Primordial and asymptotic inflation in Brans-Dicke cosmology

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Abstract

We show that rapid primordial inflation and slow asymptotic inflation is a natural consequence of Brans-Dicke theory. The ratio of the two inflation parameters is proportional to the square root of the Brans-Dicke parameter ω ($\omega \gg 1$). We also calculate the Hubble parameter H and the time variation of the time dependent Newtonian gravitational constant G for both regimes. The variation of the Hubble parameter predicted by Brans-Dicke cosmology is shown to be consistent with recent measurements.

The inflationary universe model whose key feature is a finite period of primordial rapid exponential expansion has been proposed to resolve a number of cosmological puzzles, including the horizon, flatness and monopole problems. In the original or 'old inflation model' [1], the universe supercools into a false vacuum phase and its energy density acts as an effective cosmological constant which causes an epoch of de-Sitter (exponential) expansion. In this old inflation model, the de-Sitter expansion never ends and for a generic first order phase transition, there appears an energy barrier between the false vacuum and the true vacuum phases. This problem is known as the 'graceful exit' problem. In the extended inflation [2] model, on the other hand, the spirit of the old inflation model is conserved in the sense that the universe undergoes a generic, strongly first order phase transition associated with some (unspecified) high temperature particle physics phenomenon (e.g., gauge symmetry braking). As in the old inflation model, the energy scale for the transition can be much lower than the Planck scale so that quantum gravity effects can be ignored.

In our work, we start up with a strong link between inflation and Brans-Dicke [3] theory of gravity. The theory is parameterized by a dimensionless constant ω, where $\omega \to \infty$ as Brans-Dicke theory goes over to the Einstein theory [4]. Present limits of the constant ω based on time-delay [5] experiments require $\omega > 500 \gg 1$. We assume that the scalar potential is composed only of the scalar field mass term and that ϕ evolves with time. The proposed model in this work is simple in that no other phenomenon such as the domination of the false vacuum over the scalar field energy density as in the extended inflation model is assumed. The action is the following;

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{8\omega}\phi^2 R + \frac{1}{2}g^{\mu\upsilon}\partial_\mu\phi\partial_\nu\phi - V\left(\phi\right) + L_M\right] \tag{1}$$

where ϕ represents the Brans-Dicke scalar field and ω denotes the dimensionless Brans-Dicke parameter taken to be much larger than 1 ($\omega \gg 1$) and $V\left(\phi\right) = \frac{1}{2}m^2\phi^2$ contains only a mass term and L_M, on the other hand, is the matter lagrangian such that the scalar field ϕ does not couple with it. R is the Ricci scalar. For simplicity we also restrict our analysis to the Robertson Walker metric to emphasize that ϕ is necessarily spatially homogeneous;

$$ds^2 = dt^2 - a^2\left(t\right)\frac{d\vec{x}^2}{\left[1 + \frac{k}{4}\vec{x}^2\right]^2} \tag{2}$$

where k is the curvature parameter with $k = -1, 0, 1$ corresponding to open, flat, closed universes respectively and $a\left(t\right)$ is the scale factor of the universe. After applying the variational procedure to the action and assuming $\phi = \phi\left(t\right)$ and energy momentum tensor of matter and radiation excluding ϕ is in the perfect fluid form of $T^\mu_\nu = diag\left(\rho, -p, -p, -p\right)$ where ρ is the energy density and p is the pressure and also noting that the right hand side of the ϕ equation must be zero in accordance with our previous argument on L_M being independent of ϕ, the field equations reduce to (dot denotes $\frac{d}{dt}$)

$$\frac{3}{4\omega}\phi^2\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) - \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 + \frac{3}{2\omega}\frac{\dot{a}}{a}\dot{\phi}\phi = \rho \tag{3}$$

$$\frac{-1}{4\omega}\phi^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) - \frac{1}{\omega}\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{1}{2\omega}\ddot{\phi}\phi - \left(\frac{1}{2} + \frac{1}{2\omega}\right)\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 = p \tag{4}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\phi + \left[m^2 - \frac{3}{2\omega}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\right]\phi = 0 \tag{5}$$

Primordial inflation: In the first step of this analysis, we start to solve the field equations (3-5) for an empty-static universe by setting $\dot{a} = 0$ and $p = \rho = 0$ and get the following solutions;

$$\phi = \phi_o e^{\alpha t} \tag{6}$$

$$a = a_* = cst \tag{7}$$

$$k = 1 \text{ (closed universe)} \tag{8}$$

where

$$\alpha^2 = m^2\left(\frac{\omega}{3 + 2\omega}\right) \tag{9}$$

$$a_*^2 = \frac{1}{m^2}\left(\frac{3+2\omega}{3+3\omega}\right)\left(\frac{3}{2\omega}\right). \tag{10}$$

From these (6-8) solutions we see that ϕ evolves exponentially with parameter α and on the other hand a_* is the constant size of this static universe. We regard that only the closed universe solution is possible as a positive aspect of this solution since homogeneity of the universe only makes sense if a closed universe undergoes big-bang. Let us note that, since two variables $\phi(t)$ and $a(t)$ satisfy the three equations (3-5), these solutions (6, 7) are expected to be stable. To prove this stability we impose the size of the universe a and the field ϕ to be a function of time t as follows;

$$a = a_*\left(1 + \varepsilon b(t)\right) \tag{11}$$

$$\phi = e^{\alpha t}\left(1 + \varepsilon \psi(t)\right) \tag{12}$$

where ε is the perturbation factor ($\varepsilon \ll 1$) and $b(t)$, $\psi(t)$ are perturbation functions of $a(t)$ and $\phi(t)$ respectively. We get the following differential equations by using (3-5) and equalities (9, 10)

$$\dot{\psi}(t) - \left(\frac{3}{2\omega}\right)\dot{b}(t) + \left(\frac{3}{2\omega\alpha a_*^2}\right)b(t) = 0 \tag{13}$$

$$\ddot{\psi}(t) + (4+2\omega)\alpha\dot{\psi}(t) + \ddot{b}(t) + 2\alpha\dot{b}(t) - \frac{b(t)}{a_*^2} = 0 \tag{14}$$

$$\ddot{\psi}(t) + 2\alpha\dot{\psi}(t) + 3\alpha\dot{b}(t) - \frac{3}{2\omega}\ddot{b} + \left(\frac{3}{\omega a_*^2}\right)b = 0 \tag{15}$$

Solving (13-15) simultaneously gives us that $\dot{b} = 0$ is the only solution which implies $\ddot{b} = 0$ and $\dot{\psi} = 0$ and $\ddot{\psi} = 0$. Namely, this closed universe vacuum solution where $a = a_* = cst$ and $\phi \sim e^{\alpha t}$ is stable. We now investigate how the presence of radiation changes the behavior of the universe compared to this stable solution. To this end we solve (5) for $a(t)$ by keeping $\phi \sim e^{\alpha t}$. By changing to the variable $a^2(t) = \theta(t)$, (5) turns out to be the following second order differential equation,

$$\frac{-3}{4\omega}\ddot{\theta} + \frac{3\alpha}{2}\dot{\theta} + \left(\alpha^2 + m^2\right)\theta = \frac{3}{2\omega} \tag{16}$$

with the following solution for $\theta(t)$,

$$\theta(t) = a^2(t) = \left[\left(\frac{3}{2\omega}\right)\left(\frac{1}{\alpha^2+m^2}\right) + c_1 e^{\beta_1 t} + c_2 e^{\beta_2 t}\right] \tag{17}$$

where

$$\beta_{1,2} = \frac{\left[\frac{3\alpha}{2} \pm \sqrt{\frac{9}{4}\alpha^2 + \frac{3}{\omega}\left(\alpha^2+m^2\right)}\right]}{\frac{3}{2\omega}} \tag{18}$$

and $\beta_1 < 0$, $\beta_2 > 0$, c_1 and c_2 are integration constants. Now, if we define the big-bang time as the limit when $t \to 0$ and $\omega \gg 1$, we get $\alpha \simeq \frac{m}{\sqrt{2}}$, $\beta_1 \simeq -2\alpha$, $\beta_2 \simeq 2\omega\alpha$. Equation (17) when expanded about $t = 0$ becomes

$$\theta(t) = a^2(t) = a_*^2 \left(1 + c_1 + c_2 - 2c_1\alpha t + 2c_2\alpha\omega t\right) \tag{19}$$

with the constraint $1 + c_1 + c_2 = 0$ since $a^2 \sim t$ as $t \to 0$. Thus, we end up with the general solution for the scale size of the universe in the primordial inflation regime with $\omega \gg 1$ as;

$$a^2(t) = a_*^2 \left[1 - (1 + c)\, e^{-2\alpha t} + c e^{2\alpha\omega t}\right]. \tag{20}$$

We also check that when we substitute $\phi \sim e^{\alpha t}$ and $a \sim \sqrt{t}$ into (3-5) then the equation of state $p = \frac{1}{3}\rho$ is satisfied automatically as expected. In the light of this analysis and the results, we see that the stable-empty universe solution (6) and the general solution for the scale size of the universe (20) in the primordial regime are consistent in the sense that as $t \to 0$, $a(t) \sim \sqrt{t}$ and also as $t \gtrsim 0$ immediately primordial rapid inflation starts up with $a(t) \sim e^{\alpha\omega t}$.

Asymptotic inflation: In this section, we analyze how much today's universe is far from asymptotic inflation by considering the case of slowly expanding empty universe ($\rho = p = 0$) except the ϕ field in it. Since the considered universe should be big, we ignore the curvature parameter k/a^2 as $a(t)$ increases with the expansion of the universe. Under these considerations, in analogy with the previous section, we put $a = e^{\tilde{\beta}t}$ and $\phi = e^{\tilde{\alpha}t}$ into (3-5) where $\tilde{\beta}, \tilde{\alpha}$ are new constants to be determined and search for a stable solution. We get the following coupled equations for $\tilde{\beta}$ and $\tilde{\alpha}$;

$$\tilde{\beta}^2 - \frac{2}{3}\omega\tilde{\alpha}^2 + 2\tilde{\beta}\tilde{\alpha} - \frac{2\omega}{3}m^2 = 0 \tag{21}$$

$$\tilde{\beta}^2 + \left(\frac{4}{3} + \frac{2}{3}\omega\right)\tilde{\alpha}^2 + \frac{4}{3}\tilde{\beta}\tilde{\alpha} - \frac{2\omega}{3}m^2 = 0 \tag{22}$$

$$\tilde{\beta}^2 - \frac{\omega}{3}\tilde{\alpha}^2 - \omega\tilde{\beta}\tilde{\alpha} - \frac{\omega}{3}m^2 = 0. \tag{23}$$

These equations have the solution;

$$\tilde{\beta} = 2m(1 + \omega)\left(\frac{\omega}{6\omega^2 + 17\omega + 12}\right)^{1/2} \approx \sqrt{\frac{4\omega}{3}}\alpha \tag{24}$$

$$\tilde{\alpha} = m\left(\frac{\omega}{6\omega^2 + 17\omega + 12}\right)^{1/2} \approx \sqrt{\frac{1}{3\omega}}\alpha \tag{25}$$

where the approximations are again for $\omega \gg 1$ so that $m \approx \sqrt{2}\alpha$. We see that although the primordial inflation parameter is $\omega\alpha$, the asymptotic inflation parameter is $\sqrt{\frac{4\omega}{3}}\alpha$, a factor $\sqrt{\omega}$ less than the primordial inflation parameter.

Note that although there is an experimental lower bound on ω, there is no upper bound [5]. Hence in Brans-Dicke cosmology, the asymptotic inflation can be as small as one wishes compared to the primordial inflation. Now, we consider the case where the universe is closed ($k = 1$) and matter dominated $p \approx 0$. Defining the rate of change in ϕ as $F(a) = \dot{\phi}/\phi$ and Hubble parameter as $H(a) = \dot{a}/a$, we rewrite the right hand side of the field equations(3-5) in terms of H, F and their derivatives with respect to a (prime denotes $\frac{d}{da}$)

$$H^2 - \frac{2\omega}{3}F^2 + 2HF + \frac{1}{a^2} - \frac{2\omega}{3}m^2 = (\frac{4\omega}{3})\ \frac{\rho}{\phi^2} \tag{26}$$

$$H^2 + \left(\frac{4}{3} + \frac{2\omega}{3}\right)F^2 + \frac{4}{3}HF + \frac{2a}{3}\left(H\acute{H} + F\acute{F}\right) + \frac{1}{3a^2} - \frac{2\omega}{3}m^2 = (\frac{-4\omega}{3})\frac{p}{\phi^2} = 0 \tag{27}$$

$$H^2 - \frac{\omega}{3}F^2 - \omega HF + a\left(\frac{H\acute{H}}{2} - \frac{\omega}{3}F\acute{F}\right) + \frac{1}{2a^2} - \frac{\omega}{3}m^2 = 0. \tag{28}$$

Expanding $F(a)$ and $H(a)$ in powers of $\left(\frac{1}{a}\right)$ up to a third order,

$$H(a) = H_\infty + \frac{H_2}{a^2} + \frac{H_3}{a^3} \tag{29}$$

$$F(a) = F_\infty + \frac{F_2}{a^2} + \frac{F_3}{a^3} \tag{30}$$

and putting them into (26-28) we get the perturbation constants defined above for ($\omega \gg 1$) :

$$H_\infty \approx \tilde{\beta} \approx \sqrt{\frac{4\omega}{3}}\alpha \tag{31}$$

$$H_2 \approx \frac{-1}{2H_\infty} \approx -\sqrt{\frac{3}{16\omega}}\frac{1}{\alpha} \tag{32}$$

$$F_\infty \approx \tilde{\alpha} \approx \sqrt{\frac{1}{3\omega}}\alpha \tag{33}$$

$$F_2 \approx \frac{-1}{\omega H_\infty} \approx -\sqrt{\frac{3}{4\omega^3}}\ \frac{1}{\alpha} \tag{34}$$

$$H_3 \approx \frac{2\omega}{3}F_3. \tag{35}$$

To estimate H_3 and F_3, we use the classical Friedman formula which is used for fitting observations of Hubble parameter to density parameter Ω;

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_R\left(\frac{a_0}{a}\right)^2 + \Omega_M\left(\frac{a_0}{a}\right)^3 \tag{36}$$

where Ω_Λ is the vacuum density parameter, Ω_R is the curvature density parameter, Ω_M is the matter density parameter, a_0 is the present scale size of the universe and H_0 is the present Hubble parameter. Using (31, 32), we rearrange (29) leaving H_3 as a free parameter and put into (36) to get

$$\Omega_\Lambda \approx \frac{4}{3}\omega\alpha^2\left(\frac{4}{3}\omega\alpha^2 - \frac{1}{a_0^2} + \frac{4\sqrt{3}}{3a_0^3}\sqrt{\omega}\alpha H_3\right)^{-\frac{1}{2}} \tag{37}$$

$$\Omega_R \approx -\frac{1}{a_0^2}\left(\frac{4}{3}\omega\alpha^2 - \frac{1}{a_0^2} + \frac{4\sqrt{3}}{3a_0^3}\sqrt{\omega}\alpha H_3\right)^{-\frac{1}{2}} \tag{38}$$

$$\Omega_M \approx \frac{4\sqrt{3}\alpha H_3\sqrt{\omega}}{3a_0^3}\left(\frac{4}{3}\omega\alpha^2 - \frac{1}{a_0^2} + \frac{4\sqrt{3}}{3a_0^3}\sqrt{\omega}\alpha H_3\right)^{-\frac{1}{2}}. \tag{39}$$

Using the present observational result [6] $\Omega_M \approx 0.25$ and $\Omega_\Lambda \approx 0.75$, we find H_3 and F_3 to be,

$$H_3 \approx \sqrt{\frac{\omega}{27}}a_0^3\alpha, \ \ (\omega \gg 1) \tag{40}$$

$$F_3 \approx \sqrt{\frac{1}{12\omega}}a_0^3\alpha, \ \ (\omega \gg 1). \tag{41}$$

After finding the perturbation constants explicitly for H and F, we also find it worthy to determine how the Hubble parameter $H(a) = \dot{a}/a$ and the time variation of G, where G is the time dependent value of the gravitational constant, change in the primordial and asymptotic regimes compared to their present values. To do so, we use the fact that since Brans-Dicke gravity becomes identical to Einstein gravity as ω approaches infinity , the kinetic term for the scalar field $\frac{1}{8\omega}\phi^2$ in the action (1) will be the same as that of the term $\frac{1}{16\pi G}$ in the Hilbert-Einstein action. Using this fact we get the relation between the scalar field ϕ and G as

$$G^{\ -1} = \frac{2\pi\phi^2}{\omega}. \tag{42}$$

Then, putting equations (31), (32), (40) into (29) for the present value of the Hubble constant $H = H_0$ and for the present value of the scale factor of the universe $a = a_0$, gives us the expansion parameter α, in the magnitude of when $\omega \gg 1$ as,

$$\alpha \approx \frac{0.7}{\sqrt{\omega}}H_0. \tag{43}$$

Similarly, using (42) and putting equations (33), (34), (41) into (30) for the present value of the scale factor of the universe $a = a_0$, gives us the magnitude of the present value of the parameter $\dot{G}/G$ when $\omega \gg 1$ as,

$$\left(\frac{\dot{G}}{G}\right)_0 \approx -3\alpha \approx \frac{-2.2}{\sqrt{\omega}}\ H_0. \tag{44}$$

On the other hand, since $\phi \approx e^{\alpha t}$, $a \approx e^{\alpha\omega t}$ and $\phi \approx e^{\sqrt{\frac{1}{3\omega}}\alpha t}$, $a \approx e^{\sqrt{\frac{4\omega}{3}}\alpha t}$ in the primordial and asymptotic regimes respectively, using (42) we get the parameter $\dot{G}/G$ and Hubble parameter $H = \dot{a}/a$ in these regimes as

$$\left(\frac{\dot{G}}{G}\right)_{primordial} \approx -2\alpha \approx \frac{-1.5}{\sqrt{\omega}} H_0 \tag{45}$$

$$\left(\frac{\dot{G}}{G}\right)_{asymptotic} \approx \frac{-2}{\sqrt{3\omega}}\alpha \approx \frac{-0.9}{\omega} H_0 \tag{46}$$

$$(H)_{primordial} \approx \omega\alpha \approx 0.7\sqrt{\omega} H_0 \tag{47}$$

$$(H)_{asymptotic} \approx \sqrt{\frac{4\omega}{3}}\alpha \approx 0.9 H_0. \tag{48}$$

We note that the newest measurement [6] of Ω_Λ and Ω_M has been used as input to derive these results.One interesting feature is that the predicted present day and primordial values of $|\dot{G}/G|$ are comparable whereas the asymptotic value is much smaller. In any case a measurement of $\dot{G}/G$ will be crucial in determining the Brans-Dicke parameter ω.

The fact that the ratio of the primordial and asymptotic inflation parameters is proportional to $\sqrt{\omega}$ is the appealing feature of Brans-Dicke cosmology. Thus the recent measurements which imply that in today's universe $\Omega_\Lambda \neq 0$ require $1/\omega \neq 0$ and make this model attractive.

References

[1] A. H. Guth, Phys. Rev. D 23, 347 (1981).

[2] D. La and P. J. Steinhardt, Phys. Rev. Lett. 62, 376 (1989).

[3] C. Brans and C. H. Dicke, Phys. Rev. 24, 925 (1961).

[4] C. M. Will, Phys. Rep. 113, 345 (1984).

[5] R. D. Reasenberg et al., Astrophys. J. 234, 925 (1961)

[6] R. A. Knop et al., ApJ 598:102-137, 2003