Some Results on a Subclass of alpha-quazi Spirallike Mappings

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Abstract

Let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $D = \{z \in C : |z| < 1\}$. A sense preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation $\mathcal{T}_z = w(z)f_z(\overline{f})$ where $w(z) \in H(D)$ is the second dilatation of $f$ such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if $f$ is a non-vanishing log-harmonic mapping, then $f$ can be expressed as $f(z) = h(z)g(z)$, where $h(z)$ and $g(z)$ are analytic in $D$ with the normalization $h(0) \neq 0$, $g(0) = 1$. If $f$ vanishes at $z = 0$ but it is not identically zero, then $f$ admits the representation $f = z|z|^{2\beta} h(z)g(z)$, where $\text{Re}\beta > -\frac{1}{2}$ and $h(z)$, $g(z)$ are analytic in $D$ with the normalization $h(0) \neq 0$, $g(0) = 1$. [1], [2], [3]. The class of all logharmonic mappings is denoted by $S^*_{LH}$.

The aim of this paper is to give an application of the subordination principle to the class of spirallike logharmonic mappings which was introduced by Z. Abdulhadi and W. Hengartner. [1]

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1 Introduction

Let $H$ be the linear space of all analytic functions defined in the open unit disc $D = \{z \in C : |z| < 1\}$. A sense preserving log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\mathcal{T}_z = w(z)f_z(\overline{f}),$$

(1)
Where \( w(z) \) the second dilatation of \( f \) and \( w(z) \in H(D) \), \( |w(z)| < 1 \) for every \( z \in D \). It has been shown that if \( f \) is non vanishing logharmonic mapping, then \( f \) can be expressed as

\[
f(z) = h(z)g(z)
\]

(2)

where \( h(z) \) and \( g(z) \) are analytic in \( D \) with the normalization \( h(0) \neq 0 \), \( g(0) = 1 \). On the other hand if \( f \) vanishes at \( z = 0 \), but it is not identically zero, then \( f \) admits the following representation

\[
f = z, |z|^2 \beta h(z)g(z)
\]

(3)

where \( \text{Re} \beta > -\frac{1}{2} \), \( h(z) \) and \( g(z) \) are analytic in the open disc \( D \) with the normalization \( h(0) \neq 0, g(0) = 1 \). Also we note that univalent logharmonic mapping have been studied extensively. \[1\], \[2\], \[3\] and the class of univalent logharmonic mappings is denoted by \( S_{LH} \). Let \( f = zh(z)g(z) \) be a univalent logharmonic mapping. We say that \( f \) is a starlike logharmonic mapping if

\[
\frac{\partial \arg f(re^{i\theta})}{\partial \theta} = \text{Re} \frac{zf \bar{z} - \bar{z}f \bar{z}}{f} > 0
\]

for all \( z \in D \), and the class of all starlike logharmonic mappings is denoted by \( ST_{LH}^* \).

Let \( \varphi(z) \) be analytic in \( D \) and let \( \alpha \) be a real number such that \( |\alpha| < \frac{\pi}{2} \). If \( \varphi = 0, \varphi'(0) \neq 0 \) and if

\[
\text{Re}(e^{i\alpha}z\frac{\varphi'(z)}{\varphi(z)}) > 0
\]

(4)

then \( \varphi(z) \) is univalent \[5\] and is said to be spirallike. Under these conditions we have

\[
e^{i\alpha}z\frac{\varphi'(z)}{\varphi(z)} = Q(z)
\]

(5)

where \( \text{Re}Q(z) > 0 \) and \( Q(0) = e^{i\alpha} \). Defining \( P(z) = Q(z) \sec \alpha - i \tan \alpha \) we may write

\[
z\frac{\varphi'(z)}{\varphi(z)} = e^{-i\alpha}[P(z) \cos \alpha + i \sin \alpha]
\]

(6)

where \( \text{Re}P(z) > 0, P(0) = 1 \). The class of spirallike functions is denoted by \( S_{LH}^* \). In particular with \( \alpha = 0 \), \( S_{LH}^* \) coincides with the class of normalized starlike functions. The relationship between \( S_{LH}^* \) and \( S_{LH}^* \) is indicated in the following lemma.

**Lemma 1.1** \( f(z) \in S_{0,p} \) if and only if there is a \( g(z) \in S_{0,p} \) such that

\[
\left[ \frac{f(z)}{z} \right]^{\exp(i\alpha)} = \left[ \frac{g(z)}{z} \right]^{\cos \alpha}
\]

(7)
where the branches are chosen so that each side of the equation has the value 1, when $z = 0$.

On the other hand Z. Abdulhadi and Y. Abu Muhanna was proved the following theorem.

**Theorem 1.2** Let $f(z) = z h(z) \overline{g(z)}$ be a logharmonic mapping in $D$, $0 \notin h g(D)$. Then $f \in ST_{LH}^*$ if and only if $\varphi(z) = z \frac{h(z)}{g(z)} \in ST^*$.

Finally let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $D$ and satisfying the conditions $\phi(0) = 0 \ | \phi(z)| < 1$ for every $z \in D$ and let $s_1(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, $s_2(z) = z + b_2 z^2 + b_3 z^3 + \ldots$ be analytic functions in $D$. We say that $s_1(z)$ is subordinate to $s_2(z)$ if $s_1(z) = s_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in D$ and denote by $s_1(z) \prec s_2(z)$.

## 2 Main Results

Considering Lemma (1.1) and Theorem (1.2) together we obtain the following lemma.

**Lemma 2.1** $\phi(z) \in S_\alpha^*$ if and only if there is a $f(z) = z h(z) \overline{g(z)} \in ST_{LH}^*$ such that

$$\left( \frac{\phi(z)}{z} \right)^{e^{i\alpha}} = \left( \frac{h(z)}{g(z)} \right)^{\cos \alpha}$$

(8)

where the branches are chosen so that both sides of the equation has the value 1, when $z = 0$.

**Theorem 2.2** Using Lemma 2.1 then we have the following equality,

$$e^{i\alpha} z. \frac{\phi'(z)}{\phi(z)} = \cos \alpha [1 + z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)}] + i \sin \alpha$$

(9)

We have;

$$f = z. |z|^{2\beta} h(z) \overline{g(z)} \Rightarrow \{ z f_{\beta} f = \beta + 1 + z \frac{h'(z)}{h(z)}, \frac{\overline{f_{\beta} f}}{f} = \beta + z \frac{g'(z)}{g(z)} \}$$

(10)

$$w(z) = \frac{\overline{f_{\beta} f}}{f_{\beta} f} = \frac{\overline{\beta} + z \frac{g'(z)}{g(z)}}{1 + \beta + z \frac{h'(z)}{h(z)}}$$

(11)

In the equality (10) if we take $\beta = 0$ then we obtain;

$$w(z) = \frac{z \frac{g'(z)}{g(z)}}{1 + z \frac{h'(z)}{h(z)}}$$

(12)
Therefore we have $w(0) = 0$, $|w(z)| < 1$ then we can say that $w(z)$ satisfies the conditions of Schwarz Lemma, and

$$1 - w(z) = \frac{1 + z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)}}{1 + z \frac{h'(z)}{h(z)}}$$  \hspace{1cm} (13)$$

$$\frac{w(z)}{1 - w(z)} = \frac{z \frac{g'(z)}{g(z)}}{1 + z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)}}$$  \hspace{1cm} (14)$$

Using the equality (12), (11) equalities (13) and (14) can be written in the following form,

$$1 - w(z) = \frac{1}{\cos \alpha} \left[ z \frac{\phi'(z)}{\phi(z)} - i \sin \alpha \right]$$  \hspace{1cm} (15)$$

$$\frac{w(z)}{1 - w(z)} = \frac{z}{\cos \alpha} \left[ e^{i\alpha} z \frac{\phi'(z)}{\phi(z)} - i \sin \alpha \right]$$  \hspace{1cm} (16)$$

Using the subordination principle the equalities can be written

$$\left| \frac{1}{\cos \alpha} \left[ z \frac{\phi'(z)}{\phi(z)} - i \sin \alpha \right] \right| - c_1(r) \prec \rho_1(r)$$  \hspace{1cm} (17)$$

$$\left| \frac{z}{\cos \alpha} \left[ e^{i\alpha} z \frac{\phi'(z)}{\phi(z)} - i \sin \alpha \right] \right| - c_2(r) \prec \rho_2(r)$$  \hspace{1cm} (18)$$

Because the transformations $\rho_1(r)$ and $\rho_2(r)$ map $|z| = r$ on to the discs with the centres

$$c_1(r) = \left[ \frac{m^4(1 - a) + \overline{a}(m^2 - m^2 r^2)}{m^4 - (\overline{a})^2 r^2}, 0 \right]$$

$$c_2(r) = \left[ \frac{m^4 a(1 - a) + m^2(m^2 - \overline{a})r^2}{m^4(1 - a)^2 - (m^2 - \overline{a})^2 r^2}, 0 \right]$$

and the radius

$$\rho_1(r) = \left| \frac{m^2(m^2 - \overline{a}) + \overline{a}m^2(1 - a)}{m^4 - (\overline{a})^2 r^2} \right| r$$

$$\rho_2(r) = \left| \frac{m^4(1 - a) - m^2a(m^2 - \overline{a})}{m^4(1 - a)^2 - (m^2 - \overline{a})^2 r^2} \right| r$$

respectively using the subordination principle on the expressions (17), (18) then we get the following theorem.
Theorem 2.3 Let \( f = zh(z)g(z) \) be a log-harmonic quazi spirallike function then

\[
F_1(r, \alpha) \leq \frac{z^2}{f} \leq F_2(r, \alpha)
\]

\[
F_3(r, \alpha) \leq \frac{z^2}{f} \leq F_4(r, \alpha)
\]

Since the transformations

\[
(m^2 - \overline{\alpha})z + (m^2 - m^2a) - \overline{\alpha}z + m^2
\]

and

\[
- m^2z + m^2a
\]

\[
(m^2 - \overline{\alpha})z + m^2(1 - a)
\]

map \( |z| = r \) onto the discs with centres

\[
c_1(r) = \left( \frac{m^4(1 - a) + \overline{\alpha}(m^2 - \overline{\alpha})r^2}{m^4 - (\overline{\alpha})^2r^2}, 0 \right)
\]

\[
c_2(r) = \left( \frac{m^4a(1 - a) + m^2(2 - \overline{\alpha})r^2}{m^4(1 - a)^2 - (m^2 - \overline{\alpha})^2r^2}, 0 \right)
\]

and the radius

\[
\rho_1(r) = \frac{|m^2(2 - \overline{\alpha}) + \overline{\alpha}m^2(1 - a)|r}{m^4 - (\overline{\alpha})^2r^2}
\]

\[
\rho_2(r) = \frac{|- m^4(1 - a) - m^2a(m^2 - \overline{\alpha})|r}{m^4(1 - a)^2 - (m^2 - \overline{\alpha})^2r^2}
\]

After simple calculations from Theorem 2.2 and using inequalities (17), (18) we get the result easily.

\[
F_1(r, \alpha) = \frac{m^4 - |\overline{\alpha}|^2r^2}{m^4(1 - a) + \overline{\alpha}(m^2 - \overline{\alpha})r^2 + |m^2(2 - \overline{\alpha}) + \overline{\alpha}m^2(1 - a)|r} \cdot \frac{1}{\cos \alpha} \left[ e^{i\alpha} \frac{z}{\phi(z)} - \frac{\phi'(z)}{\phi(z) - i \sin \alpha} \right]
\]

\[
F_2(r, \alpha) = \frac{m^4 - |\overline{\alpha}|^2r^2}{m^4(1 - a) + \overline{\alpha}(m^2 - \overline{\alpha})r^2 - |m^2(2 - \overline{\alpha}) + \overline{\alpha}m^2(1 - a)|r} \cdot \frac{1}{\cos \alpha} \left[ e^{i\alpha} \frac{z}{\phi(z)} - \frac{\phi'(z)}{\phi(z) - i \sin \alpha} \right]
\]

\[
F_3(r, \alpha) = \frac{m^4a(1 - a) + m^2(2 - \overline{\alpha})r^2 - |m^4(1 - a) + m^2a(m^2 - \overline{\alpha})|r}{m^4(1 - a)^2 - (m^2 - \overline{\alpha})^2r^2} \cdot \frac{1}{\cos \alpha} \left[ e^{i\alpha} \frac{z}{\phi(z)} - \frac{\phi'(z)}{\phi(z) - i \sin \alpha} \right]
\]

\[
F_4(r, \alpha) = \frac{m^4a(1 - a) + m^2(2 - \overline{\alpha})r^2 + |m^4(1 - a) + m^2a(m^2 - \overline{\alpha})|r}{m^4(1 - a)^2 - (m^2 - \overline{\alpha})^2r^2} \cdot \frac{1}{\cos \alpha} \left[ e^{i\alpha} \frac{z}{\phi(z)} - \frac{\phi'(z)}{\phi(z) - i \sin \alpha} \right]
\]
References


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