(^\text{18}\), and we will partly build on their techniques in the construction of our algorithm for capacitated \( k \)-median.

Despite the simple formulation, \( k \)-median appears to be the most difficult to handle of the three problems. The first approximation for the uncapacitated \( k \)-median was the \( 6/5 \) approximation algorithm by Charikar et al.\(^\text{9}\). For a long time, the best approximation ratio of \( 3 + \varepsilon \) was obtained by a local-search method\(^\text{4}\). Recently Charikar and Li\(^\text{10}\) gave a 3.25-approximation algorithm by directly rounding the fractional solution to the standard LP. Next, Li and Svensson gave an LP-based \((1 + \sqrt{3} + \varepsilon)\)-approximation algorithm\(^\text{21}\), in which they turn a pseudo-approximation algorithm opening a few too many facilities into an algorithm opening at most \( k \) facilities. Very recently two ingredients of this algorithm were optimized

\(^{1}\)This can be shown using the integrality of the min cost flow solution which can be used to produce the assignment.
by Byrka et al. \cite{Byrka08} to obtain a 2.61-approximation algorithm for $k$-median.

Our understanding of the capacitated versions of $k$-median is less complete. For the uniform soft capacitated $k$-median already Charikar et al. \cite{Charikar03} obtained a cost increase bounded by a factor of 16 and a capacity violation bounded by a factor of 3. Later, Chuzhoy and Rabani \cite{Chuzhoy05} gave a 40-approximation algorithm with capacity violation 50 for the general soft capacitated $k$-median. There is an integrality gap example for the standard LP, where an integral solution must violate the capacities by at least a factor of $2 - \varepsilon$ in order to have connection cost within a constant of the cost of the optimal solution to the standard LP, even for soft capacities.

Recently, Gijswijt and Li \cite{GijswijtLi14} studied the capacitated $k$-facility location problem (the common generalization of facility location and $k$-median). They designed a $(7 + \varepsilon)$-approximation algorithm for the capacitated $k$-facility location problem with nonuniform capacities using at most $2k + 1$ facilities. Next, Li \cite{Li14} broke the barrier of 2 in violating the number of facilities and obtained an algorithm for uniform capacitated $k$-median, that uses only $k(1 + \varepsilon)$ facilities.

In contrast to \cite{GijswijtLi14, Li14} we focus on not violating the number of open facilities. In the early version of this paper \cite{ChekuriH07} we gave the first approximation algorithm for the uniform hard capacitated $k$-median problem. Our algorithm rounds a fractional solution to the standard LP relaxation. We show that the obtained integral solution violates the capacities by a factor of at most $(2 + \varepsilon)$ and the connection cost of the integral solution is at most $O(1/\varepsilon^2)$ factor away from the cost of the initial fractional solution. This was very recently improved by Li \cite{Li14}, who proposed a more combinatorial algorithm and has cost at most $O(1/\varepsilon)$ times the optimal cost. First, in Section 3 we show that violating capacities by a slightly bigger factor of $3 + \varepsilon$ is sufficient to obtain constant factor approximation of the connection cost also in the case of the non-uniform hard capacitated $k$-median problem. Second, in Section 4 we extend the $2 + \varepsilon$ capacity violating algorithm for $k$-median with uniform capacities to the more general $k$-facility location problem with uniform capacities, where opening facilities incurs a location specific opening cost. Both these results are built on the idea from \cite{ChekuriH07} to consider single-point-demand instances, which we exploit in Section 2.2

\section{Bundles and Star Instances}

Given a capacitated $k$-facility location instance $(\mathcal{C}, \mathcal{F}, k, d, u)$, we will partition the facilities of $\mathcal{F}$ into bundles (similar to Charikar and Li \cite{Charikar03}). For this, we first solve the following natural LP relaxation denoted by $C_k$-FL LP, where variable $y_i$ encodes the opening of facility $i$, and variable $x_{ij}$ encodes the assignment of client $j$ to facility $i$. Throughout this work, we fix an integral parameter $\ell \geq 2$ and an optimal fractional solution $(x^*, y^*)$ to $C_k$-FL LP.

\begin{align*}
\text{minimize} & \sum_{i \in F, j \in C} d(i,j)x_{ij} + \sum_{i \in F} y_if_i & \text{such that} \sum_{i \in F} y_i \leq k; \\
\sum_{i \in F} x_{ij} = 1, & \text{ for each } j \in C; & x_{ij} \leq y_i, & \text{ for each } i \in F, j \in C; \\
\sum_{j \in C} x_{ij} \leq u_i y_i, & \text{ for each } i \in F; & x_{ij}, y_i \geq 0, & \text{ for each } i \in F, j \in C;
\end{align*}

**Bundles.** Define $d_{av}(j) := \sum_{i \in F} x_{ij}^* d(i,j)$ as the connection cost of client $j$. We select a subset $\mathcal{C}' \subseteq \mathcal{C}$ of clients that are far away from each other with respect to their connection cost. Beginning with $\mathcal{C}' = \emptyset$ and $\mathcal{C}'' = \mathcal{C}$, we select a client $j \in \mathcal{C}''$ with minimum $d_{av}(j)$ (ties are broken arbitrarily) and remove it from $\mathcal{C}''$ along with every client $j'$ with $d(j,j') \leq 2\ell d_{av}(j')$, and add $j$ to $\mathcal{C}'$. We repeat this procedure until $\mathcal{C}''$ is empty. We call the clients in $\mathcal{C}'$ bundle centers.

**Lemma 2.1** The following holds:
1. For any \( j, j' \in \mathcal{C}, j \neq j' \), it holds that \( d(j, j') > 2\ell \max\{d_{uv}(j), d_{uv}(j')\} \).

2. For any \( j' \in \mathcal{C} \setminus \mathcal{C}' \), there is a \( j \in \mathcal{C}' \) with \( d(j, j') \leq 2\ell d_{uv}(j') \).

For each bundle center \( j \in \mathcal{C}' \), we define the bundle \( \mathcal{U}_j \subseteq \mathcal{F} \) as the set of all facilities that are nearest to \( j \), that is, \( \mathcal{U}_j = \{i \in \mathcal{F} \mid \forall j' \in \mathcal{C}' : j \neq j' \rightarrow d(j, i) < d(j', i)\} \). Assuming w.l.o.g. that all distances are distinct, the bundles partition all facilities in \( \mathcal{F} \).

**Definition 1** For any set of facilities \( \mathcal{F}' \subseteq \mathcal{F} \), we define its volume as \( \text{vol}(\mathcal{F}') := \sum_{i \in \mathcal{F}'} y^*_i \). Note that the volume of a bundle is at least \( 1 - \frac{1}{\ell} \); see [18].

**Star instances.** We will now introduce a notation that will help us locally modify facility openings. Following the approach of Levi et al. [18] we group the demand served by facilities of a bundle \( \mathcal{U}_j \) in the client \( j \) being a center of the bundle. The so obtained single-demand-node instances we call star instances.

A **star instance** (or short star) is a tuple \( S_j = (j, \mathcal{F}_j, w_j, b_j) \) consisting of a star center \( j \in \mathcal{C}' \), a set of facilities \( \mathcal{F}_j = \mathcal{U}_j \), a demand \( w_j = \sum_{i \in \mathcal{F}_j} \sum_{j' \in \mathcal{C}} x^*_i,j' \geq 0 \) and a budget \( b_j \geq 0 \). A (fractional) solution to the star instance is an opening vector \( z \) for the facilities in \( \mathcal{F}_j \) that satisfies the following constraints.

\[
\sum_{i \in \mathcal{F}_j} u_i z_i \geq w_j \quad (1)
\]

\[
\sum_{i \in \mathcal{F}_j} (f_i + d(i, j)u_i)z_i \leq b_j \quad (2)
\]

\[
0 \leq z_i \leq 1 \quad \text{for each } i \in \mathcal{F}_j \quad (3)
\]

We set \( b_j = b_j^f + b_j^c \) where \( b_j^f := \sum_{i \in \mathcal{F}_j} y^*_i f_i \) is related to the opening cost of the facilities in \( \mathcal{F}_j \) and \( b_j^c := \sum_{i \in \mathcal{F}_j} \sum_{j' \in \mathcal{C}} x^*_i,j'(d(i, j') + 2\ell d_{uv}(j')) \) is related to the connection cost of the demand served by these facilities. We can show the following for the total budget of all star instances.

**Lemma 2.2** The total budget of all star instances is bounded by \( (1 + 2\ell) \text{OPT} \).

We call a star \( S_j \) small if the volume of \( \mathcal{F}_j \) is smaller than one, and big otherwise. We define the volume of a solution \( z \) to a star instance \( S_j \) as \( \text{vol}(z) := \sum_{i \in \mathcal{F}_j} z_i \). A solution is called almost integral if \( z_i \) is fractional for at most one \( i \in \mathcal{F}_j \).

**Lemma 2.3** For any solution \( z \) to a star instance, we can construct a solution \( z' \) which has at most two fractional variables and \( \text{vol}(z') \leq \text{vol}(z) \).

**Proof:** Consider the LP for the star instance with Constraints (1)-(3) and the objective of minimizing the volume \( \sum_{i \in \mathcal{F}_j} z_i \). Clearly \( z \) is a feasible solution to this LP with objective \( \text{vol}(z) \). Now consider an optimal extreme point solution \( z' \) to this LP. Of course \( \text{vol}(z') \leq \text{vol}(z) \) is satisfied. Let \( l \) be the number of positive variables in \( z' \). Since \( z' \) is an extreme point solution, at least \( l \) many of the Constraints (1)-(3) are tight. This means that at least \( l - 2 \) of the Constraints (3) are tight.

\[\square\]

**Lemma 2.4** For any solution \( z \) to a star instance with uniform capacities, we can construct an almost integral solution \( z' \) with \( \text{vol}(z') = \text{vol}(z) \).

**Proof:** (Sketch) Take the volume of \( z \) and transfer it greedily to the facility \( i \) which minimizes \( d(i, j)u + f_i \) among all facilities which are not yet fully open. \[\square\]
Lemma 2.5 For any star instance \( S_{j'} \) we can compute a solution \( z \) such that \( \text{vol}(z) \leq \text{vol}(F_{j'}) \).

**Proof:** (Sketch) The demand \( w_{j'} \) is given by \( \sum_{i \in F_{j'}} \sum_{j \in C} x_{ij}^* \). For each facility \( i \in F_{j'} \) we route \( \sum_{j \in C} x_{ij}^* \) units of demand to \( i \) by opening it by an amount of \( z_i = \left( \sum_{j \in C} x_{ij}^* \right) / u_i \leq y_i^* \). Then Constraints (1) and (3) are satisfied by the solution and \( \text{vol}(z) \leq \text{vol}(F_{j'}) \).

Note that the solution provided by Lemma 2.5 may have a volume strictly smaller than that of the underlying bundle and therefore also smaller than \( 1 - 1/\ell \). For technical reasons, we prove the following.

Lemma 2.6 It is possible to distribute the demand of the clients among the star centers such that each star center \( j' \in C' \) receives precisely \( w_{j'} \) units of demand and such that the total transportation cost is at most \( (2\ell + 2) \text{OPT} \).

3 Algorithm for Non-uniform Hard Capacitated \( k \)-Median

In this section, we describe a bi-criteria approximation algorithm for the \( k \)-median problem with non-uniform hard capacities. Note that this problem is a special case of Ck-FL if we set all opening costs \( f_i \) to zero. In the following we call its relaxed linear program Ck-MED LP.

During our algorithm, we will obtain step by step a series of solutions where the openings are more and more restricted until we finally arrive at an integral solution to Ck-MED LP.

**Definition 2** If \( (\tilde{x}, \tilde{y}) \) satisfies the restricted version of Ck-MED LP where we drop the constraint \( x_{ij}, y_i \geq 0 \) for every \( i \in F, j \in C \), then we call \( (\tilde{x}, \tilde{y}) \) a restricted solution.

**Definition 3** A solution \( (\tilde{x}, \tilde{y}) \) is a \( [1 - \frac{1}{\ell}, 1] \)-solution if for each \( i \in F \) we have \( y_i \in \{ 0 \} \cup [1 - \frac{1}{\ell}, 1] \). Similarly, a restricted solution \( (\tilde{x}, \tilde{y}) \) is a \( \{ 1 - \frac{1}{\ell}, 1 \} \)-restricted solution if for each \( i \in F \) we have \( y_i \in \{ 0, 1 - \frac{1}{\ell}, 1 \} \).

Let \( (x^*, y^*) \) be the optimum solution to Ck-MED LP that we fixed in Section 2. As described in Section 2.1 we partition all facilities into bundles and for each corresponding star instance we compute a solution of bounded volume as in Lemma 2.5.

In Section 3.1 we modify the solution to each star instance by rerouting demands and moving openings between facilities such that either there is only one open facility or all open facilities are integrally open. The union of these solutions help us to obtain a \([1 - \frac{1}{\ell}, 1]\)-solution \((x', y')\) where fractionally open facilities have capacity violation at most \( 1 + \varepsilon \) and integrally open facilities have capacity violation at most \( 2 + \varepsilon \).

Then, in Section 3.2 we construct a \( \{ 1 - \frac{1}{\ell}, 1 \} \)-restricted solution \((\hat{x}, \hat{y})\): By some greedy rule, we either round each opening in \( y' \) down to \( 1 - \frac{1}{\ell} \) or up to 1. Note that by this we might violate the constraints \( x'_{ij} \leq y'_i \) of Ck-MED LP. The connection cost of \((\hat{x}, \hat{y})\) remains the same as in \((x', y')\), however, the bound on capacity violation increases. Fractionally and integrally open facility has now capacity violation at most \((1 + \varepsilon) \frac{\ell}{\ell - 1} \) and \( 2 + \varepsilon \), respectively.

Finally, in Section 3.3 we round \((\hat{x}, \hat{y})\) into an integral solution. We do this by building so called facility-trees and cut them to the smaller instances which are easier to round. By this we obtain an integral solution \((\hat{x}, \hat{y})\) to Ck-MED LP with capacity violation at most \( 3 + 3\varepsilon \).

3.1 Obtaining a \([1 - \frac{1}{\ell}, 1]\)-Solution

In this section, we describe how to obtain a solution \((x', y')\) such that \( y'_i \in [1 - \frac{1}{\ell}, 1] \) for each facility \( i \) with positive opening \( y'_i \). Let \( \varepsilon > 0 \) be an arbitrary small positive number.

In the following we consider a star instance \( S_j = (j, F_j, w_j, b_j) \). By Lemmas 2.5 and 2.3 we compute a solution \( z \) to \( S_j \) with at most two fractionally open facilities and \( \text{vol}(z) \leq \text{vol}(F_j) \). We can distribute the total demand \( w_j \) on the facilities in \( F_j \) such that each facility \( i \) serves a demand \( d_i \) of volume at
most \( z_iu_i \). We fix such a distribution of \( w_j \) and define \( d_i \) as the demand that facility \( i \) has to serve. (Thus \( \sum_{i \in F_j} d_i = w_j \).) Note that in our distribution there is no capacity violation and the cost of moving \( d_i \) to facility \( i \) is bounded by \( d(i, j)d_i \leq d(i, j)z_iu_i \). Hence, by Constraint 2 of star instance, the connection cost of our distribution, that is, the cost of sending \( w_j \) from star center \( j \) to the facilities, is bounded by \( b_j \).

**Lemma 3.1** We can compute an opening vector \( z' \) for \( F_j \) of volume at most \( \forall \ell \) such that either there is only one open facility \( i \) and \( z'_i = \text{vol}(F_j) \), or all open facilities are integrally open. Further, we can distribute the demand \( w_j \) on facilities open in \( z' \) such that each fractionally open facility \( i \) serves a demand \( d_i' \leq (1 + \varepsilon)z'_i u_i \), and each integrally open facility \( i \) serves a demand \( d_i' \leq (2 + \varepsilon)z'_i u_i \). The connection cost of the distribution is bounded by \( (2 + 4/\varepsilon)b_j \).

**Proof:** (A more detailed proof is in Appendix 1). If we have no fractional facilities, we are done, and if we have only one open facility, we just increase its opening \( z' \) to \( \min\{1, \text{vol}(F_j)\} \) and we are done, too.

If the total volume of the two facilities with smallest openings is greater equal to one, then we choose the one with higher demand to be fully opened in \( z' \) and close the other one in \( z' \). The demand of the chosen facility increases by a factor of at most 2. Thus, the resulting increase in capacity violation and connection cost is also a factor of at most 2.

If the total volume of the two facilities with smallest openings is smaller than one, then one can show that we can always move the opening and demand from one facility to the other such that the resulting increase in capacity violation and connection cost is a factor of at most \( 1 + \varepsilon \) and of at most \( (1 + \varepsilon')/\varepsilon' \), respectively. If there are no other open facilities, we increase the opening of the chosen facility \( i' \) to \( \min\{1, \text{vol}(F_j)\} \). Otherwise there is a fully open facility \( i \) in the star, since we have at most two fractionally open facilities in \( z \). As the total volume of \( i \) and \( i' \) is greater one, we can use the same argument as above to close one of them and to fully open the other one. The opened facility serves the demand of the now two closed facilities. In the worst case, we opened \( i' \) again and have a total increase in capacity violation and connection cost of a factor of at most \( 2(1 + \varepsilon') \) and \( 2(1 + \varepsilon')/\varepsilon' \), respectively. Since we did not change the other facilities, their connection costs and demands do not change. Setting \( \varepsilon' = \varepsilon/2 \) in the above bounds proves the claim. □

**Corollary 3.2** For any \( \varepsilon > 0 \), we can efficiently compute a \( [1 - 1/k, 1] \)-solution \((x', y')\) such that \( \text{vol}(y') \leq k \), fractionally open facilities have capacity violation at most \( 1 + \varepsilon \), integrally open facilities have capacity violation at most \( 2 + \varepsilon \), and the total connection cost is at most \( (2 + 4/\varepsilon)(1 + 2\ell) \text{OPT} + (2 + 2\ell) \text{OPT} \).

**Proof:** For each star instance, we compute an opening vector as by Lemma 3.1. Let \( y' \) be the union of all these opening vectors. Thus, \( y' \) is \( (1 - \varepsilon/2) \)-restricted. Note that \( \text{vol}(y') \leq \sum_{i \in C'} \text{vol}(F_j) \leq \sum_{i \in F} y'_i \leq k \), where the second last inequality follows from the fact that each facility belongs to exactly one star instance.

To construct a feasible \( x' \) for Ck-MED LP, consider any facility \( i \). Let \( S_i \) be the star to which \( i \) belongs to and let \( d'_i \) be its demand in Lemma 3.1 Then consider any client \( j' \). Let \( x_{ij'}' = \sum_{j \in F_j} x_{ijj'}' \) be total demand of \( j' \) that is served by \( S_i \). We send a fraction of \( d'_i/w_j \) of \( x_{ij}' \) to \( i \), i.e. \( x_{ij}' = d'_i/w_j \cdot x_{ij}' \).

Now we show that the constraint \( x_{ijj'}' \leq y_{ij}' \) of Ck-MED LP is satisfied. If \( y'_i = 1 \), then the constraint holds. Otherwise we have by Lemma 3.1 that

\[
y'_i = \text{vol}(F_j) = \sum_{i \in F_j} z_i \geq \sum_{i \in F_j} x_{ijj'}' = x_{ijj'}' \geq x_{ijj'}'.
\]

If we define \( x' \) in the same way for every client, \( i \) serves a total amount of \( \sum_{j \in C} x_{ijj'}' = d'_i/w_j \sum_{j \in C} x_{ijj'}' = d'_i \). Thus the bounds on capacity violation of Lemma 3.1 still hold. On the other side, each client \( j' \) is fully served, since \( \sum_{i \in F_j} x_{ijj'}' = \sum_{j \in C} x_{ijj'} \), \( \sum_{i \in F_j} d_i/w_j = \sum_{j \in C} x_{ijj'} = 1 \). Thus, \((x', y')\) is a feasible solution.

Regarding the connection cost of \( x' \), we can assume that we first move the demands of the clients to the star centers and then move them to the open facilities. The cost for the first step is bounded by \( (2 + 2\ell) \text{OPT} \) (Lemma 2.6). By Lemma 3.1 we can bound the cost for the second step by \( (2 + 4/\varepsilon)b_j \) for each star center \( j \). Since we have that the total budget is bounded by \( (1 + 2\ell) \text{OPT} \) (Lemma 2.2), the total cost is at most \( (2 + 4/\varepsilon)(1 + 2\ell) \text{OPT} + (2 + 2\ell) \text{OPT} \). □
3.2 Computing a \( \{1 - \frac{1}{\ell}, 1\} \)-restricted solution

Let \((x', y')\) be a \( \{1 - \frac{1}{\ell}, 1\} \)-solution obtained by Corollary 3.2. We will now transform it into a \( \{1 - \frac{1}{\ell}, 1\} \)-restricted solution \((\hat{x}, \hat{y})\). We define \( N_1 := \{ i \in F \mid y'_i = 1 \}\) and \( N_2 := \{ i \in F \mid 1 - \frac{1}{\ell} < y'_i < 1 \}\). Similarly we define \( \hat{N}_1 := \{ i \in F \mid \hat{y}_i = 1 \}\) and \( \hat{N}_2 := \{ i \in F \mid \hat{y}_i = 1 - \frac{1}{\ell} \}\). For each facility \( i \in N_2 \), let \( d'_i := \sum_{j \in C} x'_{ij} \) be the demand served by \( i \) and let \( s(i) \) be its closest facility in \( N_1 \cup N_2 \setminus \{ i \}\) (breaking ties arbitrary).

**Lemma 3.3** We can efficiently compute a \( \{1 - \frac{1}{\ell}, 1\} \)-restricted solution \((\hat{x}, \hat{y})\) of volume \( \text{vol}(\hat{y}) = k \) and connection cost at most \((2 + 4/\varepsilon)(1 + 2\ell)\)OPT +\((2 + 2\ell)\)OPT where the capacity violation of a facility is at most \( 2 + \varepsilon \) if it is in \( \hat{N}_1 \) and at most \( (1 + \varepsilon) \cdot \ell/(\ell - 1) \) if it is in \( \hat{N}_2 \). The following holds:

\[
\sum_{i \in N_2} d'_i (1 - \hat{y}_i) d(s(i), i) \leq \sum_{i \in N_2} d'_i (1 - y'_i) d(s(i), i) .
\]

**Proof:** For each facility \( i \in N_1 \) set \( \hat{y}_i = 1 \). Then sort all facilities in \( N_2 \) non-increasingly by \( d'_i d(s(i), i) \). For the first \( k\ell - |N_1|\ell - |N_2| (\ell - 1) \) facilities set \( \hat{y}_i = 1 \) and for the rest set \( \hat{y}_i = 1 - 1/\ell \). Note that by this we have exactly \( \text{vol}(\hat{y}) = k \geq \text{vol}(y') \). Given \( y'_i \geq 1 - 1/\ell \) for all \( i \in N_2 \) we have \( \sum_{i \in N_2} d'_i y'_i d(s(i), i) \leq \sum_{i \in N_2} d'_i \hat{y}_i d(s(i), i) \).

We set \( \hat{x} = x' \) and have thus the same cost as \( x' \). Since the openings of the facilities in \( \hat{N}_2 \) have decreased by a factor of at most \((\ell - 1)/\ell \), we can bound their capacity violation by \( \ell/(\ell - 1) \cdot (1 + \varepsilon) \).

The capacity violation of other facilities did not change. Integrally open facilities have in worst case still capacity violation \( \max\{1 + \varepsilon, 2 + \varepsilon\} = 2 + \varepsilon \).

\( \square \)

3.3 Rounding \( \{1 - \frac{1}{\ell}, 1\} \)-restricted solution \( \hat{y} \) to integral solution \( \bar{y} \).

Recall that \((\hat{x}, \hat{y})\) is a \( \{1 - \frac{1}{\ell}, 1\} \)-restricted solution obtained by Lemma 3.3. In this section, we describe how to round this solution to an integral solution \((\bar{x}, \bar{y})\). For the sake of easier presentation, we assume that the demand of the clients is moved to the facilities via solution \((\hat{x}, \hat{y})\) so that facility \( i \) carries demand \( d'_i \). We will describe how to obtain an integral opening vector \( \bar{y} \) and how to reroute the demand from the facilities opened by \( \bar{y} \) to the facilities opened by \( \hat{y} \). We will give an upper bound of \( 3 + 3\varepsilon \) on the capacity violation and analyze the cost of rerouting. Altogether, this leads to the solution \((\bar{x}, \bar{y})\) for the original instance (where the demand is in the clients) with capacity violation \( 3 + 3\varepsilon \). The cost of this solution is the cost of \((\hat{x}, \hat{y})\) plus the cost of rerouting.

**Building facility-trees.** We construct the set of facility-trees spanning facilities in \( \hat{N}_2 \) as in 3.2. Recall that \( s(i) \) is the closest facility to \( i \) in \( \hat{N}_1 \cup \hat{N}_2 \setminus \{ i \} \). We draw a directed edge from \( i \in \hat{N}_2 \) to \( s(i) \). To eliminate cycles we choose any node from a cycle as the root of the respective facility-tree and delete the edge emanating from this root. If there is an edge from \( i \) to \( s(i) \) we call \( s(i) \) a parent of \( i \).

**Decomposing facility-trees to rooted facility stars.** We cut each facility-tree into facility stars consisting of a root and a group of leaves. To this end, we greedily choose the leaf node \( i \) that has the biggest number of edges on the path to the root of the tree. Then we remove the subtree rooted at \( s(i) \). See pseudo-code in Appendix D.

**Rounding facility stars.** Each leaf of a facility star belongs to \( \hat{N}_2 \). From Lemma 3.3 we know that clients from \( \hat{N}_1 \) and \( \hat{N}_2 \) have capacity violation at most \( 2 + \varepsilon \) and \((1 + \varepsilon) \ell/(\ell - 1)\), respectively.

Let \( Q_t \) denote the facility star rooted at node \( t \). For each facility \( i \in \hat{N}_1 \setminus \bigcup_t Q_t \) we set \( \hat{y}_i = 1 \). Demand \( d'_i \) is served by \( i \) itself. The capacity violation of this facility is at most \( 2 + \varepsilon \).

\( ^2 \)Recall that \( \ell \) is an integer.
Definition 4 Facility star \( Q_t \) is called even facility star if \( |Q_t \cap \hat{N}_2| = 2q \) for \( q \in \mathbb{N} \setminus \{0\} \) and odd facility star otherwise.

Consider facility star \( Q_t \). We can treat the root \( t \) as a leaf of \( Q_t \) whose distance to the root is equal to zero. The below procedure Round\((Q_t)\) opens at most \( \left| \sum_{i \in Q_t} \hat{y}_i \right| \) facilities in \( Q_t \).

Case 1: Even facility star. Let \( i_1, i_2, \ldots, i_{2q} \) be a sequence of all fractionally open facilities in \( Q_t \) in non-decreasing order of the demand. For \( l = 1, \ldots, q \), open facility \( i_{2l} \) and reroute demands of facility \( i_{2l-1} \) and \( i_{2l} \) to it. The capacity violation of each open facility is at most \( 2 \cdot (1 + \varepsilon) \cdot \frac{L_t}{L_t - 1} \cdot \frac{L_t}{L_t - 1} = 2 + 2\varepsilon \).

Case 2: Odd facility star with \( \hat{y}_t = 1 - \frac{1}{t} \). Let \( i_1, i_2, \ldots, i_{2q+1} \) be a sequence of all fractionally open facilities in \( Q_t \) in non-decreasing order of demand volume. Open facility in each \( i_{2l+1} \), for \( l = 1, \ldots, q \) and reroute demands of facility \( i_{2l} \) and \( i_{2l+1} \) to it. Moreover reroute the demand of \( i_1 \) to \( i_3 \). The capacity violation of each open facility, except \( i_3 \), is at most \( 2 \cdot (1 + \varepsilon) \cdot \frac{L_t}{L_t - 1} \cdot \frac{L_t}{L_t - 1} = 2 + 2\varepsilon \). Facility \( i_3 \) has capacity violation equal to \( 3 + 3\varepsilon \).

Case 3: An odd facility star with \( \hat{y}_t = 1 \). Let \( i_1, i_2, \ldots, i_{2q+1} \) be a sequence of all leafs in \( Q_t \) in non-decreasing order of demand volume. Open facility in each \( i_{2l+1} \), for \( l = 1, \ldots, q \) and reroute demands of facility \( i_{2l} \) and \( i_{2l+1} \) to it. The capacity violation of each open facility, except \( i_1 \) and \( t \), is at most \( 2 \cdot (1 + \varepsilon) \cdot \frac{L_t}{L_t - 1} \cdot \frac{L_t}{L_t - 1} = 2 + 2\varepsilon \). If \( d_t^l \geq d_i^l \) then open facility in \( i \) and move the demand of \( i_t \) to \( i_1 \). Otherwise we have \( d_t^l < 2d_i^l \). Then open facility in \( i_t \) and move the demand of \( t \) to \( i_1 \). The capacity violation is \( \max\{(2 + \varepsilon) \cdot \frac{L_t}{L_t - 1}, (1 + \varepsilon) \cdot 3\} \leq 3 + 3\varepsilon \).

Lemma 3.4 In the integral solution \( \bar{y} \) at most \( k \) facilities are open. The biggest capacity violation in solution \( \bar{y} \) is \( 3 + 3\varepsilon \).

Lemma 3.5 The cost of rounding fractional solution \( (\bar{x}, \bar{y}) \) to integral solution \( (\bar{x}, \bar{y}) \) can be bounded by \( 2\sum_{i \in \hat{N}_2} d_i^l d(s(i), i) \).

Proof: In all cases of procedure Round\((Q_t)\) we send demand of volume at most \( 2d_i^l \) along the edge from \( i \) to \( s(i) \). Summing over all nodes in \( \hat{N}_2 \) we get the lemma. \( \square \)

Lemma 3.6 The total cost of facility stars \( \sum_{i \in \hat{N}_2} d_i^l d(s(i), i) \) is bounded by \( 2\ell(2 + 4/\varepsilon)(1 + 2\ell) \text{OPT} (2 + 2\ell) \text{OPT} \).

Theorem 3.7 Our algorithm is \( O(\frac{1}{\varepsilon^2}) \), \( 3 + O(\varepsilon) \)-approximation algorithm for non-uniform hard capacitated \( k \)-median problem. Especially for \( \ell = 2 \) we get a \( (96 + 180/\varepsilon, 3 + 3\varepsilon) \)-approximation algorithm.

Proof: From Lemma 3.3 we know that the cost of solution \( (\bar{x}, \bar{y}) \) is at most \( (2 + 4/\varepsilon)(1 + 2\ell) \text{OPT} (2 + 2\ell) \text{OPT} \). Moreover, using Lemmas 3.5 and 3.6, we can bound the cost of rounding \( (\bar{x}, \bar{y}) \) to \( (\bar{x}, \bar{y}) \) by \( 4\ell((2 + 4/\varepsilon)(1 + 2\ell) (2 + 2\ell)) \text{OPT} \). Summing this up we get \( (4\ell + 1)((2 + 4/\varepsilon)(1 + 2\ell) + (2 + 2\ell)) \text{OPT} \). \( \square \)

4 Algorithm for Uniform Hard Capacitated \( k \)-Facility Location

Theorem 4.1 For any \( \ell \geq 2 \), there is a factor \( (32\ell^2 + 28\ell + 7, 2 + \frac{3}{\ell - 1}) \)-approximation algorithm for \( k \)-facility location with uniform hard capacities.

Due to lack of space we present a weaker result with factor 6 capacity violation and constant approximation of the cost. The \( 2 + \varepsilon \) capacity violation algorithm builds on the same Star trees construction, but requires move involved rounding procedure which is given in Appendix A.
4.1 Star trees

We now describe a structure called star tree which can be derived from a solution to the LP relaxation of (uniformly) capacitated k-facility location. We show that in order to obtain a bi-factor approximation algorithm for capacitated k-facility location it suffices to appropriately “round” a star tree.

**Definition 5** Let $C_a$ be a subset of clients, $F_s$ be a set of facilities, $d_a$ be a metric on $C_s \cup F_s$, and $u$ a positive integer. A star tree is an $r$-rooted in-tree $T = (C_a, E)$ that satisfies the following properties:

(i) Each $j \in C_a$ is associated with a star instance $S_j = (j, F_j, w_j, b_j)$ with $F_j \subseteq F_s$ and an almost integral solution $z_j$ to this instance.

(ii) The family $\{F_j \mid j \in C_a\}$ partitions $F_s$.

(iii) For any $j \in C_a$, the volume of solution $z_j$ is at least $1 - 1/\ell$.

(iv) Let $j' \neq r$ be a client whose star instance $S_{j'}$ has only one facility $i$ with positive opening $z_i$. Let $(j', j'') \in E$ be the outgoing arc of $j'$. Then we have that $(1 - z_i) d_a(j', j'') \leq 16 b_{j'}$.

(v) Each $j \in C_a$ has in-degree $\deg^-(j) \leq 2$ and root $r$ has in-degree $\deg^-(r) \leq 1$.

(vi) For consecutive edges $(j, j'), (j', j'')$ we have that $d_a(j, j') \geq d_a(j', j'')$.

The budget $b(T)$ of the star tree $T$ is $b^c(T) + b^f(T)$, where $b^c(T) = \sum_{j \in C_a} b_j^c$ and $b^f(T) = \sum_{j \in C_a} b_j^f$. The volume $\text{vol}(T)$ of the star tree is given by $\sum_{j \in C_a} \text{vol}(z_j)$.

Consider a star tree $T$. A solution to $T$ with capacity violation $\gamma$ is a set $F' \subseteq F_s$ and an assignment $(z_{ij})_{i \in F', j \in C_a}$ satisfying the following constraints:

$$\sum_{i \in F'} z_{ij} \geq w_j \quad \text{for each } j \in C_a$$

$$\sum_{j \in C_a} z_{ij} \leq \gamma u \quad \text{for each } i \in F'$$

$$z_{ij} \geq 0 \quad \text{for each } i \in F', j \in C_a.$$

The cost of the solution is given by $\sum_{i \in F'} f_i + \sum_{i \in F'} \sum_{j \in C_a} z_{ij} d_a(i, j)$. A star forest $H$ is a collection of disjoint star trees. Budget, cost and volume of a star forest are given by the sum of budgets, costs and volumes of its star trees, respectively. A solution to a star forest provides a solution to each of its star trees and additionally satisfies that the total number of open facilities is no more than $\text{vol}(H)$.

**Short Center Trees.** Below, we build a directed forest $G$ with node set $C'$. Here $C'$ is the set of bundle centers as described in Section 2. The connected components of this forest are in-trees, which we call short center trees. Procedure Short-Trees($C'$) adds an edge from each $j \in C'$ to the node $j' \in C' \setminus \{j\}$, which is closest to $j$ (recall that we assumed all distances to be distinct). Afterwards, it removes one edge from each cycle of length two. The pseudo-code of the procedure can be found in Appendix C.

**Lemma 4.2** Let $i, j, l \in C'$. If $(j, i), (i, l) \in E(G)$ then $d(j, i) > d(i, l)$.

**Lemma 4.3** The graph returned by procedure Short-Trees($C'$) is a forest of in-trees.

**Definition 6** If $(i, j)$ is an edge in $G$ then we call $i$ a son of $j$, and $j$ a father of $i$. Moreover, any node with out-degree zero is called a root.
Creating Star Trees. The in-degrees of nodes in short center trees may be unbounded. Therefore, we change the structure of each short center tree to obtain a binary center tree, in which the in-degree of each node is at most two. Figure 1 depicts this process. We associate each node \( j \) of the binary center tree with the star instance \( S_j \) (see next paragraph). By showing that all properties of Definition 5 are fulfilled, we prove that the constructed binary center tree is a star tree.

Consider a procedure Binary-Trees\((G)\) (for the pseudo-code see Appendix [G]) that takes as input the forest \( G \) of short center trees returned by Short-Trees\((C')\). Each short center tree \( T \) in \( G \) is separately modified as follows. For each node \( i \in V(T) \), we sort all incoming edges of \( i \) by non-decreasing length and remove all of them except the shortest one. In the next step, the procedure adds for each \( j \) (son of \( i \)) an edge from \( j \) to its left brother (if there exists one). Note that no edge is added to the leftmost son of \( i \). The resulting forest of binary center trees is denoted by \( H \).

For technical reasons, we will use a new metric \( d_s \) on the binary center trees. More specifically, consider a node \( j \) in some short center tree \( T \) with father \( i \). Let \( i' \) be the father of \( j \) in the star tree \( T' \) arising from \( T \). Then we set \( d_s(j,i') := 2d(j,i) \). The distances within star instances associated with the nodes of the binary center tree are as in the original \( k \)-facility location instance.

**Lemma 4.4** Let \( T' \) be a binary center tree. Then \( \deg^- (r) \leq 1 \) for root \( r \), \( \deg^- (j) \leq 2 \) for any node \( j \) and \( d(j,i') \leq d_s(j,i') \) for any edge \((j,i')\), hence \( T' \) has Property (i). Moreover, \( T' \) satisfies (ii) and (iii).

**Star Instances.** For each \( j' \in C' \) the associated star instance \( S_{j'} = (j', F_{j'}, w_{j'}, b_{j'}) \) is constructed as described in Section 2. As the facility set \( F_{j'} \) is the bundle \( U_{j'} \) of \( j' \) we can conclude that Property (iii) is satisfied. By Lemma 2.3 and the fact that \( \text{vol}(F_{j'}) \geq 1 - \frac{1}{4} \), Properties (i) and (iii) are satisfied, respectively.

We have now shown that all properties for star trees as required in Definition 5 are actually satisfied. The following corollary is a consequence of properties (iv) and (v) of star trees.

**Corollary 4.5** Let \( j' \neq r \) be a node in a star tree \( T' \) such that the star instance \( S_{j'} \) has only one facility \( i \) with positive opening \( z_i \). Let \( j'' \) be a node such that there is a \( j' - j'' \) path consisting of \( h \) edges. Then we have that \((1 - z_i)d_s(j',j'') \leq 16hb_{j'}^c\).

**Theorem 4.6** Assume there is an efficient algorithm that computes for a given star forest \( H \) a solution of cost at most \( c \cdot b(H) \) for some constant \( c > 0 \) with capacity violation \( \gamma \). Then there is a \((2\ell + 2 + c (2\ell + 1))\)-approximation algorithm for capacitated \( k \)-facility location with capacity violation \( \gamma \).

### 4.2 An (O(1),6)-Approximation Algorithm

The algorithm creates a matching on the nodes of a star forest. Then it opens at least one facility in each matched pair of stars associated with a matched node pair. The last step is to establish connections between clients and open facilities, which allows us to analyze connection cost and upper-bound the capacity violation. In this algorithm we set \( \ell \) to 2.

**Matching on a star tree.** We construct a matching for each star tree \( T \). To obtain the matching we need to remove all nodes \( j \) (along with incident edges) that are centers of big stars. This operation can split \( T \) to smaller trees \( T_1, \ldots, T_s \) for which we compute matchings separately. We use the following notation: \( l(j) \) and \( r(j) \) gives the left and right son of \( j \), respectively, or NULL if there is no such son. If \( j \) has only one son we call it left son of \( j \).
The procedure Make-Matching\((j)\) works as follows: Node \(j\) (the argument) is matched with its left son. If \(j\) has no son it is not matched. Next the procedure makes a recursive call on \(r(j)\). If both sons of \(l(j)\) are leafs of the tree, we match them, otherwise we do a recursive call on each of them. We run procedure Make-Matching\((j)\) on the root of each tree \(T_1, \ldots, T_s\). The procedure adds edges to the (initially empty) set \(M(T)\) which forms a matching on \(T\). The pseudo-code of the procedure can be found in Appendix G.

**Randomized facility opening and routing.**

For each big star we close the fractionally open facility (if such a facility exists) and reroute all its demand to some other facility in this star, which is open. This step causes a violation of two in capacity for big stars. The small stars are handled by means of dependent rounding (see Appendix A). We open a facility in a small star with probability equal to its volume. By Property (iii) and \(\ell = 2\), the total volume of each pair in the matching is at least one. Applying dependent rounding we achieve that at least one facility is opened in each matched pair. We then apply dependent rounding to all unmatched small stars that have fractional volume.

**Lemma 4.7** The number of facilities opened by the above procedure is at most \(k\). Expected cost of opened facilities is bounded by \(\sum_{i \in \mathcal{F}} y_i f_i\).

**Lemma 4.8** There is an assignment of the demand to open facilities such that the capacity violation is at most six. The expected cost of this assignment is bounded by \(38 \cdot b^c(H)\) where \(H\) is the star forest.

**Proof:** (Sketch) It is not hard to give an assignment of the demand to open facilities such that for any closed small star its demand is routed to its father, left brother or grandfather. The demand of any big star is served within this star. Moreover, the assignment can be constructed so that the worst case capacity violation is at most 6. The worst case scenario for the capacity and also the assignment is depicted in Figure 2. A complete exposition can be found in the appendix. □

Combining this result with Theorem 4.6 we obtain.

**Theorem 4.9** There is an \((196, 6)\)-approximation algorithm for \(k\)-facility location with uniform capacities.

5 Concluding Remarks and Open Questions

In our algorithm for \(k\)-facility location with \(2 + \varepsilon\) capacity violation we could use one extra facility per group instead of violating the capacities. Suppose the star forest would contain only trees of size \(\Omega(l)\) we could partition the star trees into \(O(k)\) separate groups, which would imply an algorithm using \(k(1 + \varepsilon)\) facilities with no capacity violation. However, there exist instances leading to star trees of size \(O(1)\), for which this approach can only give \(2 + \varepsilon\) violation of the number of facilities.

We show algorithms for either non-uniform capacities or non-uniform facility opening cost. It remains open to construct an algorithm for non-uniform capacitated \(k\)-facility location problem, which is a common generalization of the above two settings.
References


Appendix

A The Dependent Rounding Approach

The dependent rounding approach of Gandhi et al. [14] iteratively rounds a given vector \( y = (y_1, y_2, \ldots, y_N) \in [0, 1]^N \) until all components are in \( \{0, 1\} \). It works as follows. Suppose the current version of the rounded vector is \( v = (v_1, v_2, \ldots, v_N) \in [0, 1]^N \); \( v \) is initially \( y \). When we describe the random choice made in a step below, this choice is made independently of all such choices made thus far. If each \( v_i \) lies in \( \{0, 1\} \), we are done, so let us assume that there is at least one \( v_i \in (0, 1) \). The first (simple) case is that there is exactly one \( v_i \) that lies in \( (0, 1) \); we round \( v_i \) in the natural way – to 1 with probability \( v_i \), and to 0 with complementary probability of \( 1 - v_i \); letting \( V_i \) denote the rounded version of \( v_i \), we note that

\[
\mathbb{E}[V_i] = v_i.
\]

This simple step is called a Type I iteration, and it completes the rounding process. The remaining case is that of a Type II iteration: there are at least two components of \( v \) that lie in \( (0, 1) \). In this case we choose two such components, \( v_i \) and \( v_j \), in an arbitrary manner. Let \( \varepsilon \) and \( \delta \) be the positive constants such that: (i) \( v_i + \varepsilon \) and \( v_j - \varepsilon \) lie in \( [0, 1] \), with at least one of these two quantities lying in \( \{0, 1\} \), and (ii) \( v_i - \delta \) and \( v_j + \delta \) lie in \( [0, 1] \), with at least one of these two quantities lying in \( \{0, 1\} \). It is easily seen that such strictly-positive \( \varepsilon \) and \( \delta \) exist and can be easily computed. We then update \((v_i, v_j)\) to a random pair \((V_i, V_j)\) as follows:

- with probability \( \delta / (\varepsilon + \delta) \), set \((V_i, V_j) := (v_i + \varepsilon, v_j - \varepsilon)\);
- with the complementary probability of \( \varepsilon / (\varepsilon + \delta) \), set \((V_i, V_j) := (v_i - \delta, v_j + \delta)\).

The main properties of Type II iteration that we need are:

\[
\Pr[V_i + V_j = v_i + v_j] = 1; \quad \mathbb{E}[V_i] = v_i \quad \text{and} \quad \mathbb{E}[V_j] = v_j; \quad \mathbb{E}[V_i V_j] \leq v_i v_j.
\]

We iterate the above iteration until all we get a rounded vector. Since each iteration rounds at least one additional variable, we need at most \( N \) iterations.

Note that the above description does not specify the order in which the elements are rounded. Observe that we may use a predefined laminar family of subsets to guide the rounding procedure. That is, we may first apply Type II iterations to elements of the smallest subsets, then continue applying Type II iterations for smallest subsets among those still containing more than one fractional entry, and eventually round the at most one remaining fractional entry with a Type I iteration. One may easily verify that executing the dependent rounding procedure in this manner, we almost preserve the sum of entries within each of the subsets of our laminar family.

In our \((2 + \varepsilon)\)-violation algorithm, inside each of the groups we first round the topmost two fractional entries on a bottom-up directed path of the group. This guarantees that in any suffix of such a path of length \( \alpha \geq 2 \), we will always have enough open facilities to serve the demand from these \( \alpha \) nodes.

B An \((O(1/\varepsilon^2), 2+\varepsilon)\)-Approximation Algorithm

We now describe a rounding procedure which is tuned to minimize the capacity violation while allowing a large but constant approximation factor for connection cost. The algorithm in this section first forms groups of \( \ell \geq 2 \) nodes in each star tree. In the next step, at the cost of loosing some accuracy with distances,
we simplify the graph structure within each of the groups. Eventually we use a dependent rounding routine to decide the actual openings of facilities, and give a flow-cut based argument that (deterministically) there is sufficient capacity open up the tree, and hence there exists a feasible routing of demand on the star tree.

Building groups. The nodes of a star tree $T$ will be grouped by a top-down greedy procedure starting from the root $r$; see Figure 3. When forming a new group, a single node $j$ (having all its descendants yet not grouped) will be selected as a root of the new group. Then new nodes will be added to the group in a greedy fashion until either the group has reached the size of $\ell$ nodes, or all descendants of $j$ are already included. The greedy choice of the next node to include will be to take one which is connected to the already included nodes by a cheapest tree edge. When a group is complete, we exclude the selected nodes from participating in the later formed groups. As long as not all nodes of the tree are grouped, we select a top-most one $j$ and build a group $G_j$ rooted at $j$.

Definition 7 Group $G_j$ is a parent of group $G_{j'}$ if there is a directed edge in $T$ from $j'$ to a node in $G_j$.

Observation 1 If $G_j$ has at least one child, then it has size equal $\ell$, otherwise (if it has no children) $G_j$ may have smaller size. Moreover, each group has at most $\ell + 1$ children.

The next lemma follows by Definition 3 and the way in which the algorithm selects nodes to a group.

Lemma B.1 Consider a group $G_j$ and its child group $G_{j'}$. Let $e_{j'}$ be the tree edge from $j'$ (the root of $G_{j'}$) to its father in $G_j$. For any edge $e$ in $G_j$ and any edge $e'$ in $G_{j'}$ we have $d_s(e) \leq d_s(e_{j'}) \leq d_s(e')$.

Group modification. To facilitate rounding of facility openings within groups we will modify the tree structure within groups to obtain a new in-tree $T'$ from the initial star tree $T$. The partition of nodes into groups will stay unchanged and the parent-child relation between groups will also be preserved. The modification within a single group is as follows.

Consider a group of nodes $G_j$ and the order in which the nodes were added to the group by the greedy procedure. In the modified tree $T'$, group $G_j$ will form a chain graph directed towards its root $j$, with the nodes closer to $j$ being those selected earlier by the group forming algorithm. Finally, for any group $G_{j'}$ which is a child of $G_j$, let the edge outgoing from $j'$ point to the lowest vertex in $G_j$ in $T'$.

Clearly, such modification of the tree structure may interfere with routing demand along edges of the used tree. Nevertheless, we will argue that we may bound this influence to only a constant multiplicative growth in the routing distance.

Recall that the lengths of edges of $T$ were monotone non-increasing on any directed path towards the root node $r$. We will no longer have this property in $T'$, but we will now exploit the monotonicity of $d_s$ on edges of directed paths in $T$ to bound distances on $T'$.

Lemma B.2 Let $(i, j)$ be an edge in $T$ and let $j'$ be a node that lies above $i$ in the same group as $i$ in $T'$. Then $d_s(i, j') \leq (\ell - 1) \cdot d_s(i, j)$.

Proof: Since $j'$ lies above $i$ in the same group as $i$ in $T'$, we have that $i$ was added later to this group than $j'$. Hence all edges on the path (ignoring edge directions) from $i$ to $j'$ in $T$ have length at most $d_s(i, j)$. Since no more than $\ell - 1$ edges lie on this path and since $d_s$ is a metric, the claim follows. \qed
Rounding the facility openings. To decide the eventual openings of facilities we now use a dependent rounding procedure based on pipage rounding [11, 14]. The procedure considers pairs of still fractional variables. In such a pair it pumps one up and the other down randomly choosing the one to increase (see Appendix A for more details). We use that the procedure preserves the sum of entries, hence we will open exactly $k$ facilities. Moreover, the probability of eventually opening facility $i$ equals its initial fractional opening $z_i$. The expected cost of opened facilities is $\sum_{i \in F} z_i f_i$.

On top of these standard properties, we will also exploit that we may guarantee to almost preserve the sum of entries in a number of chosen subsets of entries, provided that the subsets form a laminar family. Here, rather than explicitly defining the family of subsets we will directly say in which order should the pairs of fractional entries be chosen.

The rounding will proceed first within the groups until at most one fractional entry is left in each of the groups. Within each group the rounding procedure will always select the top-most pair of currently fractional entries. Please note, that by modifying the shape of the tree inside the groups into chain graphs we have made the choice of the top-most pair unambiguous. When there is at most one fractional entry left in each group, the rounding may be continued in an arbitrary order.

Routing and analysis. Once the facilities are open, a min-cost assignment of clients to facilities can be found, e.g., by a min-cost flow computation in the original graph. Nevertheless, for the purpose of the analysis we will consider a suboptimal assignment computed by a min-cost flow computation in the tree $T'$ subject to a limit on capacity violation.

We will argue that the demand of a node $j$ of $T'$ will be satisfied not farther than at the root of the group of the parent of $j$, which by Lemmas B.2 and B.1 is not too far. Recall that by Property (iii) each of the stars has volume at least $1 - 1/\ell$. Each non-leaf group of tree nodes has $\ell$ nodes each corresponding to a bundle. Observe that by first rounding the facility openings within each group we make sure that at least $\ell - 1$ facilities get open in each non-leaf group. We will show that after scaling up the capacity of each facility by a factor of $2 + \frac{3}{\ell - 1}$ each non-root group will send up at most $u$ units of demand. Then we will argue that the excess capacity of at least $(\ell + 1)u$ in a group is sufficient to service the demand sent up from the child group of the considered group. The tricky part is to control the demand transportation within groups.

Note that once we let a unit of demand travel along an edge of $T'$, by paying only $\ell$ times more we may let it travel longer as long as it stays within the same group. Therefore, it is essential to make sure that inside each group sufficient capacity is provided by the open facilities above a node to collect its demand.

Lemma B.3 Consider a group $G_j$. Then, after scaling the capacity by a factor of $2 + \frac{3}{\ell - 1}$ we can assign the demand of $G_j$ to open facilities in $G_j$ such that the following holds.

(i) All demand in $G_j$ gets assigned except possibly $u$ units of demand in the root star $S_j$.

(ii) No demand gets assigned to a facility below in $G_j$.

(iii) The demand of each big star $S'_j$ gets completely assigned to facilities in $S'_j$.

Proof: Let $j_1, \ldots, j_l$ be the star centers as they are ordered in the group $G_j$. We prove the claim by induction for every subgroup $j_1, \ldots, j_m$ where $m = 1, \ldots, l$.

Consider first the case $m = 1$. If $S_{j_1}$ is a small star then the claim is trivially true as the demand of the facility in this star does not have to be served. If $S_{j_1}$ is a big star then the claim also holds: if the single fractional facility in $S_{j_1}$ is closed then its demand can be served by one of the integral facilities as we scale up the capacities by factor at least 2.

Consider now the case that $m \geq 2$. There exists an assignment $\sigma$ satisfying the required properties for the subgroup $j_1, \ldots, j_{m-1}$. We will extend this assignment to $S_{j_m}$. Let $v$ be the total volume of the stars $S_{j_1}, \ldots, S_{j_m}$. By sum preservation of dependent rounding the total capacity provided by the open facilities
in the stars $S_{j_1}, \ldots, S_{j_m}$ is $2u \cdot |v|$. The total demand located at these stars is $u \cdot v$. Since $m \geq 2$ and since each star has volume at least $1 - 1/\ell \geq 1/2$, we have that $v \geq 1$ and therefore $2u \cdot |v| \geq uv$. Let $v'$ be the volume in the stars $S_{j_1}, \ldots, S_{j_{m-1}}$. Since we are going to extend the assignment $\sigma$, the leftover capacity in the stars $S_{j_1}, \ldots, S_{j_m}$ is $2u \cdot |v| - uv'$ which is larger than the demand $uv - uv'$ of $S_{j_m}$. Therefore, if $S_{j_m}$ is a small star, we can assign the demand of $S_{j_m}$ in an arbitrary manner to the leftover capacity. If $S_{j_m}$ is a big star, then its volume $v''$ is larger than 1. Again, using the fact that $2u \cdot |v''| \geq uv''$ we can see that the demand of this star can be served by its own facilities. □

The above lemma shows that the demand of any node, except perhaps the root of a group may be satisfied from an open facility within the group. It remains to handle the demand from the root node $j$ of a group $G_j$. If the star $S_j$ in $j$ is big then by Lemma B.3 all demand of $S_j$ is served within $S_j$. Suppose now that $S_j$ is small. If $j$ is the root of the tree $T'$, then we can assume that a facility in $S_j$ is open. To see this, let $j'$ be the child node of $j$ in $T'$ (by Property (iv) there is at most one). Observe that at least one facility will be opened in $S_j$ or $S_{j'}$. Moreover, $j$ and $j'$ formed a cycle of length two when we created the short tree containing $j$ and $j'$. Therefore, $j$ and $j'$ can both act as the root of $T'$ depending on which of the two edges $(j, j')$ and $(j', j)$ we removed when creating the short tree. If $j$ is not the root of $T'$, then possibly the demand of $j$ will be forwarded to the parent group. Recall that the total capacity of the open facilities in a non-leaf group $G_j$ after scaling is at least \((2 + \frac{3}{2\ell-1})(\ell - 1)u = (2\ell + 1)u\), and at most $u\ell$ is used for demand from $G_j$. Hence, at least $(\ell + 1)u$ capacity remains to be potentially used by demand forwarded from the child groups. Since there are at most $\ell + 1$ child groups and each of them forwards at most $u$ of demand, the remaining capacity is sufficient.

Lemma B.4 The expected cost is \((16\ell + 5) \cdot b(H)\), where $H$ is the original star forest.

Proof: To give an upper bound on the expected connection cost, we will first only consider the cost of connecting to star centers. For bounding the connection cost from the demand collected at star centers to the actual facilities, we will use a different argument.

We have seen above that for each outcome of the randomized opening procedure we can route the demand such that the demand of each big star is satisfied within this star.

Now consider a small star $S_j$ with facility $i$. Let $j'$ be the father of $j$ in the original star forest $H$. In case $i$ is closed the demand of $S_j$ gets rerouted to a star which lies either above $j$ in the same group as $j$ or in the parent group. By the above lemmas we know that the distance of $j$ to the center of this star is at most $\ell d_{av}(j, j')$. Since $i$ is closed with probability $1 - z_i$ we can conclude by the properties of a star tree that the expected assignment cost for $j$ is at most $(1 - z_i)u\ell d_{av}(j, j') \leq 16b j'$. Summing over all small stars shows that the total assignment cost of the star tree is at most $16b^c(H)$.

We now give an upper bound on the expected cost for redistributing the demand collected at star centers to the actual facilities. To this end consider some star $S_j$. For each facility $i \in F_j$ the probability of being opened is precisely $z_i$. Hence we can upper bound the expected connection and opening cost within $S_j$ by the quantity $z_i f_i + (2 + \frac{3}{2\ell-1})u \sum_{i \in F_j} z_i d_{av}(i, j)$. By property of star instances this quantity is at most $(2 + \frac{3}{2\ell-1})b_j$. Summing over all stars gives expected redistribution cost of at most $(2 + \frac{3}{2\ell-1})b(H) \leq 5b(H)$. □

Combining this result with Theorem 4.6 we obtain Theorem 4.1

C Missing Proofs from Section 2

Proof: (Lemma 2.1) Let $j, j' \in C'$. W.l.o.g., $j$ was added before $j'$ into $C'$ and thus we have $d_{av}(j) \leq d_{av}(j')$. Since $j'$ was not removed from $C''$ when $j$ was added to $C'$, we also have $2\ell d_{av}(j') < d(j, j')$, and the first claim follows.

Next, if $j' \not\in C'$, then $j'$ was removed from $C''$ when a client $j$ was added to $C'$. Thus we have $d(j, j') \leq 2\ell d_{av}(j')$ and the second claim follows.
Proof: (of Lemma 2.2)

\[
\sum_{j' \in C'} b_{j'} = \sum_{j' \in C'} \sum_{i' \in F_{j'}} y_{i'} f_i + \sum_{j' \in C} x_{ij}^*(d(i, j) + 2\ell d_{av}(j)) = \sum_{i \in F} y_i f_i + \sum_{j' \in C'} \sum_{j \in C} x_{ij}^*(d(i, j) + 2\ell d_{av}(j))
\]

\[
\sum_{j' \in C'} b_{j'} = \sum_{i \in F} y_i f_i + \sum_{j \in C} x_{ij}^* d_{av}(j) \leq \text{OPT} + 2\ell \sum_{j \in C} d_{av}(j) \sum_{i \in F} x_{ij}^* = \text{OPT} + 2\ell \sum_{j \in C} d_{av}(j) \leq (1 + 2\ell) \text{OPT}
\]

Proof: (of Lemma 2.5) The demand \( w_{j'} \) is given by \( \sum_{i \in F_{j'}} \sum_{j \in C} x_{ij}^* \). For each facility \( i \in F_{j'} \) we route \( \sum_{j \in C} x_{ij}^* \) units of demand to \( i \) by opening it by an amount of \( z_i = (\sum_{j \in C} x_{ij}^*)/u_i \leq y_i \). Then Constraints (1) and (3) are satisfied by the solution and \( \text{vol}(z) \leq \text{vol}(F_{j'}) \).

Now we prove that also Constraint (2) is satisfied. Let \( j \in C \) be an arbitrary client and let \( j'' \) be the bundle center in \( C' \) closest to \( j \) (possibly \( j = j'' \)). Note that \( d(j, j'') \leq 2\ell d_{av}(j) \) by Lemma 2.1. Since \( i \in F_{j'} \) we have that \( d(i, j') \leq d(i, j'') \leq d(i, j) + d(j, j'') \leq d(i, j) + 2\ell d_{av}(j) \). So we have

\[
\sum_{i \in F_{j'}} (f_i + d(i, j')u_i)z_i = \sum_{i \in F_{j'}} z_i f_i + \sum_{i \in F_{j'}} \sum_{j \in C} d(i, j')x_{ij}^* \leq \sum_{i \in F_{j'}} y_i f_i + \sum_{i \in F_{j'}} \sum_{j \in C} (d(i, j) + 2\ell d_{av}(j))x_{ij}^* = b_{j'}. \]

Proof: (of Lemma 2.4) Let \( j \in C \) be an arbitrary client and \( i \) be a facility lying in a star \( S_{j'} \) for some \( j' \in C' \). By Lemma 2.1 there is a \( j'' \in C' \) with \( d(j, j'') \leq 2\ell d_{av}(j) \). Since \( d(i, j') \leq d(i, j'') \leq d(i, j) + d(j, j'') \), it holds that \( d(j, j') \leq d(i, j) + d(i, j') \leq 2d(i, j) + 2\ell d_{av}(j) \). We ship precisely \( x_{ij}^* \) units of flow from \( j \) to \( j' \). Performing this operation for any client-facility pair we ensure (by construction of the star instances) that star center \( j'' \in C' \) collects precisely \( w_{j''} \) units of demand. The total cost of this flow can be bounded by

\[
\sum_{j' \in C'} \sum_{j \in C} \sum_{i \in F_{j'}} x_{ij}^* (2d(i, j) + 2\ell d_{av}(j)) = 2 \sum_{j \in C} \sum_{i \in F} x_{ij}^* d(i, j) + 2\ell \sum_{j \in C} \sum_{i \in F} x_{ij}^* d_{av}(j) \leq 2 \text{OPT} + 2\ell \sum_{j \in C} d_{av}(j) \sum_{i \in F} x_{ij}^* = 2 \text{OPT} + 2\ell \sum_{j \in C} d_{av}(j) \leq (2 + 2\ell) \text{OPT}. \]

D Missing proofs from Section 3

Proof of Lemma 3.1

In dependence of \( \text{vol}(z) \), we compute a new opening vector \( z' \) for the facilities in \( F_j \) where either there is only one open facility or all open facilities are integrally open. In parallel we assign each facility a demand \( d'_i \) such that the new distribution of \( w \) has cost in \( O(b_j) \) and capacity violation \( O(1 + \varepsilon) \).
**Small Volume** First we consider the case when $\text{vol}(z) < 1$, which is always true for small stars and might sometimes also hold for big stars. If there is only one fractionally open facility $i$, we just set $z'_i := \min\{1, \text{vol}(F_j)\}$ (thus $z'_i \geq z_i$) and $d'_i = d_i$. If there are exactly two fractionally open facilities $i, i'$, then we close one of them and move all their demand and opening to the other one. In fact, we can do so without any increase of capacity violation as the next lemma shows.

**Lemma D.1** For at least one $i'' \in \{i, i'\}$ there is no capacity violation if we set the opening of $i''$ to $z_i + z_i'$ and its demand to $d_i + d_i'$.

**Proof:** Choose $i'' \in \{i, i'\}$ with $u_{i''} = \max\{u_i, u_i'\}$ and observe that $(z'_i + z_i)u_i \geq d_i + d_i'$. Unfortunately, the connection cost might be unbounded in the above lemma. However, if we allow a slightly capacity violation, we can control the costs.

**Lemma D.2** Let $\varepsilon' > 0$. For at least one $i'' \in \{i, i'\}$ it holds that if we set the opening of $i''$ to $z_i + z_i'$ and its demand to $d_i + d_i'$, then its capacity violation is at most $1 + \varepsilon'$ and its demand at most $(1 + \varepsilon')/\varepsilon' \cdot d_{i''}$.

**Proof:** If for both choices of $i''$ the resulting capacity violation is at most $1 + \varepsilon'$, we choose $i'' \in \{i, i'\}$ with minimum $d(i'', j)$.

Now assume that for one of the choices, say $i'$, we have capacity violation greater than $1 + \varepsilon'$. Then by Lemma [D.1] the other choice for $i''$ has no capacity violation at all. Thus we chose $i'' = i$. Regarding the demand, observe that the violation factor of $i'$ is given by $z_i/(z_i + z_i') \cdot (d_i + d_i)/d_i > 1 + \varepsilon'$. Together with $1 - d_i/(d_i + d_i') = d_i/(d_i + d_i')$ it follows that $(d_i + d_i')/d_i < (1 + \varepsilon')/\varepsilon'$.

Thus, by Lemma [D.2] and $\varepsilon' = \varepsilon$, we select an appropriate facility $i''$, set its opening $z_{i''} := \min\{1, \text{vol}(F_j)\}$ (thus $z_{i''} \geq \text{vol}(z)$ and the capacity bound still holds), set its demand $d_{i''} := w_j$ and close all other facilities. Given that $d_{i''} \leq (1 + \varepsilon)/\varepsilon \cdot d_{i''}$ and given $d_{i''}d(i'', j) \leq z_{i''}u_{i''}d(i'', j) \leq b_j$ (Constraint 2), the connection cost is at most $(1 + \varepsilon)/\varepsilon \cdot b_j$.

**Big Volume** Next, we consider the case when $\text{vol}(z) \geq 1$, which is only true for big stars. If there are no fractionally open facilities, then we just set $z' = z$ and $d'_i = d_i$ for each $i \in F_j$.

Otherwise consider two open facilities $i'$ and $i''$ with smallest opening. Since we have at most two fractionally open facilities, the remaining open ones are integrally open. If $z_{i'} + z_{i''} \geq 1$, we choose one of them by Lemma [D.3] to be fully open in $z'$ and close the other one. If $z_{i'} + z_{i''} < 1$, then there is an integrally open facility $i$ in the star. By Lemma [D.4] we chose one of the three facilities to be fully open in $z'$ and close the other two. In both cases we route the demand of the closed facilities to the chosen one. The resulting capacity violation is at most $2 + \varepsilon$. Since we do not change the openings and demands of the other facilities, and only increased the demand of one facility by a factor at most $2 + 4/\varepsilon$, the total connection cost is at most $\sum_{i \in F_j} d(i, j)(2 + 4/\varepsilon)d_i \leq (2 + 4/\varepsilon)b_j$. Note that $\text{vol}(z') \leq \text{vol}(z) \leq \text{vol}(F_j)$.

**Lemma D.3** Let $i', i'' \in F_j$ with $z_{i'} + z_{i''} \geq 1$. For at least one them it holds that if we integrally open it and set its demand to $d_{i'} + d_{i''}$, then its capacity violation and demand increases by a factor of at most 2.

**Proof:** Choose a facility in $\{i', i''\}$ with demand $\max\{d_{i'}, d_{i''}\}$ and observe that the claim holds.

**Lemma D.4** Let $i, i', i'' \in F_j$ with $z_i = 1$ and $z_{i'} + z_{i''} < 1$. For at least one the three it holds that if we integrally open it and set its demand to $d_i + d_i' + d_i''$, then its capacity violation increases by a factor of at most $2 + \varepsilon$ and its demand increases by a factor of at most $2 + 4/\varepsilon$.

**Proof:** By Lemma [D.2] and $\varepsilon' = \varepsilon/2$, we choose one of $\{i', i''\}$, say $i'$, such that its capacity violation is at most $1 + \varepsilon'$ and its demand is at most $(1 + \varepsilon')/\varepsilon' \cdot d_{i'}$. We set $z_{i'} = z_{i'} + z_{i''}$ and $d_{i'} = d_{i'} + d_{i''}$ and
apply Lemma \[D.3\] on \(i\) and \(i'\). In the worst case we choose to open \(i'\) and get capacity violation of at most \(2(1 + \varepsilon') = 2 + \varepsilon\) and an total increase of demand by a factor of at most \(2(1 + \varepsilon')/\varepsilon' = 2 + 4/\varepsilon\). □

The discussion of the two cases of \(\text{vol}(z)\) leads to Lemma \[3.1\]

---

**Procedure Decompose(T)**

\[
\text{Procedure Decompose(T)}
\]

\[
\text{while there are at least two nodes in } T \text{ do}
\]

\[
\quad \text{choose a leaf node } i \text{ with the biggest number of edges on the path from } i \text{ to the root;}
\]

\[
\quad \text{consider the subtree rooted at } s(i) \text{ as a rooted facility star, and remove this subtree;}
\]

\[
\text{if only one node } i \text{ (root) left and } \hat{y}_i < 1 \text{ then}
\]

\[
\quad \//\text{there exists a facility star with root } t, \text{ where } d(i, s(i)) \geq d(i, t)
\]

\[
\quad \text{add } i \text{ to the facility star rooted at } t \text{ as a child}
\]

---

**Proof:** (of Lemma \[3.4\]) For each kind of facility star \(Q_t\), we open at most \(\sum_{i \in Q_t} \hat{y}_i\) facilities. Each facility \(i \in \hat{N}_1 \setminus \bigcup_t Q_t\), for which we have \(\hat{y}_i = 1\), is open in the integral solution. So we have \(\sum_{i \in F} \hat{y}_i \leq \sum_{i \in F} \hat{y}_i \leq k\). Which implies that the number of open facilities in solution \(\bar{y}\) is at most \(k\).

The fact that the capacity violation of each open facility can be bounded by \(3 + \varepsilon\) easily follows from the case analysis of Round\(Q_t\) and the fact that each facility \(i \in \hat{N}_1 \setminus \bigcup_t Q_t\), which is open in \(\bar{y}\), has capacity violation at most \(2 + \varepsilon\). □

**Proof:** (of Lemma \[3.6\]) For each \(i \in Q_t \cap \hat{N}_2\), we have \(1 - \frac{1}{\ell} = \hat{y}_i\), so

\[
d'_t d(s(i), i) = \ell d'_t(1 - \hat{y}_i) d(s(i), i) .
\]

We sum \(\ell d'_t(1 - \hat{y}_i) d(s(i), i)\) over all \(i \in \hat{N}_2\) to get an upper bound for the total cost of facility stars. Note that \(\hat{N}_2 \subseteq \hat{N}_2\), and \(\hat{y}_i \leq 1\) for each \(i \in \hat{N}_2\). Thus,

\[
\ell \sum_{i \in \hat{N}_2} (1 - \hat{y}_i) d'_t d(s(i), i) \leq \ell \sum_{i \in \hat{N}_2} (1 - \hat{y}_i) d'_t d(s(i), i) .
\]

By Property \[3.3\], we know that

\[
\ell \sum_{i \in \hat{N}_2} d'_t(1 - \hat{y}_i) d(s(i), i) \leq \ell \sum_{i \in \hat{N}_2} d'_t(1 - y'_i) d(s(i), i) .
\]

By the definition of \(d'_t\), we have

\[
\ell \sum_{i \in \hat{N}_2} d'_t(1 - y'_i) d(s(i), i) = \ell \sum_{j \in C} \sum_{i \in \hat{N}_2} x'_{ij}(1 - y'_i) d(s(i), i) .
\]

We will show that for all \(j \in N\) that

\[
\sum_{i \in \hat{N}_2} x'_{ij}(1 - y'_i) d(s(i), i) \leq 2 \sum_{i \in N_1 \cup N_2} x'_{ij} d(i, j) .
\]

Using the fact that \(1 - y'_i \leq 1 - x'_{ij} \leq \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'dj}\) we have

\[
\sum_{i \in \hat{N}_2} x'_{ij}(1 - y'_i) d(s(i), i) \leq \sum_{i \in \hat{N}_2} x'_{ij} \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'dj} d(s(i), i) .
\]
From the definition of $s(i)$ we know that $d(s(i), i) \leq d(i', i)$ for each $i' \in N_1 \cup N_2 \setminus \{i\}$, so
\[
\sum_{i \in N_2} x'_{ij} \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'j} d(s(i), i) \leq \sum_{i \in N_2} x'_{ij} \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'j} d(i', i)
\]
Using the triangle inequality, we have
\[
\sum_{i \in N_2} x'_{ij} \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'j} d(i', i) \leq \sum_{i \in N_2} x'_{ij} \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'j} (d(i', j) + d(i, j)) = \sum_{i \in N_2} x'_{ij} \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'j} d(i', j) \leq \sum_{i' \in N_1 \cup N_2} x'_{i'j} d(i', j) \sum_{i \in N_2} x'_{ij} + \sum_{i \in N_2} x'_{ij} d(i, j) \sum_{i' \in N_1 \cup N_2 \setminus \{i\}} x'_{i'j}.
\]
For each $j \in C$ the inequality $\sum_{i \in N_1 \cup N_2} x'_{ij} \leq 1$ holds. Using this fact we can upper bound the above expression by the following
\[
\sum_{i' \in N_1 \cup N_2} x'_{i'j} d(i', j) + \sum_{i \in N_2} x'_{ij} d(i, j) \leq 2 \sum_{i \in N_1 \cup N_2} x'_{ij} d(i, j).
\]
From Corollary 3.2 we have
\[
2\ell \sum_{j \in C} \sum_{i \in N_1 \cup N_2} x'_{ij} d(i, j) \leq 2\ell (2 + 4/\varepsilon)(1 + 2\ell) \text{OPT} + (2 + 2\ell) \text{OPT}.
\]

\section*{E Missing proofs from Section 4}

\textbf{Proof:} (Lemma 4.2) We have that $i \neq l$ because the algorithm removes one edge from each cycle of length two. By construction of the edge set we have that $d(i, l) \leq d(j, i)$. The claim follows since no two client pairs have the same distance.

\textbf{Proof:} (Lemma 4.3) By construction, every node in the graph has out-degree at most one, and by Lemma 4.2 there are no cycles.

\textbf{Proof:} (Lemma 4.4) Let $T'$ be a binary center tree.

Tree $T'$ satisfies Property (vi), the proof is the following.
The node $j$ can have at most two incoming edges in $T'$: One from its closest son in $T$ and one from its right brother in $T$. Since a root has no brother, its in-degree is at most 1 in $T'$.

Consider the edge $(j, i') \in E(T')$. Moreover, let $i$ be the father of $j$ in the short center tree $T$ from which $T'$ was derived. We show that $d(j, i') \leq 2d(j, i)$. There are two cases: Either $i'$ is the father of $j$ in $T$, or it isn’t. The first case is trivial as $d(j, i') \leq 2 \cdot d(j, i')$. In the second case, node $i'$ is the left brother of $j$ in $T$ and $i$ is the common father of $j$ and $i'$ in $T$. Since $i'$ is the left brother of $j$ in $T$ we have $d(i', i) \leq d(j, i)$. Hence $d(j, i') \leq d(j, i) + d(i, i') \leq 2 \cdot d(j, i)$.

Tree $T'$ satisfies Property (vi). The proof is the following.
Let $(j,j'),(j',j'')$ be edges in $T'$. We claim $d_s(j, j') \geq d_s(j', j'')$. 

\nonumber
Let $T$ be the short center tree from which $T'$ was derived and let $i, i'$ be the fathers of $j, j'$, respectively. We show that $2d(j, i) \geq 2d(j', i')$. If $j'$ is father of $j$ in $T$ then the claim holds by Lemma 4.2. If $j'$ is the left brother of $j$ in $T$ then the claim also holds by the construction of the star tree.

Tree $T'$ satisfies Property (iv). The proof is the following.

We redefine $j'' \in C' \setminus \{j\}$ to be the bundle center distinct from $j'$ that is closest to $j'$. We then show that $(1 - y_i)ud(j', j'') \leq 8b_{j'}^y$ which implies the claim by the definition of $d_s$.

Recall that the demand $w_{j'}$ is given by $\sum_{i \in F_j} \sum_{j \in C} x_{i,j}$. Clearly, this demand must be equal to $y_i u$. We will show that $(1 - y_i)y_i u d(j', j'') \leq 4b_{j'}^y$. This implies the claim since we know by Property (ii) that $1/y_i \leq 1/(1 - 1/\ell) \leq 2$.

Let $i' \in F_j'$ and $j \in C$. The contribution of the pair $(i', j)$ to the quantity $(1 - y_i)y_i u d(j', j'')$ is $x_{i',j}(1 - y_i)d(j', j'')$. The contribution of $(i', j)$ to the budget $b_{j'}^y$ is $x_{i',j}(d(i', j) + 2\ell d_{av}(j))$.

In what follows we will show that $(1 - y_i)d(j', j'')$ is bounded by $4(d(i', j) + 2\ell d_{av}(j))$. Summing over all such pairs $(i', j)$ completes the proof. We distinguish the two cases where the $d(j, j')$ is smaller or larger than $R := d(j', j'')/2$, respectively.

First, assume that $d(j, j') \leq R$. This means that $j'$ is the closest client in $C'$ to $j$. Hence $d(j, j') \leq 2\ell d_{av}(j)$. Also $d_{av}(j) \geq d_{av}(j')$. Note that at most $d_{av}(j')/R$ of the demand of $j'$ can be served by facilities $i'$ with $d(i', j') > R$. Hence at least $1 - d_{av}(j')/R$ of the demand of $j'$ is served by facilities within a radius of $R$ around $j'$. Since all such facilities lie in the bundle $F_{j'}$ the volume of $F_{j'}$ is at least $1 - d_{av}(j')/R$. Thus $y_i \geq 1 - d_{av}(j')/R$. Hence $(1 - y_i)d(j', j'')$ is bounded by $d_{av}(j')/R \cdot 2R = 2d_{av}(j') \leq 2d_{av}(j)$.

Second, suppose that $d(j, j') \geq R$; see Figure 4. Let $j''' \in C'$ be the bundle center closest to $j$. Then $d(j, j''') \leq 2\ell d_{av}(j)$. Since $i$ lies in the bundle $U_j$ we have that $d(i, j') \leq d(i, j'') \leq d(i, j) + d(j, j'') \leq d(i, j) + 2\ell d_{av}(j)$ and $d(j, j') \leq d(j, i) + d(i, j') \leq 2d(i, j) + 2\ell d_{av}(j)$. We therefore have that $(1 - y_i)d(j', j'') \leq 2R \leq 2d(j, j') \leq 4d(i, j) + 4\ell d_{av}(j) \leq 4(d(i, j) + 2\ell d_{av}(j))$.

\[ \square \]

**F Missing Lemmas and Proofs of the (O(1),6)-Approximation Algorithm**

**Proof:** (of Theorem 4.6) Given a solution to the star forest, we can compute an optimal assignment of the clients to facilities opened by this solution by solving a minimum-cost flow problem.

To bound the cost of this solution, we give a sub-optimal fractional flow which uses the edges in the star forest and which satisfies the claimed cost bound. The flow is constructed in two steps. First the demand of the clients is transported to the star centers so that each node $j' \in C'$ collects precisely $w_{j'}$ units of demand. By Lemma 2.6 this can be accomplished at cost at most $(2 + \ell) \cdot \text{OPT}$. To transport the demand collected at the star centers to the actual facilities we use the assignment provided by the solution to the star forest. By definition, this assignment transports precisely $w_{j'}$ units of demand to the facilities.
opened by the solution. The cost of this assignment is \(c \cdot b'(H) \leq c \cdot (2\ell + 1) \text{OPT}\) by Lemma 2.5.

**Proof:** (Lemma 4.7) Let \(\mathcal{F}_B\) denote the set of all facilities in big stars and let \(\mathcal{F}_S\) denote the set of all facilities in small stars. The number of open facilities in big stars is at most \(\lfloor \text{vol}(\mathcal{F}_B) \rfloor\). One property of the dependent rounding we use is sum preservation. This ensures that the number of open facilities in small stars is equal to either \(\lfloor \text{vol}(\mathcal{F}_S) \rfloor\) or \(\lceil \text{vol}(\mathcal{F}_S) \rceil\). It follows from \(\text{vol}(\mathcal{F}_B) + \text{vol}(\mathcal{F}_S) \leq k\) that \(\lfloor \text{vol}(\mathcal{F}_B) \rfloor + \lceil \text{vol}(\mathcal{F}_S) \rceil \leq k\).

All fractional facilities in a big bundle are closed by our procedure. Each fractionally open facility \(i\) in a small star will be open with probability \(y_i\). The expected cost of opened facilities could be bounded by \(\sum_{i \in \mathcal{F}} y_i f_i\).

In the next paragraphs we will complete the proof of Lemma 4.8 and show that the capacity violation is a most 6. We begin by describing the routing of the demand between stars.

**Routing between stars.** The demand of clients associated with big stars is served within their stars. For a small star, however, it is necessary to reroute the demand to other stars if this star happens to be closed by the randomized procedure. In what follows, we describe how to reroute the demand of closed small stars. For the sake of simplicity we only describe to which star center the demand is rerouted. We specify in some other paragraph how the demand is redistributed from star centers to facilities.

**Lemma F.1** The root \(r\) of a star tree or its left son \(l(r)\) or both contain an open facility.

**Proof:** If \(r\) and \(l(r)\) are paired in the matching, then at least one of them will be opened. Otherwise at least one of them is a big star.

We send demand from root \(r\) to \(l(r)\) if \(r\) is closed and vice versa. Any node which has a father and a grandfather in a star tree is called an internal node. The proof of the following lemma is an easy case analysis, so we omit it.

**Lemma F.2** Consider an internal node \(j\) of a star tree with all facilities closed in its star instance. If \(j\) is a left son, then either its father or grandfather has an open facility. If \(j\) is a right son, then its left brother, father or grandfather has an open facility.

For each internal node that has no open facility in its star, we send its demand to its father, left brother or grandfather. We consider these three candidates in the above order and choose the first that has an open facility. This rule allows us to make the following observations.

**Proposition F.3** For each \(j \in \mathcal{C}',\) there is at most one non-descendant node that might send a demand to \(j:\) the father of \(j,\) or the brother of \(j.\)

**Proof:** The only node that might send a demand to its left son, is the root. Recall that a root does not have a right son (Property (v) of star trees), so a node can’t get a demand simultaneously from its father and right brother.

We define a node to be above another one, if the first node is a non-descendant of the second one.

**Proposition F.4** Consider nodes \(j'\) and \(l(j')\) and suppose that the edge \((j', l(j'))\) is in a matching. For any resulting rounding of facility openings, node \(j'\) and all its descendants can send together at most one unit of demand to nodes above \(j'.\)
Proof: By the routing rule, the only nodes that can send demand above \( j' \) are \( j' \) and its sons. If at least one facility is open in the star associated with \( j' \) after the rounding of facility openings, then neither \( j' \) nor its descendants will send a demand to nodes above \( j' \). Otherwise there is one open facility in a star associated with \( l(j') \) and \( r(j') \) (if it exits) routes all its demand to this facility. The only facility that sends its demand above \( j' \) is \( j' \).

\[ \square \]

Proposition F.5 Consider nodes \( j' \) and \( l(j') \) and suppose that at least one of the stars associated with these nodes is a big star. For any resulting rounding of facility openings, node \( j' \) and all its descendants can send together at most one unit of demand to nodes above \( j' \).

Proof: If \( j' \) has an open facility after the rounding, then it satisfies all the demand from \( j' \) and its descendants. Otherwise, \( j' \) must be a small star and \( l(j') \) a big star. Then \( l(j') \) contains an open facility and sends no demand to \( j' \) or above. Further, any demand coming from \( r(j') \) is routed to \( l(j') \) (see the routing rule), thus only \( j' \) can send a demand above \( j' \).

\[ \square \]

We call a node \( j \) a gate if it has no sons or if \( j \) and \( l(j) \) form a configuration as required in Proposition F.4 or F.5. Thus a gate has the property that at most one unit of demand is sent above it. Now we are ready to prove the bound of capacity violation.

Proof: (of Lemma 4.8)

Full argument of the routing cost in as follows. Here, we give a complete analysis of the expected routing cost. We first bound the cost of routing the demand of a closed star to another star center and temporarily ignore the cost for routing this demand from there to the actual facilities. Consider a small star \( S_j \) with fractional facility \( i \). The probability of closing \( i \) is \((1 - y_i)\). Let \( j' \) be the star center to which we reroute in case \( i \) is closed. Then by Lemma 4.3 the expected routing cost can be bounded by \((1 - y_i)ud_s(j, j') \leq 2 \cdot 16b'j'\) since \( j' \) is the grandfather (or left brother) of \( j \) in the worst case. Summing this over all small stars the expected cost of routing the demand of a star to some star center which has an open facility is at most \( 32b(H) \).

We now analyze the redistribution cost. As stated above it is possible to redistribute the total demand collected at star centers to their facilities by violating the capacities by a factor of at most \( 6 \). Consider some star instance \( S_j \). Every facility \( i \) in \( S_j \) is opened with probability at most \( y_i \). Hence the expected redistribution cost in \( S_j \) is bounded by \( y_if_i + 6u \sum_{i \in F_j} y_id(j, i) \). By Constraint (2) for star instances this quantity is bounded by \( 6b \). Summing over all \( j \in C \) shows that the total expected redistribution cost is bounded by \( 6b(H) \).

Full argument of the capacity violation in as follows. The proof is an easy case analysis. In each case we suppose that there is at most one open facility in each star instance. Otherwise we can split the demand, which the star has to serve, between two (or more) open facilities and decrease the capacity violation. Our goal is to show that in each case the capacity violation of the facility that we consider is at most six.

Consider any facility \( i \) that is open after the rounding of facility openings. Suppose that facility \( i \) is associated with a root \( r \) of a star tree. A node that is a root of a star tree does neither have a right son nor a right brother. In consequence all nodes that might send one unit of their demand to \( r \) are the left son of \( r \) and both children of \( l(r) \) (if they exist). Further, if the star associated with \( r \) is a big star, then it might cause one extra unit of demand that needs to be served. Thus capacity violation is at most five in the worst case.

Now suppose that node \( j \) to which facility \( i \) belongs is not a root. We distinguish two cases. Either \( j \) is in a matching or it isn’t. If \( j \) is in a matching, then there are two possibilities. Node \( j \) is in the matching with its father or with one of its sons. In the first case, nodes \( l(j) \) and \( r(j) \) (if they exist) are gates, and by Propositions F.5 and F.4 they send at most two units of demand to \( j \). Further, either the father or the
right brother might send one additionally unit of demand to \( j \) (see Proposition \[F.3\]). So capacity violation is at most four in that case. In the second case, let \( j' \) be the son that is in a matching with \( j \) and let \( j'' \) be the other son (if it exists). Then \( j'' \) and each son of \( j' \) is a gate and they send altogether at most three units of demand. At most two units can be sent together by \( j' \) and one by the father or right brother of \( j \). In the second case capacity violation is at most six.

If \( j \) is not in a matching, then it is either associated with a big star or has no sons associated with small stars. In both cases each son of \( j \) is a gate, thus the sons send together at most two units of demand (if they exist). Also the father and the right brother of \( j \) (if it exists) can send (together) at most one unit of demand. If the star associated with \( j \) is big, we also have to serve one extra unit of demand. Thus the total demand which \( j \) has to serve is at most five units.

\[ \square \]

G Pseudo-codes of Procedures

<table>
<thead>
<tr>
<th>Procedure Short-Trees((C'))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create ( G = (C', \emptyset) );</td>
</tr>
<tr>
<td>( \textbf{forall the} \ j \in C' \ \textbf{do} )</td>
</tr>
<tr>
<td>( \quad \text{select} \ j', \text{which is the closest node to} \ j \ \text{in} \ C' \setminus {j}; )</td>
</tr>
<tr>
<td>( \quad \text{add directed edge} \ (j, j') \ \text{to} \ G; )</td>
</tr>
<tr>
<td>( \textbf{forall the} \ (j, j'), (j', j) \in E(G) \ \textbf{do} )</td>
</tr>
<tr>
<td>( \quad \text{remove edge} \ (j, j') \ \text{from} \ G; )</td>
</tr>
<tr>
<td>return ( G; )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Procedure Binary-trees((G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = \emptyset; )</td>
</tr>
<tr>
<td>( \textbf{forall the} \ \text{short trees} \ T \ \text{in} \ G \ \textbf{do} )</td>
</tr>
<tr>
<td>( \quad \textbf{forall the} \ \text{nodes} \ i \in V(T) \ \textbf{do} )</td>
</tr>
<tr>
<td>( \quad \text{sort all sons of} \ i \ \text{from left to right by non-decreasing distance to} \ i; )</td>
</tr>
<tr>
<td>( \quad \text{remove all incoming edges of} \ i \ \text{from} \ T \ \text{except the shortest one;} )</td>
</tr>
<tr>
<td>( \quad \text{add a directed edge from each son of} \ i \ \text{to its left brother (if there exists one);} )</td>
</tr>
<tr>
<td>( \quad \text{add} \ T \ \text{to} \ H; )</td>
</tr>
<tr>
<td>return ( H; )</td>
</tr>
</tbody>
</table>
Procedure Make-Matching($j$)

$$j' = l(j);$$

if $j' == NULL$ then

    return;

add edge ($j', j$) to $M(T)$;

if $r(j) != NULL$ then

    Make-Matching ($r(j)$);

if $l(j')$ and $r(j')$ are leafs of a tree $T$ then

    add ($l(j'$), $r(j')$) to $M(T)$;

else

    if $l(j') != NULL$ then

        Make-Matching ($l(j')$);

    if $r(j') != NULL$ then

        Make-Matching ($r(j')$);

return;