

Review

# Uncertainty Relations and Possible Experience

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**Abstract:** The uncertainty principle can be understood as a condition of joint indeterminacy of classes of properties in quantum theory. The mathematical expressions most closely associated with this principle have been the uncertainty relations, various inequalities exemplified by the well known expression regarding position and momentum introduced by Heisenberg. Here, recent work involving a new sort of “logical” indeterminacy principle and associated relations introduced by Pitowsky, expressable directly in terms of probabilities of outcomes of measurements of sharp quantum observables, is reviewed and its quantum nature is discussed. These novel relations are derivable from Boolean “conditions of possible experience” of the quantum realm and have been considered both as fundamentally logical and as fundamentally geometrical. This work focuses on the relationship of indeterminacy to the propositions regarding the values of discrete, sharp observables of quantum systems. Here, reasons for favoring each of these two positions are considered. Finally, with an eye toward future research related to indeterminacy relations, further novel approaches grounded in category theory and intended to capture and reconceptualize the complementarity characteristics of quantum propositions are discussed in relation to the former.

**Keywords:** quantum mechanics; uncertainty relations; quantum logic; boolean logic; complementarity

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## 1. Introduction

Quantum mechanics is a theory supplying probabilities of outcomes of measurements on physical systems and is most commonly employed at the atomic scale. These probabilities are strictly less than one in cases where pairs of properties represented by noncommuting operators are measured successively. In these cases, the quantities also cannot be measured simultaneously, that is, are strictly complementary. Moreover, it has been shown that a formulation of the joint indeterminacy hypothesis of Heisenberg [1] together with an axiomatic formulation of complementarity rigorously imply the existence of incompatible observables [2] and so exclude the classical mechanical description of a physical system within the quite general,  $(O, S, p)$  formulation of general physical theory considered by Mackey [3,4].

In the  $(O, S, p)$  formulation, one assumes that, for each physical system, one can associate with the set of all its observables  $O$  and the set of all its states  $S$  as a function  $p: O \times S \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ , where  $\mathcal{B}(\mathbb{R})$  is the set of all Borel subsets of the real line that provides a probability connecting theory with experience. Quantitative relations expressing the relationship between indeterminacy and limitations on joint measurements are commonly referred to as uncertainty relations. The quantum indeterminacy principle, also known as the “uncertainty principle” is a statement about the associated indeterminacy of incompatible sharp properties (projection valued measures, PVMs) in quantum mechanics rather than, say, merely the epistemic uncertainty regarding an independent quantity. In particular, it is a statement to the effect that the associated properties, jointly considered, are

objectively indefinite [5]. The indeterminacy principle can be contrasted to the determinacy principle, that the magnitude of each continuous quantity is determined in reality by a real number, as is typically assumed in classical mechanics [6].

The trade-off relationships between the definiteness of the preparation or measurement of two quantities are consistent with the notion that simultaneous or sequential determination of their values requires a nonzero amount of unsharpness in the quantum observables involved, where unsharp observables are those represented by positive operator valued measures (POVMs) that are not projection-valued, and unsharpness quantifies the extent to which such an observable differs from the corresponding sharp one [7]. These relationships are thus more than simply formal expressions of complementarity; they represent precise limits within which joint measurements can be accomplished [8] and can be seen as consequences of the noncommutativity of these observables. They are thus typically viewed as highlighting “an important nonclassical feature of quantum mechanics” cf. [9].

Three forms of traditional uncertainty relations or, more precisely, indeterminacy relation related to that introduced by Heisenberg [1] have been identified, namely state preparation indeterminacy, joint-measurement inaccuracy and accuracy–disturbance trade-off, which have often been confused or conflated [9]. It was the last that Heisenberg initially engaged, in the case of continuous position and momentum observables, by considering a sequence of measurements in which a measurement of sharp momentum is followed by an approximate position measurement that finally yields an unsharp momentum, which, for clarity, can be written

$$\delta q D p \geq \hbar/2 \quad (1)$$

where  $\delta q$  indicates the accuracy of the position measurement and  $D p$  is a measure of the disturbance of an initially sharp momentum value. More generally, the disturbance of the distribution of an observable  $B$  through a measurement of observable  $A$  can be expressed as the change in the variance of  $B$ , which could be arranged initially to be zero, after the selective measurement of  $A$  [10].

In recent years, an additional perspective on joint indeterminacy has been explored that also yields inequalities, which, in contrast to traditional uncertainty relations, directly involve probabilities of sharp measurement outcomes. These inequalities have been shown to arise from what Boole called “conditions on possible experience” [11]. They can be considered indeterminacy relations because, among other things, they express quantitatively the complementarity of some pairs of alternatively measurable quantities [12]. Their formulation depends by construction on the existence of classical probability distributions [13] of the sort traditionally associated with a lack of information about propositions regarding possible events [14] but are obtained for systems in *pure* states. This raises the question of the basis of the quantum character of these inequalities, unlike the original indeterminacy relations, which have always been seen as distinctly quantum in nature, no matter the form in which they are presented.

In this brief review of recent work, it is seen that these novel indeterminacy inequalities are nonclassical despite their relationship to Boole’s conditions of possible experience because the propositions involved are restricted by the geometry of quantum physical state space, which can be presented as a Hilbert space. Finally, recent work reconceptualizing quantum states spaces via category theory is described that offers a promising new direction of research differing from the quantum logical approach in which the indeterminacy inequalities, which are our main focus here, were first presented.

## 2. Indeterminacy and Possible Experience

Although the indeterminacy relations have most often, though not exclusively, been considered for continuous variables, as in the archetypical example, Inequality (1), here, we are concerned primarily with discrete quantities. The newer quantum indeterminacy inequalities considered here are

also ones naturally constructed for individual systems and are not violated by microscopic systems, in contrast with Bell-type inequalities, which are of a classical character, involve extended compound systems, and can be violated by the behavior of microscopic systems.

The novel sort of indeterminacy relation of interest here was first set out by Itamar Pitowsky who sought to connect with quantum mechanics the result of Boole that identifies necessary and sufficient conditions for a set of rational numbers  $p_1, p_2, \dots, p_n$  to represent properly the probabilities, considered (relative) frequencies, of the occurrence of a set of  $n$  logically connected events  $E_1, \dots, E_n$  [11]. Pitowsky noted, in particular, that Boole [13] had identified necessary and sufficient conditions for his “conditions of possible experience”

$$p_i = \text{prob}(E_i) \quad i = 1, 2, \dots, n \quad (2)$$

The most important characteristic of these conditions is that they are entirely of the form of linear inequalities or equalities in  $p_1, p_2, \dots, p_n$ . As others have, e.g., [15], Pitowsky also noted that the (inherently classical) Bell-type inequalities, e.g., [16], can be shown to follow from such conditions on joint probability distributions that are based entirely on basic elementary assumptions of classical probability theory and/or propositional logic, presumably indicating their independence of some or most of the particulars of physics [17,18]. Quantum mechanics cannot provide the classical probability distributions for the expectation values of all quantum observables required for satisfaction of Bell’s inequality for joint systems. This is one reason that these probabilities, considered as relative frequencies of microphysical events and corresponding to distinct samples, sometimes violate some of Boole’s conditions associated with these events [19]. In obtaining indeterminacy relations, Pitowsky turned his attention also to the application of the result of Boole to single-component systems.

Boole had noted that, if the events under consideration are entirely independent, then the fractions corresponding to probabilities might be constrained only by the conditions:

$$p_i \geq 0, p_i \leq 1 \quad (3)$$

Boole showed, however, that the expression for sets within possible experience must take the simple form:

$$a + \sum_i^N a_i p_i \geq 0 \quad (4)$$

where  $a, a_i$  are constants involving the logical relations that constrain them [13]. It is this set of classical conditions on probabilities that were shown by Pitowsky, in the case of probabilities of correlation, to lie within  $n$ -dimensional polytopes [17]. Recall that a polytope is the convex hull  $\text{conv}(v_1, \dots, v_n)$  of a finite number of points in  $\mathbb{R}^d$ , that is, the set of all convex combinations of its points. These polytopes are of dimensions given by the number of the events involved and have facets determined by these equations. Any violation of these conditions is manifested geometrically by the location of points (corresponding to probabilities) outside of the polytope dictated by the conditions (4).

Pitowsky first noted how the conditions on possible experience can be methodically constructed from the logical relations among sets of possible events, thereby revealing the relationship between Boole’s original problem concerning probabilities and propositional logic, as follows [11]. Pitowsky considered, as the simplest example, a pair of events  $E_1, E_2$  having relative frequencies  $p_1, p_2$ , taking  $p_{12}$  to denote the frequency of the joint event  $E_1 \cap E_2$ . He showed that one can then find Boole’s conditions on the numbers  $p_1, p_2$  and  $p_{12}$ , first seeing that these have the following relations:

$$p_1 \geq p_{12}, p_2 \geq p_{12}, p_{12} \geq 0 \quad (5)$$

The frequency of  $E_1 \cup E_2$  is then  $p_1 + p_2 - p_{12}$  with

$$p_1 + p_2 - p_{12} \leq 1 \quad (6)$$

He then pointed out that the relations (5) and (6) together are also necessary and sufficient for these numbers to be the frequencies of two events and their joint event, that is, Boole's conditions of possible experience there.

One has a corresponding three-dimensional space of vectors  $(p_1, p_2, p_{12})$  that can be viewed as a convex polytope with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(1, 1, 1)$  as well. This is so because every convex polytope in a Euclidean space is describable via its facets; in such a description, a given vector is an element of the polytope if and only if its coordinates satisfy a set of linear inequalities representing the spaces the intersections of which defines the polytope. The connection with logic was made by Pitowsky through the truth table for the two propositions corresponding to the two events, which has rows corresponding to the vectors for these vertices [11].

The above is the simplest example of Pitowsky's general algorithm for arriving at Boole's conditions in any given case; the algorithm for the general case is the following. Given the probabilities of a number of logically connected events, for example, the relative frequencies as exemplified above, one considers the logical connections of the propositional (that is, Boolean) formulas and its corresponding truth table. The convex hull of the rows of this table provides a polytope for which there are corresponding inequalities (possibly including equalities), which are exactly Boole's conditions of possible experience for the set of events considered. The existence of this method for obtaining inequalities involving individual probabilities and correlations, because the inequalities are derived beginning from basic principles of classical, that is, Boolean logic might suggest that the resulting uncertainty-type relations are classical results. However, so long as the probabilities entering into the conditions of possible experience derive from a common sample, the resulting expressions will not conflict with the behavior of quantum systems [11]. These points regarding the nature of these inequalities are discussed further below.

In order to provide a specific example of this new class of indeterminacy relation, Pitowsky considered a scenario in which there is a set of measurements known to have as outcomes 0 and 1, such as the practical example of measurements on a squared value  $S_i^2$  of the component of spin along orthogonal spatial directions (for a spin-1 system) [12]. In this case, the basic operators  $S_i$  do not commute and so cannot be precisely measured simultaneously while their squares  $S_i^2$  do. Note that these observables play a prominent role in the "free will theorems" of Conway and Kochen [20,21]—cf. [22], Chapter 4, for an extended discussion of these theorems in relation to probability as indeterminacy.  $S_i^2 = 2I$ , where  $I$  is the identity, so that in a simultaneous measurement of these spin-squared operators, one and only one of these observables will have the value 0, while the others take value 1. To connect this with experience, note that there is a measurement that can be performed in practice via the measurement of the observable  $H = S_x^2 - S_y^2$  using an electrostatic field with possible outcomes 1, 0, and  $-1$ ; these three values correspond to the values of  $S_y^2$ ,  $S_z^2$  and  $S_x^2$ , respectively, being 0.

This spin-squared measurement example is an illustration of the general situation corresponding to measurements with a triple of possible outcome events; the possible results are:

$$\langle E1, E2, F2 \rangle , \langle E1, E3, F3 \rangle , \langle E2, E4, E6 \rangle \quad (7)$$

$$\langle E3, E5, E7 \rangle , \langle E6, E7, F \rangle , \langle E4, E8, F4 \rangle \quad (8)$$

$$\langle E5, E8, F5 \rangle \quad (9)$$

where the events that appear in more than one measurement are indicated by  $E_i$  and those that appear in only one triple are indicated by  $F_i$  [12]. When an event  $E_j$  is shared, its complement  $\bar{E}_j$  in the set is also implicitly shared from the logical point of view. Each of these situations corresponds to a Boolean algebra. Recall that, in general, a *Boolean algebra*  $B_n$  is an algebraic structure given by the collection of  $2^n$  subsets of the set  $I = \{1, 2, \dots, n\}$  and three operations under which it is closed: the two binary operations of union ( $\cup$ ) and intersection ( $\cap$ ), and a unary operation, complementation ( $\neg$ ). In addition to there being complements (and hence the null set  $\emptyset$  being an element), the following conditions hold of a Boolean algebra: (i) commutativity:  $S \cup T = T \cup S$  and  $S \cap T = T \cap S$ ; (ii) associativity:

$S \cup (T \cup U) = (S \cup T) \cup U$  and  $S \cap (T \cap U) = (S \cap T) \cap U$ ; (iii) distributivity:  $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$  and  $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$ ; (iv)  $\neg\emptyset = I$ ,  $\neg I = \emptyset$ ,  $S \cap \neg S = \emptyset$ ,  $S \cap \neg S = I$ ,  $\neg(\neg S) = S$ , for all its elements  $S, T, U$ . In the current setting,  $\neg E$  corresponds to  $\bar{E}$ ,  $I$  corresponds to 1 and  $\emptyset$  corresponds to 0. Thus, above, a logical complement, e.g.,  $\bar{E}_1$ , within a triple, refers simply to the disjunction of the remaining two possible outcomes in any triple in which the event occurs, so that  $\bar{E}_1 = E_2 \cup F_2$ , etc. To connect this situation with experience, note that if  $x'$  and  $y'$  are two orthogonal directions such that  $x, y, z$  and  $x', y', z'$  form two orthogonal triples having a common  $z$ -direction, then operators  $H$  (above) and  $H' = S_{x'}^2 - S_{y'}^2$  will not commute, but the Hilbert-eigenspace corresponding to the eigenvalues 0 of the two will be the same.

In accordance with the principle of *noncontextuality of probability*—the requirement that probability assignments do not depend on the outcomes of measurements of other observables that might be measured at the same time [23,24]—the same probability is to be assigned to each event above in all cases it which it appears. Following Pitowsky, one notes that the sum of the probabilities in each triple of possible outcomes must sum to unity, so that

$$p(E_1) + p(E_2) + p(F_2) = 1 \quad (10)$$

$$p(E_1) + p(E_3) + p(F_3) = 1 \quad (11)$$

$$p(E_2) + p(E_4) + p(E_6) = 1 \quad (12)$$

$$p(E_3) + p(E_5) + p(E_7) = 1 \quad (13)$$

$$p(E_6) + p(E_7) + p(F_1) = 1 \quad (14)$$

$$p(E_4) + p(E_8) + p(F_4) = 1 \quad (15)$$

$$p(E_5) + p(E_8) + p(F_5) = 1 \quad (16)$$

Then, given the requirements on probability shown above as the Inequalities (3), Equations (10)–(16) imply, for example, the indeterminacy inequality [12]

$$p(E_1) + p(E_8) \leq 3/2 \quad (17)$$

As one of the outcomes  $E_1$  and  $E_8$ —which cannot arise as alternative outcomes of the same measurement—becomes more certain (as expressed by an increased probability of occurrence), the other becomes less certain (as expressed by an decreased probability of occurrence). Inequality (17) is seen to express an indeterminacy (uncertainty) relation; it expresses quantitatively that the alternative measurements likelihoods are complementary [12].

Thus, one sees that an indeterminacy relation arises beginning from the consideration of a collection of sets of alternative events in Boolean logic within single measurements with the two events involved in the inequality resulting from a collection of such measurements as outcomes of different measurements.

### 3. Logic and Experience in Quantum Mechanics

The novel type indeterminacy inequality given as Inequality (17) differs *prima facie* from the traditional type shown in Inequality (1) in that it was derived beginning from rules of Boolean logic applied to sets of measurements rather than, for example, a derivation involving the noncommutativity of the Hermitian operators representing the observables of quantum mechanics. However, the mathematical formalism of quantum mechanics has a natural translation into logic, as has been demonstrated throughout the long development of logics in the quantum setting originating in the work of Birkhoff and von Neumann [25], which follows from the fact that every one-dimensional subspace of the Hilbert space of a system corresponds to an atomic proposition describing a physical possibility and orthogonality in Hilbert space of subspaces corresponds to negation, cf. [26–31].

The axiom used by the Birkhoff and von Neumann differing from those in classical logic is *irreducibility*, that is, for some event  $z$  and all events  $x$ :

$$x = (x \cap z) \cup (x \cap z^\perp) \Rightarrow z = 0 \text{ or } z = 1 \quad (18)$$

where  $\perp$  indicates negation; in contrast, in classical logic, it is assumed for *all* pairs of events  $x$  and  $z$  that

$$x = (x \cap z) \cup (x \cap z^\perp) \quad (19)$$

(reducibility). This difference can be seen as underlying quantum indeterminism, in that, intuitively, irreducibility reflects the uncertainty relations: when  $x$  cannot be presented as the union of its intersection with  $z$  and its intersection with the complement of  $z$ , then  $x$  and  $z$  cannot be assigned definite values at the same time [32].

Pitowsky viewed the basis of the novel type of indeterminacy relation as involving logic in the quantum setting in the following way. The starting point of his broad view of the foundations of quantum mechanics was to connect the axioms of Birkhoff and von Neumann axioms with probability via Gleason's theorem. This theorem is the result that the only probability measure that can be defined noncontextually on a Hilbert space of dimension three or greater yields precisely Born's probability rule [33]. For the three-dimensional case, the generalized probability measures considered in relation to the closed linear subspaces of Hilbert space involve a map,  $f$ , from the closed linear subspaces of  $\mathcal{H}^3$  to the closed unit interval, satisfying the conditions:

$$fa + fb \leq 1 \quad (20)$$

when  $a \perp b$  and

$$fa + fb + fc = 1 \quad (21)$$

when the rays  $a, b, c$  are all mutually orthogonal. A generalized two-valued measure takes values in  $\{0, 1\}$ ; such measures can be interpreted as probability measures when the rays are taken to represent propositions, which, if 0 and 1 are interpreted as false and true, respectively, provide a generalized truth-value assignment. Pitowsky argued that because "quantum 'logic' dictates the probabilistic structure of quantum mechanics" so that the structure of quantum states can be viewed as determined by the algebraic structure of the closed subspaces of Hilbert space, the relationship between the truth values of one proposition and those of another depends on it.

In this picture, quantum states correspond to probability distributions over the set of atomic propositions that appear in indeterminacy inequalities as described above. Thus, the indeterminacy inequalities of Pitowsky are to be viewed as examples of a quantum "logical indeterminacy principle" in that they depend on the negation relation between propositions on which quantum logic is developed, whatever the logical novelties accompanying it. Pitowsky also noted that Gleason's construction can be considered in terms of orthogonality graphs, which allowed him to demonstrate that Gleason's theorem can be considered combinatorial in nature. Recall that a graph, in the sense relevant here, is an ordered pair  $G = (V, E)$  comprising a set  $V$  of vertices, nodes or points together with a set  $E$  of edges, arcs or lines, which are two-element subsets of  $V$ . Pitowsky showed, in particular, that Gleason's demonstration depends only on a finite graph structure [34].

Given any two distinct nonorthogonal rays, one can construct a finite set of rays that contains them and is such that no probability distribution on that set assigns to both of them probability zero or one unless they are both false. Pitowsky formalized the corresponding logical indeterminacy principle as follows:

*The Indeterminacy Principle:* Let  $a$  and  $b$  be two distinct nonorthogonal unit vectors.

Then, any quantum state  $f$  must satisfy  $fa + fb \in \{0, 1\}$  only if  $fa = fb = 0$ .

This indicates that no quantum state can determine two propositions corresponding to different Hilbert space eigenbases unless they are both attributed probability 0. For the cases of Hilbert space dimension greater than 2, this principle is a corollary of Gleason's theorem [34]. This shows that the principle of indeterminacy, a relationship between probabilities, can also be converted into a theorem of propositional logic in the above quantum setting, often called quantum logic.

William Demopoulos criticized Pitowsky's fundamentally logic-based approach to possible experience by focusing on explanation. Demopoulos argued that, in the case of quantum mechanics, it is not the state as a collection of true propositions about a quantum system that explains its statistical predictions but, instead, "the systematic representation of the 'distribution' of the propositions formulable about the system" that does the explanation [35], so that it is the latter that is of primary importance. Accordingly, his view of Pitowsky's indeterminacy principle is that its real significance lies in its showing that quantum states yield precise knowledge of the truth values of complementary propositions only when we can know neither proposition. Demopoulos argues that this is so because the quantum probability assignments depend on the geometry of the propositions. His interpretative position thus differs from the quantum logical sort of interpretation of quantum mechanics, a central idea of which is to explain the statistical relations of the theory via a logical structure as a possibility structure on the basis of which the totality of all possible statistical states is given, justified by Gleason's theorem. For Demopoulos, the "mixing of geometry and logic is a feature that sets quantum mechanics apart from classical mechanics," with an emphasis on the geometrical properties of quantum theory.

In more general terms, Demopoulos holds that quantum mechanics is to be viewed as a minimally *a priori* theory: elementary propositions are characterized by the minimal logical space (the free partial Boolean algebra) and the criterion of application for the property each contains. The criterion is an operational procedure indicating perfectly whether the property is present or absent [35]. Recall that a *partial Boolean algebra* is the union of a family of Boolean algebras having a common null and unit and obeys the following conditions: (i) the intersection of two algebras of the family is an algebra of the family; and (ii) if every pair of elements in  $\{P_1, \dots, P_n\}$  belongs to an algebra of the family, then there is an algebra containing all the  $P_i$ , for  $i = 1, \dots, n$ . It can be viewed in this way as a sort of "pasting together" of Boolean algebras. He singles out for special attention the partial Boolean algebra consisting of a family of four-element Boolean algebras with a continuum number of free generators such that any pair of algebras of the family share only their unit and null, precisely because its elementary propositions are logically independent and minimal nonzero elements of the algebra, and can be viewed as representing the possibility structure of quantum propositions, in contrast with classical ones. This is the partial Boolean algebra isomorphic to the partial Boolean algebra of rays through a point of the Euclidean plane having the following correspondences: the intersection of subspaces is the Boolean meet and the Boolean is the span of subspaces, with the complement being the orthogonal complement. *Freeness* is the property that every function on a free set of generators is uniquely extendible to a homomorphism into an arbitrary partial Boolean algebra.

Demopoulos uses the notion of freeness to connect knowledge and information with this geometry, invoking the following temporal "analogy". A quantum system is to be thought of as starting from a state that admits the possibility of possession of complete information of it and then to evolve into a state of information given by some higher dimensional partial Boolean algebra of propositions provided by the Hilbert-space formalism of the theory. That is, the initial state of information is given by a partial Boolean algebra that represents all the elementary propositions as being independent in the same way as algebra of subspaces of the two-dimensional case. He argues that, in the general case, the quantum probability assignments depend on the "geometry" of the propositions given by their representation in  $B(H^n)$  because "complete knowledge of the truth values of possible propositions regarding a system is not possible"; the information about one proposition regarding a quantum system that is relevant to that of another has a special geometric dependence which is encoded in the theory. Although this is logical, in that it has a mathematical representation that can be viewed formally as a generalization of the Boolean algebraic structures, Demopoulos is arguing that the new

indeterminacy relations depend on the underlying geometry of quantum probability and involve more than logic.

The new sort of indeterminacy relation produced by Pitowsky can then be considered quantum in character fundamentally because its instances are derived on the basis of orthogonality relations of Hilbert space, and this space differs geometrically from classical state space. Demopoulos considers the true significance of these relations to be as illustrations of the nature of the distribution of information about the properties of the world constituted by quantum systems that is reflected in this geometry. Returning then to the question first considered far above, namely that of why the logical uncertainty relations are always obeyed in the quantum domain, whereas the Bell inequalities can be violated there, one can see an answer in the fact that the Bell inequalities are fully classically derived and involve neither the relationship between propositions nor the constraints imposed on quantum mechanical probabilities by the geometry of the complex Hilbert space: Bell-type inequalities allow for the existence of probability distributions that are automatically precluded by the orthogonality relations of Hilbert space that enter into the quantum probabilities appearing in Pitowsky's relations.

Finally, let us look beyond the main topic of this review toward newer, related mathematical work, noting that Demopoulos' reaction to Pitowsky's new indeterminacy relations can be seen as part of an emerging re-engagement of the relationship between logic and experience at the fundamental level and noting that it is not the only manner of doing so. Most recently, others have explored this mathematical situation along two other, differing avenues by involving category theory.

Within the topos approach begun by Chris Isham and Jeremy Butterfield and continued with Andreas Doering and others, it was noted that there is an associated intuitionist logic that can be used that has the potential to avoid difficulties encountered in the quantum logical approach, one considering the formal language of quantum theory via the notion of Boolean contexts [36,37]. A topos is a category that can be considered a generalized universe of sets with an internal intuitionist logic offering the possibility of distributivity that promises greater structural similarity with classical theory. For example, one can proceed by defining an arrow from the state object (state space) to a quantity-value object generalizing the real line that parallels the traditional functions that map states to real values and introducing contexts as follows. One considers noncommutative spaces, for example  $\mathcal{A}$ , a noncommutative, unital  $C^*$ -algebra (or von Neumann algebra) of physical quantities of system  $S$  and the set  $\mathcal{V}(\mathcal{A})$  of nontrivial counital commutative  $C^*$  (or von Neumann) subalgebras, partially ordered under inclusion having elements that are the contexts, with  $\mathcal{V}(\mathcal{A})$  being the context category. It has been shown that it is possible to provide further structures such that, for each context, one has a local state space (a compact, Hausdorff space) on which all physical quantities can be given as continuous, real-valued functions. These local state spaces are then arranged so as to constitute a single global object, the spectral presheaf, which is a noncommutative space that serves as the full state space of the quantum system which, however, has no global elements, as one would expect given the Kochen–Specker theorem [38].

The second of these two new approaches is the categorial semantics approach engaged by Elias Zafiris and Vassilios Karakostas, where one considers Boolean frames: Boolean algebras are pasted together so as to reconstruct geometrically structures associated with quantum logic [39]. The main idea of this approach is the introduction of a topological covering scheme of a quantum event algebra of families of local Boolean logical frames; these frames provide local covers of a quantum event algebra via complete Boolean algebras. The local Boolean covers capture individually complementary features of a quantum algebra of events and collectively provide its categorical local decomposition in the terms of Boolean logical frames while requiring compatibility between overlapping local Boolean covers. In particular, one sets up a categorical adjunction between sheaves of variable local Boolean frames, which constitute a topos, and the category of quantum event algebras. It was shown by these investigators that the quantum event algebras can be provided object of truth values with which to assign truth values to propositions describing quantum systems [40].

#### 4. Conclusions

The introduction of quantum indeterminacy relations derived from conditions of possible experience is a significant step in the exploration of fundamental notions of quantum mechanics and their relationships to experience, logic and geometry. These “logical indeterminacy relations” were grounded by Pitowsky in such conditions, which were first formulated by Boole. Differently from the sort of reciprocity first captured by the Heisenberg-type uncertainty relations, and later explored by others in relation to unsharp measurement, the latter are used to find inequalities directly between probabilities of various outcomes of sharp measurements.

It was seen here that these novel inequalities are always obeyed, even though the Bell-type inequalities, also derivable via Boolean conditions of possible experience, can be violated by virtue of the relationship of the propositions involved in the former to Hilbert space geometry. It was also seen that, although these novel indeterminacy relations might be thought to be, in some sense, more classical than the Heisenberg relations due to the Boolean elements of their derivation, this is not so; again, the logical and geometrical properties of the Hilbert space formalism, within which quantum states and observables are definable, differ from those of classical states and properties sufficiently to provide the indeterminacy relations a nonclassical character.

Finally, with an eye toward future developments related to these indeterminacy relations, two novel approaches intended to capture and reconceptualize the complementarity between quantum propositions, grounded in category theory, were pointed out. These are the topos approach, which involves intuitionist logic, offering the possibility of distributivity and bearing greater structural similarity with classical theory, and the categorical semantical approach, which invokes Boolean frames with Boolean algebras being used to reconstruct geometrically structures associated with quantum logic.

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