Unified Subdivision Generalizing 2- and 4-Direction BoxSplines

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Abstract

This paper applies a modified composite subdivision framework to an extensive family of box splines, and therefore generalizes these box splines to irregular control meshes. Particularly, a variant of the quad subdivision of Peters and Shiue (2004) is shown as a special case of the new framework. In addition, a new dual subdivision scheme is also derived, as a generalization of some special box splines. Towards the practical use, the unified framework is also extended for modelling boundary and crease features.

1 Introduction

Composite subdivisions provide an important solution for unifying a variety of existing subdivisions and for extending them to higher order subdivisions in a unified form. A number of frameworks have been investigated so far based on different splitting operators. Some of them are induced from box splines and some others are not. Zorin & Schröder [1] and Stam [2], for example, independently established a unified framework for generalizing bi-degree tensor-product B-splines, a two-direction box splines, using the 1-4 splitting operator over quadrilateral meshes. Stam also brought 3-direction box splines over triangular meshes into a unified framework [2], Doo-Sabin [3], Catmull-Clark [4] and Loop [5] schemes become special cases of these unified subdivisions. On the other hand, composite subdivision and composite subdivision are two examples that are not derived from parametric surfaces [6, 7].

In a recent paper, Li and Ma [8] reported a unified scenario based on the splitting operator, which can cover most of the existing subdivisions based on quadrilateral meshes. However, the original universal setting only covers tensor-product B-splines, a kind of box splines associated with horizontal and vertical vectors, and 4-directional box splines based on horizontal, vertical, diagonal and skew-diagonal directions. Schemes like A4-subdivision, associated with different numbers of pairs of ((1,0), (0,1)) and ((1,1), (1,-1)), are not discussed in [8]. The A4-subdivision was reported by Peters and Shiue [9] for integrating 3- and 4-direction subdivisions. It was declared to be the only subdivision with a 3 × 3 stencil that is symmetric and ripple-free in all of the four directions. Another limitation of the composite subdivision is that it fails to deal with generalization of B-splines and 4-directional box splines in a completely uniform setting. To include the former, a so-called partial averaging operator has to be introduced and thus extra state flag for each vertex is necessary. In addition, it is also indispensable to fix the schemes with boundary and crease modeling for practical use. As a further extension, this paper addresses the aforementioned issues. Specifically, this paper focuses on the following topics:

(1). To facilitate composite subdivision to include a larger class of box spline surfaces which are associated with 4-direction vector sequence \((d_1d_2)^m(d_3d_4)^n = ((1,0)(0,1))^{2m}((1,1)(1,-1))^{2n}\) for arbitrary integers \(m\) and \(n\). Present a completely uniform setting for all aforementioned subdivisions instead of the original two kinds of refinement operators defined in [8].

(2). To extend composite subdivision by adapting its atomic operators for modeling boundary and crease features. Namely, boundary/crease vertices are also iteratively generated by using averaging operations such that they are compatibly refined similar to internal vertices.

(3). To introduce two new subdivision schemes that can be included in the composite subdivision. The first one is a dual subdivision scheme associated with \((1,0)(0,1)((1,1)(1,-1))\) and the other one is a variant of A4-subdivision.

In the rest of this paper, Section 2 briefly introduces the composite subdivision framework, we then modify the composite framework to a more general setting as the generalization of box splines associated with four directions in Section 3 and further investigate the development of boundary/crease modeling in Section 4. A variant of the A4-subdivision and a new dual subdivision are derived using the elementary operators in Section 5. Section 6 presents some examples and then Section 7 draws conclusions.
2 Preliminaries

To explain the foundation where the proposed unified scheme originated from, we briefly introduce the composite $\sqrt{2}$ subdivision [8] which is a unified framework based on the $\sqrt{2}$ splitting operator and a variety of atomic smoothing operators. As two special cases, a family of bi-degree B-splines and a class of 4-direction box splines [10, 11] are properly included in the framework, respectively. In Table 1, we list a portion of operators defined in [8] and briefly describe their behaviors for later use.

### Table 1. Topological and geometric operators used in [8]

<table>
<thead>
<tr>
<th>Operator</th>
<th>Type of operator</th>
<th>Descriptions</th>
</tr>
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<tbody>
<tr>
<td>$T_{\sqrt{2}}$</td>
<td>Topological</td>
<td>The $\sqrt{2}$ splitting operator which inserts a new face vertex (F-vertex), whose position is zero, for each face and generates a new quad for each edge by connecting the two endpoints of the edge and its two neighboring F-vertices.</td>
</tr>
<tr>
<td>$T_d$</td>
<td>Topological</td>
<td>The dual operator which creates a new mesh by converting the vertices and faces of the old mesh into faces and vertices of the new mesh, respectively.</td>
</tr>
<tr>
<td>$S_F(\lambda)$</td>
<td>Geometric</td>
<td>A scale operator for scaling mesh vertices by $\lambda$.</td>
</tr>
<tr>
<td>$A_{FV}$</td>
<td>Geometric</td>
<td>An averaging operator from vertex to face which renews the centroid of a given face with the average of the corner positions.</td>
</tr>
<tr>
<td>$A_{VF}$</td>
<td>Geometric</td>
<td>An averaging operator from face to vertex, which is a dual operator of $A_{FV}$, which renews the position of a given vertex with the average of the centroids of its neighboring faces.</td>
</tr>
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</table>

3 Unified framework combining 2- and 4-direction box splines

In this section, we modify the composite $\sqrt{2}$ subdivision framework in [8] in order to generalize box splines with direction sequence $\Lambda(m,n) = (d_1,d_2)^m(d_3,d_4)^n = ((1,0),(0,1))^m((1,1),(1,-1))^n$ to irregular meshes. We first present the unified definition of the modified composite subdivision.

**Definition 1** Unified framework combining 2- and 4-direction box splines is defined by the following subdivision operator:

$$ R_{\Lambda}(m,n) = (T_d,A_{FV})^{m/2} (A_{VF},A_{FV})^{n/2} S_F(2)T_{\sqrt{2}} $$

Note that the operators in Eq. (1) are performed from right to left. According to [8], the right half sequence of $R_{\Lambda}(m,n)$ acts as the repeated averaging process related to the diagonal and skew-diagonal vector sequence of $(d_3,d_4)^n$. Specifically, $A_{FV}$ and $A_{VF}$ correspond to averaging steps associated with $d_3$ and $d_4$, respectively, in the subdivision of box splines. Similarly, the former part of $R_{\Lambda}(m,n)$ performs repeated averaging operations associated with $(d_1,d_2)^m$, while $A_{PV}$ and $A_{VP}$ correspond to averaging steps associated with $d_1$ and $d_2$, respectively. Henceforth, $R_{\Lambda}(m,n)$ must be equivalent to the subdivision reproducing box splines associated with $\Lambda(m,n)$ for regular meshes. This concludes

**Proposition 1** Operator $R_{\Lambda}(m,n)$ induces a unified subdivision generalizing box splines surfaces associated with vector sequence $\Lambda(m,n)$.

Obviously, Proposition 1 is stronger than Theorems 2 and 3 derived in [8]. Particularly, we are able to establish the relationship between Proposition 1 and the results described in [8]. For the definition of $R_{\sqrt{2}}(m)$, $R_{\sqrt{2}}(m)$, $R_{1-4}(m)$ and $R_{1-4}(m)$ appearing in the following, we refer the reader to [8].

**Corollary 1** Subdivision surfaces induced by $R_{\Lambda}(m,n)$ generalize box splines with respect to direction sequence $((1,0),(0,1),(1,1),(1,-1))^m$, i.e. we have $R_{\Lambda}(2m,2m) = (R_{\sqrt{2}}(m))^2$ and $R_{\Lambda}(2m - 1,2m - 1) = (R_{\sqrt{2}}(m))^2$, where $m \geq 1$.

**Corollary 2** Subdivision surfaces induced by $R_{\Lambda}(2m,0)$ are generalization of box splines with respect to direction sequence $((1,0),(0,1))^2m$, i.e. $R_{\Lambda}(2m + 1,0) = R_{1-4}(m)$, $R_{\Lambda}(2m + 2,0) = R_{1-4}(m)$, where $m \geq 1$.

In Corollary 1, $R_{\Lambda}(m,n)$ interrelates vector sequences which have the same number of horizontal/vertical direction pairs and diagonal/skew-diagonal direction pairs. The averaging operations are evenly distributed in each round of $T_{\sqrt{2}}$ refinement. $R_{\Lambda}(2m,0)$ in Corollary 2 reduces to subdivision generalizing B-splines associated with horizontal/vertical directions. It should be noticed that multiresolution based on the $\sqrt{2}$ splitting is unavailable in this case as all averaging operations

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are clustered in even steps of refinement according to the setting of Eq.(1) though it is still available based on the 1-4 splitting operator.

Corollaries 3 and 4 in the following touch the cases that can be regarded as the general form of that of Corollaries 1 and 2. Averaging operations are unevenly distributed in odd and even refinements here. This case has not been addressed yet in [8].

**Corollary 3** Subdivision surfaces induced by $R_A(2m, m)$ generalize box splines with respect to $((1,0)(0,1)(−1,0)(0,−1)(1,1)(1,−1))^m$.

**Corollary 4** Subdivision surfaces induced by $R_A(m, 2m)$ generalize box splines with respect to $((1,0)(0,1)(1,1)(1,−1)(−1,1)(−1,−1))^m$.

To verify Corollary 3, we only need to notice that vector sequence $((1,0)(0,1)(−1,0)(0,−1)(1,1)(1,−1))^m$ and $((1,0)(1,0)(1,0)(1,1)(1,−1))^m$, which is equal to $((1,0)(0,1))^m((1,1),(1,−1))^m$, induce the same subdivision. The treatment of Corollary 4 is similar.

## 4 Boundary and crease modeling

Though Zorin and Schröder presented some examples with boundaries and creases in [1], it is not clear how their method works and whether it adapts to subdivisions as generalization of B-spline surfaces of arbitrary degree while other literature did not touch the problem. Generally, it is expected that boundary curves are independent of inner vertices of control meshes [15]. Though interior and boundary/crease cases may be dealt with independently, however, synchronized refinement of smooth and boundary/crease regions should be sustained in order to achieve compatible results. It is necessary to extend atomic operators in Eq. (1) one by one in order to generate reasonable boundary/crease features.

For convenience of description, an edge is called special if it is either a boundary or a crease edge, or normal otherwise. Similarly, boundary or crease vertices are called special vertices while other vertices are normal vertices. Considering that the behavior of the topological operators is quite different in odd and even subdivision steps, we address the extensions respectively. As the behaviors of the operators have been described in [8] for the normal case, we focus on how boundary/crease features are addressed. When interpreting the behavior of these operators, we always take creases as examples. The treatment of boundaries is similar. As we found that it’s much more difficult and complex to deal with darts and corner features, we just focus on the boundary and crease features in this paper.

### 4.1 Odd steps of refinement

**Splitting operator $T_{S}$** Firstly, one vertex is inserted for each face (F-vertex) and each special edge (E-vertex), and the position of the vertex is always set to zero. The vertex created for a crease edges is a new crease vertex. The crease edge splits into two. Secondly, a normal edge or a boundary edge always yields one quadrilateral face at this stage while a crease edge would generates two faces, each at one side of the edge. The vertex set of a new face associated with a special edge consists of two endpoints of the old edge (the two V-vertices), the new E-vertex of the edge, and the new F-vertices of the old faces sharing the edge as shown in Fig. 1b. This kind of new faces, which always contain a vertex of valence 2 (also called an E-vertex), is named special faces. Other new faces are normal faces.

**Dual operator $T_{D}$** The vertices of the dual mesh consist of two portions. The first portion contains centroids of all normal faces of the primal mesh except for those special faces containing special edges. The second portion is the set of all special vertices, including the E-vertices and ordinary crease vertices. Attention should be paid that both the positions of F-vertex and E-vertices come from the AVF operator. All vertices of the primal mesh except for those E-vertices contribute a face to the dual mesh as shown in Fig. 1c.

![Fig. 1. Topological operations for meshes with crease (and boundary) features (odd steps of refinement): (a) Initial mesh with heavy black lines for crease edges; (b) The refined mesh drawn in black after odd step of $T_{S}$ splitting, crease V-vertices in black, and crease E-vertices in hollow whose valence is always 2; (c) Dual result by performing $T_{D}$ over (b) in which all special vertices remain special, but turn to be of valence 2.](image)

**Geometric operators** $S_V(\lambda)$, $A_{VF}$, and $A_{FV}$

The scaling operator $S_V(\lambda)$ scales all vertices by $\lambda$. The VF-type averaging operator $A_{VF}$ evaluates the centroid of normal faces as the average of their vertices while centroid of special edges as their midpoint. The centroid of a special edge then replaces the centroid of the special face pointed by this special edge. The FV-type averaging operator $A_{FV}$ is performed in a few steps. Firstly, it updates each special vertex with the average of the midpoints of the two special edges sharing that vertex followed by the replacement of the centroid of faces...
containing special edges, in order to keep the operation pair $A_{VF}A_{VF}$ compatible in special regions and normal regions. Secondly, $A_{VF}$ evaluates normal vertices with centroids of their adjacent faces (some faces are special). Note that $A_{VF}$ before $T_D$ needs special treatment when $n=1$. The centroid of all normal faces are first calculated. All the crease vertices are then scaled by 0.5, which cancels the effect of $S_V(2)$. Thirdly, the position of the E-vertex is set to the average of the two end points of the crease edge. Remember that the valence of an E-vertex is always 2.

### 4.2 Even steps of refinement

**Splitting operator** $T_{VZ}$ Firstly, one vertex is inserted into each normal face, whose position is set to zero. Secondly, a quadrilateral face is still created for each normal edge, but special edges do not produce a new face. It should be noticed that as the vertex with valence 2 of a special face actually acts as the centroid of the face as shown in Fig. 2a, a new face associated with the normal edge of an old special face always contains a special edge. Also, we call this kind of faces special faces, and normal otherwise. Furthermore, applying $T_{VZ}$ to the dual mesh generated in odd steps results in a mesh with the same topology as that of the primal one.

**Dual operator** $T_D$ The vertices of the dual mesh also consist of two categories: the centroids of all normal faces and the centroids of all special edges. In addition, a new face is created for each normal vertex. The vertices of a new face consist of the centroids of old faces adjacent to the vertex. It should also be noticed that we substitute the centroids of special edges for the centroids of old faces adjacent to the vertex. The vertices of a new face consist of the centroids of all normal vertices with the average of centroids of its two adjacent special faces while special vertices with the average of centroids of its two adjacent special edges.

Operation $A_{VF}$ is relatively simple. It updates normal vertices with the average of centroids of its neighboring faces while special vertices with the average of centroids of its two adjacent special edges.

![Fig. 3. The centroid for f and the midpoint for special edge of the first VF operation.](image)

### 4.3 Remark

It is not difficult to understand that the extensions discussed in Subsections 4.1 and 4.2 generate boundary and crease curves of B-splines. Through an elaborated verification, we may find that subdivision operators $R_A(3,0)$, $R_A(4,0)$, $R_A(1,1)$ and $R_A(2,2)$ generate quadratic, cubic, quadratic and quintic B-spline boundary/crease curves, respectively. This indicates that the proposed boundary generation approach is coincident with previous results [12, 13, 14]. Certainly, one may also employ different strategies to do this as long as they can maintain balanced convergence between normal regions and crease or boundary regions.

### 5 Two novel subdivisions

By fixing a non-negative integer pair $(m, n)$ we can derive a specific subdivision using Eq. (1). For example, there are three schemes $R_A(2,0), R_A(1,1)$ and $R_A(0,2)$ for $m+n=2$ among which $R_A(2,0)$ is the generalized linear B-spline and $R_A(1,1)$ is the mid-edge subdivision [16]. However, $R_A(0,2)$ has not been reported so far. When $m+n=3$ we have $R_A(0,3), R_A(1,2), R_A(2,1)$ and $R_A(3,0)$ in which $R_A(3,0)$ and $R_A(2,1)$ or their variants have been reported in relation to Doo-Sabin subdivision and the A4-subdivision [9], respectively, while the other two cases are new. For $m+n=4$, $R_A(4,0)$ and $R_A(2,2)$ are respectively the variants of Catmull-Clark and the 4-8 subdivision schemes, while the others are also new. As for higher order, only a variant of $R_A(5,0)$, the biquartic subdivision, has been addressed [1]. We present explicit subdivision masks for subdivisions $R_A(1,2)$ and $R_A(2,1)$ in this section in details.

#### 5.1 $R_A(1,2)$—a new dual subdivision

The new dual subdivision $R_A(1,2)$ is defined as

$$R_A(1,2) = T_D A_{VF} S_V(2) T_{VZ} A_{VF} A_{VF} S_V(2) T_{VZ}. \quad (2)$$
The topological rule implied by \( R_A(1, 2) \) is the same as that of a round of Doo-Sabin subdivision [3] or two steps of the midedge subdivision. However, the stencils of its geometric rules are obviously larger than those of the two aforementioned subdivisions.

### 5.1.1 Subdivision rules of \( R_A(1, 2) \)

Generally, regular and irregular masks induced by \( R_A(1, 2) \) have a uniform representation. Given a polygonal face \( f \) of \( k \) vertices, let \( v \) be a vertex with valence \( l \) in \( f \). After refinement, a small polygonal face \( f' \) with the same number of vertices as \( f \) is generated. The vertex of \( f' \) corresponding to \( v \) can then be evaluated using the mask of Fig. 4 in which polygons with solid and dotted lines are respectively \( f \) and \( f' \). Fig. 5 shows an example generated by this subdivision. The analysis of global \( C^1 \) continuity of \( R_A(1, 2) \) can be found in Appendix B.

![Fig. 4. Subdivision mask for \( R_A(1, 2) \).](image)

![Fig. 5. An example of \( R_A(1, 2) \) subdivision: (a) Original model, (b) refinement at level 4.](image)

#### 5.1.2 Remarks on improvement for \( R_A(1, 2) \)

The subdivision introduced in Section 5.1.1 is completely based on averaging operations. The leading eigenvalues tend to cluster with the increase of the edge number of irregular faces. It is possible to prevent clustering by employing different masks for irregular faces. A suggestion is given here to:

1. use the subdivision masks derived in section 5.1.1 for regular vertices, but
2. use irregular masks of Doo-Sabin or Midedge subdivisions for vertices of irregular faces.

### 5.2 \( R_A(2, 1) \)-a variant of A4-subdivision

Like the midedge and 4-8 subdivisions, the A4 subdivision proposed by Peters and Shiue [9] is also derived from 4-direction box splines. Nevertheless, the number of vector pairs of horizontal and vertical directions is no longer identified to that of diagonal and skew-diagonal directions.

Consider the following composite sequence:

\[
R_A(2, 1) = A_{FP} A_{VP} S_y(2) T_{e}^2 T_{p} R_p S_y(2) T_{e}^2. \tag{3}
\]

Following the discussions of [8], it is also easy to show that applying the above composition to an arbitrary polygonal mesh yields a new subdivision using the 1-4 splitting operator whose subdivision rules are shown in Fig. 6. An F-vertex of the refined mesh is just the centroid of the corresponding face, while the E-vertex of an edge is defined as the linear combination of vertices of its winged-edge structure. Both types of vertices share the same masks as that of Catmull-Clark subdivision. The V-vertex of a given vertex, however, is only the linear combination of its adjacent vertices and itself in which the given vertex dominates with \( 1/2 \), while each of other vertices contributes \( 1/(2k) \), where \( k \) is the valence of the given vertex. Like the A4 subdivision, the proposed variant also bears smaller masks than that of Catmull-Clark subdivision. The details of \( C^1 \) continuity are again given in Appendix A.

![Fig. 6. Subdivision masks for \( R_A(2, 1) \): (a) F-vertex mask; (b) E-vertex mask for an edge whose adjacent faces can be an arbitrary polygon; (c) V-vertex mask for a vertex of valence \( n \) with polygonal faces as neighbors.](image)

Though the new scheme is not curvature bounded, but it guarantees the convex hull property which the original subdivision in [9] does not hold. Though Peters and Shiue tried to lessen this artifact by adjusting the subdivision rule for V-vertices, the strategy does not ultimately diminish negative weights and it is not clear whether their improved masks still sustain bounded curvature. Eq. (3) indicates that the subdivision has also a variant induced by only VF-type (from vertex to face centroid) and FV-type (from face centroid to vertex).

### 6 Experimental results

The implementation of the proposed algorithms is similar to that of [8]. However, attentions must be paid to deal with boundary/crease cases following Section 4. This section presents several examples to show the modeling effects of the proposed framework. Fig. 7 shows a series of subdivision surfaces related to \( R_A(m, 0) \). Figs 8 and 9 illustrate examples for modeling sharp features and boundaries, respectively. Fig. 10 demonstrates that the new subdivision \( R_A(2, 1) \) is also ripple-free.
Fig. 7. Several subdivision surfaces produced by $R_A(m,0)$ in Section 3, they are all at level 4. From (a) to (h), original model, $m=2$ (generalize linear B-spline subdivision), $m=3$ (related to the Doo-Sabin scheme), $m=4$ (related to the Catmull-Clark scheme), $m=5$ (a biquartic scheme), $m=6$ (not reported), $m=13$ (not reported), and $m=18$ (not reported).

Fig. 8. Simultaneous illustration of crease feature and smooth region refinement: (a) Four edges of the bottom face are tagged as crease edges; From (b) to (d), each example shows the smooth-refinement and the crease-refinement in subdivision level 2.

Fig. 9. Boundary modeling: (a) Original model with 3 boundaries, (b) Subdivision surfaces by $R_A(3,0)$, (c) Subdivision surfaces by $R_A(3,1)$, (d) Subdivision surfaces by $R_A(4,0)$. 
Fig. 10. The new dual subdivision is also ripple-free in 4-directions similar to A4 subdivision. From left to right: (a) Control mesh, (b)-(d) are subdivision surfaces produced using $R_{A}(4,0)$ (variant of Catmull-Clark subdivision), $R_{A}(2,1)$, and $R_{A}(2,2)$ (variant of 4-8 subdivision) respectively.

7 Conclusions

We investigated a unified subdivision in relation to box splines with a direction sequence $(((1,0)(-1,0)(0,1)(0,-1)(1,1)(-1,1))^m$. The scheme further extends composite $\sqrt{2}$ subdivision to cover more box spline surfaces. As a special case, a new subdivision is derived by evaluating the explicit masks based on the 1-4 splitting of $R_{A}(1,2)$ which generalizes box splines defined by $((1,0)(0,1)(1,1)(-1,1)(1,-1))$. As a variant of the quad scheme in [9], explicit masks based on the 1-4 splitting operator are given for the subdivision generalizing box splines associated with the $((1,0)(0,1)(1,1)(-1,1)(1,-1))$. Subdivision surfaces generated by this scheme are $C^3$ continuous except at a finite number of extraordinary points where the surface is $C^1$. We also present a solution for adapting the framework to meshes with boundary and crease features. It allows compatible refinement of meshes in both normal and special regions.

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8. References


Appendix A. $C^1$ Continuity of $R_A(2,1)$

One can establish a subdivision matrix following the subdivision rules using an l-neighborhood stencil of a vertex of valence $k$. The dominant eigenstructure is the same for different $l$. Therefore we investigate of a vertex of valence $k$. The dominant eigenstructure imposed to the eigenvalues of the subdivision matrix. Following subdivision rules, we have the Fourier matrix of $R_A(1,2)$

$$B_l(z) = \frac{1}{16} \begin{pmatrix} 8 & 8 & 0 & 0 & 0 & 0 \\ 6 & z^{-1} + 6 + z & 1 & 0 & 0 & 0 \\ 4 & 4 + 4z & 4 & 0 & 0 & 0 \\ 2 & 2z^{-1} + 2z & 2 & 0 & 0 & 0 \\ 2 & 2z & 1 & 0 & 0 & 0 \\ 1 & 1 + 6z & 6 & z & 0 & 0 \\ 1 & 1 + 6z & 6 & z & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \end{pmatrix} \begin{pmatrix} z \bar{B}_l(z) \end{pmatrix}$$

where $\bar{B}_l(z)$ is the $6 \times 6$ sub-matrix of $B_l(z)$ consisting of 2-6 rows and 2-6 columns. $S_k$ is similar to $\text{diag}(\bar{B}_{0,k}, \bar{B}_{1,k}, \bar{B}_{2,k}, ..., \bar{B}_{k-1,k})$, where $\bar{B}_{0,k} = B(1)$ and $\bar{B}_{j,k} = B(e^{-2\pi i/k})$. Therefore the eigenstructure of $S_k$ is available by analyzing $\bar{B}_{j,k}(j = 1, ..., k - 1)$. One can prove the following

1) The eigenvalues of $\bar{B}_{0,k}$ are $1$, $1/4$, $1/8$, $1/16$, $1/16$, $0$, $0$.

2) The eigenvalues of $\bar{B}_{j,k}$ ($j = 1, ..., k - 1$) are $\mu_{j0}$, $\mu_{j1}$, $1/8$, $1/16$, $1/16$, $0$, $0$.

$$\mu_{j0} = \frac{1}{16} \left( 5 + \cos \frac{2j\pi}{k} + \frac{1}{1 + \cos \frac{2j\pi}{k}} \left( 9 + \cos \frac{2j\pi}{k} \right) \right)$$

$$\mu_{j1} = \frac{1}{16} \left( 5 + \cos \frac{2j\pi}{k} - \frac{1}{1 + \cos \frac{2j\pi}{k}} \left( 9 + \cos \frac{2j\pi}{k} \right) \right)$$

Noticing that $\mu_{j0}(j = 1, ..., k - 1)$ are monotonic functions of $\cos(2j\pi/n)$, we have $\mu_{j0} = \mu_{k-j,0} > 7/16$. Furthermore, these two eigenvalues are the largest ones among $\mu_{j0}(j = 1, ..., k - 1)$, namely $\mu_{10} = \mu_{k-1,0} > \mu_{10}(j = 2, ..., n - 2)$. On the other hand, it is easy to show $\mu_{j1} < 3/8$ for all $j$. This demonstrates that the subdominant eigenvalue of $S_k$ is two multiplicity: $\mu_{10} = \mu_{k-1,0}$. Fig A1 depicts the control meshes composing of their corresponding eigenvectors for $k=3,5,6,7,8,9,10$ and corresponding results after 4 steps of refinement. Numerical analysis demonstrates that $R_A(2,1)$ is $C^1$ continuous.

Appendix B. $C^1$ Continuity of $R_A(1,2)$

Consider a stencil of the 1-neighborhood of an irregular face. Assume all its surrounding elements are regular. Similar to Appendix A, we have the following Fourier matrices of $R_A(1,2)$:

$$B_k(l) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} \\ \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} \end{pmatrix}$$

$$B_k(z) = \begin{pmatrix} \omega + \frac{1}{4} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{6}{16} + \frac{1}{16} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{6}{16} + \frac{1}{16} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{6}{16} + \frac{1}{16} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{6}{16} + \frac{1}{16} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{6}{16} + \frac{1}{16} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{6}{16} + \frac{1}{16} (z + z^{-1}) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$z = \omega, ..., \omega^{k-1}$, $\omega = \cos(2\pi/k) + \sin(2\pi/k)$.

Numerically verification demonstrates that the eigenvalues of the above blocks satisfy the necessary condition of $C^1$ continuity. Fig. B.1 illustrates the characteristic maps for $k=3,5,6,7,8,9,10$. Fig. B.1 Characteristic maps for valences 3 and 5-10.