A general multiple attribute decision-making approach for integrating subjective preferences and objective information

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Abstract

A general multiple attribute decision-making (MADM) approach is proposed to integrate subjective preferences and objective information and to assess the relative weights of attributes, where subjective preferences are expressed either by a fuzzy preference relation on decision alternatives or by a pairwise comparison matrix on the relative weights of attributes or by both of them and objective information is expressed by a decision matrix. Three special cases of the approach are investigated, which are the weighted least deviation norm (WLDN), the weighted least-square deviation norm (WLSDN) and the weighted minimax deviation norm (WMDN) approaches. The extension of the general approach to group decision-making and the choice of decision parameters are also discussed. A numerical example is examined to show the applications of the proposed MADM approaches.

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1. Introduction

Multiple attribute decision analysis (MADA) always involves multiple decision attributes and multiple decision alternatives. In order to make a decision or choose a best alternative, a decision maker (DM) is often asked to provide his/her preferences either on alternatives or on the relative weights of attributes or on both of them. Especially in group decision analysis, a decision is made by multiple DMs or experts. Some of them may prefer to give their preferences on alternatives and some on the relative weights. Fuzzy and multiplicative preference relations are two commonly used means to characterize DM's subjective preferences on decision alternatives and on the relative weights of attributes, respectively [1–7,11,20–22]. The former constitutes a fuzzy preference matrix on decision alternatives and on the relative weights of attributes, respectively. The latter forms a well-known pairwise comparison matrix on the relative weights of attributes. Either of them can be used to determine an estimate of the relative weights of attributes. Once the relative weights of attributes are estimated, the final decision

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can be made on the basis of a decision matrix, which is thought to be objective, by using traditional multiple attribute
decision-making (MADM) approaches such as the simple additive weighting (SAW) or TOPSIS approaches or others [8]. However, there is no guarantee that these two different kinds of preference relations can lead to the same estimate of the relative weights of attributes. Sometimes, the two estimates may be highly conflicting. Therefore, it is very important to incorporate the two different kinds of preference relations together with the objective decision matrix information into an integrated decision model to generate an overall estimate for the relative weights of attributes and an overall decision for the MADM problem under consideration.

Ma et al. [10] developed a subjective and objective integrated approach for MADM, where the multiplicative preference relation on the relative weights of attributes was integrated together with decision matrix information into an integrated decision model. Their approach utilized the quadratic programming technique to assess the attribute weights and was further discussed by Xu [19]. Fan et al. [5] proposed an approach to MADM based on the fuzzy preference relation on decision alternatives. In their approach the fuzzy preference relation on alternatives was incorporated together with decision matrix information through a quadratic programming model. Wang and Parkan [18] integrated the fuzzy preference relation and decision matrix information together in three different ways, which used linear programming technique. Fan et al. [3] also incorporated them together using linear programming technique, but with a different integrated model. Fan et al. [4,6] developed further a linear goal programming model and a two-objective optimization model to integrate fuzzy and multiplicative preference relations both on decision alternatives. It is quite clear that there has been no effort so far to integrate the fuzzy preference relation on decision alternatives and the multiplicative preference relation on the relative weights of attributes together with decision matrix information into one integrated decision model. The purpose of this paper is to develop a general MADM approach to integrate the two kinds of preference relations and the decision matrix information into one model so that the relative weights of attributes can be estimated in an integrated manner.

The rest of this paper is organized as follows. Section 2 gives a brief description of the well-known SAW approach for MADM and the eigenvector method (EM) for modeling DM’s subjective preference relations to pave the way for Section 3, where a general MADM approach is developed to integrate both subjective preferences and objective information and three special cases are investigated in depth. This is followed by an illustrative example, which is provided to show the applications of the proposed approaches for MADM problems. The paper is concluded in Section 5.

2. The SAW approach and EMs

2.1. The SAW approach for MADM

Suppose a MADM problem has \(n\) decision alternatives \(A_1, \ldots, A_n\) and \(m\) decision attributes \(G_1, \ldots, G_m\). Each alternative is evaluated with respect to the \(m\) attributes, whose values constitute a decision matrix denoted by \(X = (x_{ij})_{n \times m}\). Due to the incommensurability among attributes, the decision matrix \(X = (x_{ij})_{n \times m}\) needs to be normalized. The most commonly used normalization method is as follows:

\[
\begin{align*}
    z_{ij} &= \frac{x_{ij} - x_{j}^{\min}}{x_{j}^{\max} - x_{j}^{\min}}, \quad i = 1, \ldots, n, \quad j \in \Omega_1, \\
    z_{ij} &= \frac{x_{j}^{\max} - x_{ij}}{x_{j}^{\max} - x_{j}^{\min}}, \quad i = 1, \ldots, n, \quad j \in \Omega_2,
\end{align*}
\]

where, \(x_{j}^{\min} = \min_{1 \leq i \leq n} \{x_{ij}\}, x_{j}^{\max} = \max_{1 \leq i \leq n} \{x_{ij}\}\), \(z_{ij}\) is the normalized attribute value, and \(\Omega_1\) and \(\Omega_2\) are, respectively, the sets of benefit attributes and cost attributes. The so-called benefit attributes are those for maximization, while the cost attributes are those for minimization.

Let \(Z = (z_{ij})_{n \times m}\) be the normalized decision matrix and \(W = (w_1, \ldots, w_m)^T\) be the normalized vector of attribute weights satisfying

\[
e^T W = 1,
\]
where $e^T = (1, \ldots, 1)$ is a vector with all elements being one. According to the SAW approach [8], the overall weighted assessment value of alternative $A_i (i = 1, \ldots, n)$ can be expressed as

$$d_i = \sum_{j=1}^{m} z_{ij} w_j, \quad i = 1, \ldots, n,$$

where $d_i$ is a linear function of weight variables $w_j (j = 1, \ldots, m)$. The greater the $d_i$, the better the alternative $A_i$. The best alternative is the one with the greatest overall weighted assessment value. For brevity, Eq. (4) can be re-expressed in vector form as

$$D = Z W,$$

where $D = (d_1, \ldots, d_n)^T$ is a vector of the overall weighted assessment values for all the alternatives.

### 2.2. EMs for modeling subjective preference relations

#### 2.2.1. EM for modeling multiplicative preference relation

To use the SAW approach to conduct decision analysis, the weight vector, $W = (w_1, \ldots, w_m)^T$, must be known. Usually, it can be estimated subjectively or objectively. Objective approaches such as the relative entropy method [8,19], mathematical programming methods [12,14–16], the principal component analysis [17] and the factor analysis [13], determine the weights of attributes using decision matrix information, but take no account of DM’s preferences on the relative importance of attributes. The weights estimated in this way may sometimes be counterintuitive. Therefore, subjective approaches are extensively used so that DM’s preferences can be considered in the determination of attribute weights. The most widely used subjective approach is the method of pairwise comparison matrix on the relative weights of attributes.

Let the multiplicative preference relation on the relative weights of attributes be represented by

$$A = \begin{bmatrix}
w_1 & w_2 & \cdots & w_m \\
 a_{11} & a_{12} & \cdots & a_{1m} \\
 a_{21} & a_{22} & \cdots & a_{2m} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mm}
\end{bmatrix},$$

where $a_{ji} = 1/d_{ij} > 0$ and $a_{ii} = 1 (i, j = 1, \ldots, m)$. According to Saaty’s EM [11], weight vector $W = (w_1, \ldots, w_m)^T$ can be estimated by solving the following eigenvalue problem:

$$AW = \lambda_{\max} W.$$

If the multiplicative preference relation $A$ is a precise/consistent comparison matrix on the relative weights of attributes, then Eq. (7) can be simplified as

$$AW = mW.$$

It is hoped that the multiplicative preference relation provided by DM should be as consistent as possible.

#### 2.2.2. EM for modeling fuzzy preference relation

In some situations, DM may prefer to provide a fuzzy preference relation on decision alternatives. This has been substantially investigated in the past [1–6,20–22]. Suppose the fuzzy preference relation on decision alternatives provided by a DM is known and expressed by

$$P = \begin{bmatrix}
A_1 & A_2 & \cdots & A_n \\
 A_1 & p_{11} & p_{12} & \cdots & p_{1n} \\
 A_2 & p_{21} & p_{22} & \cdots & p_{2n} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 A_n & p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix},$$

where $a_{ji} = 1/d_{ij} > 0$ and $a_{ii} = 1 (i, j = 1, \ldots, m)$. According to Saaty’s EM [11], weight vector $W = (w_1, \ldots, w_m)^T$ can be estimated by solving the following eigenvalue problem:

$$AW = \lambda_{\max} W.$$
where \( p_{ij} + p_{ji} = 1 \), \( p_{ii} = 0.5 \) and \( p_{ij} \geq 0 \) \((i, j = 1, \ldots, n)\). According to [3,5,6], the fuzzy preference relation \( P \) can be seen as a subjective estimate of the following pairwise comparison matrix:

\[
\begin{bmatrix}
\bar{p}_{11} & \bar{p}_{12} & \cdots & \bar{p}_{1n} \\
\bar{p}_{21} & \bar{p}_{22} & \cdots & \bar{p}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{p}_{n1} & \bar{p}_{n2} & \cdots & \bar{p}_{nn}
\end{bmatrix}
\]

\( (10) \)

where \( \bar{d}_i \) is the overall weighted assessment value of alternative \( A_i \) determined by Eq. (4) \((i = 1, \ldots, n)\). Theoretically, we have the following system of equations:

\[
\begin{align*}
\frac{d_1}{d_1 + d_2} (d_1 + d_2) + \frac{d_1}{d_2 + d_1} (d_2 + d_1) + \cdots + \frac{d_1}{d_n + d_1} (d_n + d_1) &= (n - 1)d_1, \\
\frac{d_2}{d_2 + d_1} (d_2 + d_1) + \frac{d_2}{d_2 + d_2} (d_2 + d_2) + \cdots + \frac{d_2}{d_n + d_2} (d_n + d_2) &= (n - 1)d_2, \\
&\vdots \\
\frac{d_n}{d_n + d_1} (d_n + d_1) + \frac{d_n}{d_n + d_2} (d_n + d_2) + \cdots + \frac{d_n}{d_n + d_{n-1}} (d_n + d_{n-1}) &= (n - 1)d_n.
\end{align*}
\]

(11)

If the fuzzy preference relation/matrix \( P = (p_{ij})_{n \times n} \) is the precise estimate of \( \bar{P} = (\bar{p}_{ij})_{n \times n} \) in (10), then (11) can be equivalently expressed as

\[
\begin{align*}
p_{12}(d_1 + d_2) + p_{13}(d_1 + d_3) + \cdots + p_{1n}(d_1 + d_n) &= (n - 1)d_1, \\
p_{21}(d_2 + d_1) + p_{23}(d_2 + d_3) + \cdots + p_{2n}(d_2 + d_n) &= (n - 1)d_2, \\
&\vdots \\
p_{n1}(d_n + d_1) + p_{n2}(d_n + d_2) + \cdots + p_{nn}(d_n + d_{n-1}) &= (n - 1)d_n.
\end{align*}
\]

(12)

or as

\[
\begin{align*}
(\sum_{j \neq 1} p_{1j})d_1 + p_{12}d_2 + \cdots + p_{1n}d_n &= (n - 1)d_1, \\
p_{21}d_1 + (\sum_{j \neq 2} p_{2j})d_2 + \cdots + p_{2n}d_n &= (n - 1)d_2, \\
&\vdots \\
p_{n1}d_1 + p_{n2}d_2 + \cdots + (\sum_{j \neq n} p_{nj})d_n &= (n - 1)d_n.
\end{align*}
\]

(13)

Let

\[
B = \begin{bmatrix}
\sum_{j=2}^{n} p_{1j} & p_{12} & \cdots & p_{1n} \\
p_{21} & \sum_{j=2}^{n} p_{2j} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & \sum_{j=1}^{n-1} p_{nj}
\end{bmatrix}
\]

(14)

Eq. (13) can be rewritten as [9,18]:

\[
BD = (n - 1)D.
\]

(15)

Obviously, \( D \) can be viewed as the principal right eigenvector of matrix \( B \). It has been proved [18] that Eq. (15) holds for any fuzzy preference relation no matter whether or not it is the precise estimate of \( \bar{P} = (\bar{p}_{ij})_{n \times n} \) in (10). Accordingly, DM’s subjective ranking for alternatives can be obtained by solving the eigenvalue problem (15).
If we substitute (5) into (15), then we get the following relationship between fuzzy preferences and weight vector:

\[ BZW = (n - 1)ZW. \]  

(16)

Such a relationship can be used to estimate the relative weights of attributes. The interested reader is referred to [18] for details.

3. Approaches for integrating subjective preferences and objective information

3.1. A general approach

Without loss of generality, we assume that a DM can provide both the fuzzy preference information on decision alternatives and the multiplicative preference information on the relative weights of attributes so that they can be integrated in one model. However, it is not essential for a DM to provide both of them.

Since there is no guarantee that fuzzy and multiplicative preference relations can lead to the same estimate of the relative weights of attributes, Eqs. (8) and (16) are generally difficult to hold simultaneously. We therefore introduce the following deviation vectors:

\[ E = AW - mW = (A - mI)W, \]  

(17)

\[ \Gamma = [BZ - (n - 1)Z]W, \]  

(18)

where \( E = (e_1, \ldots, e_m)^T, \Gamma = (\gamma_1, \ldots, \gamma_n)^T \) and \( I \) is an \( m \times m \) unit matrix with the elements on leading diagonal being one and all the others being zero. It is most desirable that the absolute value of each deviation variable be kept as small as possible, which enables us to construct the following optimization model based on deviation norm:

\[
\text{Minimize } J = \left( \sum_{i=1}^{n} |\gamma_i|^p + \beta \sum_{j=1}^{m} |e_j|^p \right)^{1/p},
\]

s.t. \[
\begin{align*}
[BZ - (n - 1)Z]W - \Gamma &= 0, \\
(A - mI)W - E &= 0, \\
e^TW &= 1, \\
W &\geq 0,
\end{align*}
\]

(19)

where \( \alpha \geq 0 \) and \( \beta \geq 0 \) are, respectively, the weight coefficients satisfying \( \alpha + \beta = 1 \) and \( p > 0 \) is a parameter on deviation norm. When \( \alpha = 0 \), (19) considers only the multiplicative preference relation. If \( \beta = 0 \), then (19) considers the fuzzy preference relation and the objective decision matrix information. If \( \alpha \) and \( \beta \) are both nonzero, then (19) considers both the fuzzy and multiplicative preference relations as well as the decision matrix information. For different \( p \) values, (19) may produce different sets of the relative weights of attributes. In what follows, three special cases of \( p \) will be discussed, where \( p = 1, 2 \) and \( \infty \), respectively.

3.2. The weighted least deviation norm (WLDN) approach

Let \( p = 1 \) in (19). Accordingly, the objective function becomes a WLDN. To solve model (19), we introduce the following deviation variables:

\[
\begin{align*}
\gamma_i^+ &= \frac{\gamma_i + |\gamma_i|}{2}, & i &= 1, \ldots, n, \\
\gamma_i^- &= \frac{-\gamma_i + |\gamma_i|}{2}, & i &= 1, \ldots, n, \\
e_j^+ &= \frac{e_j + |e_j|}{2}, & j &= 1, \ldots, m, \\
e_j^- &= \frac{-e_j + |e_j|}{2}, & j &= 1, \ldots, m,
\end{align*}
\]

(20)

(21)

It is evident that \( \gamma_i^+ \) and \( \gamma_i^- \) as well as \( e_j^+ \) and \( e_j^- \) are all nonnegative variables, based on which \( \gamma_i \) and \( e_j \) can be expressed as

\[
\gamma_i = \gamma_i^+ - \gamma_i^-, \quad i = 1, \ldots, n,
\]

(22)
where $\gamma_i^+ \cdot \gamma_i^- = 0$ and $e_j^+ \cdot e_j^- = 0$. Model (19) is therefore transformed into a goal programming (GP) model:

$$\text{Minimize } J = \alpha \cdot e^T (I^+ + I^-) + \beta \cdot e^T (E^+ + E^-),$$

subject to:

$$[BZ - (n - 1)Z]W - I^+ + I^- = 0,$$

$$(A - mI)W - E^+ + E^- = 0,$$

$$e^T W = 1,$$

$$W, E^+, E^-, I^+, I^- \geq 0,$$

where $I^+ = (\gamma_1^+, \ldots, \gamma_n^+)^T$, $I^- = (\gamma_1^-, \ldots, \gamma_n^-)^T$, $E^+ = (e_1^+, \ldots, e_m^+)^T$ and $E^- = (e_1^-, \ldots, e_m^-)^T$. Such a GP model is referred to as the WLDN model, which can be solved using LINDO software package or Microsoft EXCEL solver.

Let $W^* = (w_1^*, \ldots, w_m^*)^T$ be the optimal weight estimate generated from (24). Based on it, the overall weighted assessment value of each alternative can be computed using Eq. (4) and the decision can be made.

### 3.3. The weighted least-square deviation norm (WLSDN) approach

Let $p = 2$ in (19), whose objective function becomes a WLSDN. Accordingly, model (19) can be rewritten as

$$\text{Minimize } J' = \alpha \sum_{i=1}^{n} \gamma_i^2 + \beta \sum_{j=1}^{m} e_j^2,$$

subject to:

$$[BZ - (n - 1)Z]W - I = 0,$$

$$(A - mI)W - E = 0,$$

$$e^T W = 1,$$

$$W \geq 0,$$

whose objective function can be further expressed as

$$J' = \alpha \cdot I^T I + \beta \cdot E^T E$$

$$= \alpha W^T [BZ - (n - 1)Z]^T [BZ - (n - 1)Z]W + \beta W^T (A - mI)^T (A - mI)W$$

$$= W^T [\alpha [BZ - (n - 1)Z]^T [BZ - (n - 1)Z] + \beta (A - mI)^T (A - mI)] W$$

$$= W^T G W,$$

where

$$G = \alpha [BZ - (n - 1)Z]^T [BZ - (n - 1)Z] + \beta (A - mI)^T (A - mI).$$

Hence, (25) can be simplified as

$$\text{Minimize } J' = W^T G W,$$

subject to:

$$e^T W = 1,$$

$$W \geq 0.$$  \hspace{1cm} (27)

In some situations, (27) can be solved by constructing the Lagrangian function

$$L(W, \lambda) = W^T G W + 2\lambda (e^T W - 1).$$

The solution to (27) is found to be

$$W^* = \frac{G^{-1} e}{e^T G^{-1} e}$$  \hspace{1cm} (29)

iff $W^* \geq 0$, where $G^{-1}$ is the inverse matrix of $G$ and $e = (1, \ldots, 1)^T$. In other words, only in the situations when (29) gives $W^* \geq 0$, (27) can be solved by constructing the Lagrangian function. A more general solution method is
to solve the following linear programming (LP) model using the Quadratic Programming Solver in LINDO software package:

Minimize \( \sum_{i=1}^{m} w_i + \lambda \)  

s.t. \( GW + \lambda e \geq 0 \)  
\( e^T W = 1 \)  
\( W, \lambda \geq 0 \)  

(30) \hspace{1cm} (31) \hspace{1cm} (32) \hspace{1cm} (33) \hspace{1cm} (34)

Inequality constraints (31) are the first-order conditions of the Lagrangian function (28) with respect to each weight variable \( w_i (i = 1, \ldots, m) \) and statements (33) and (34) are commands required by LINDO software. Denote the optimal solution to the above model (30)–(34) by \( W^* = (w^*_1, \ldots, w^*_m)^T \), based on which the overall weighted assessment value of each alternative can be calculated using Eq. (4) and the decision can be made. Such a method is called the WLSDN approach.

3.4. The weighted minimax deviation norm (WMDN) approach

Let \( p = \infty \) in (19). The objective function becomes a WMDN and the model becomes

Minimize \( J = x \cdot \gamma + \beta \cdot \varepsilon, \)

\[
\begin{align*}
-BZ + (n - 1)Z & \leq \gamma \cdot e, \\
(A - mI)W + \varepsilon \cdot e & \geq 0, \\
e^T W & = 1, \\
W, \gamma, \varepsilon & \geq 0,
\end{align*}
\]

(35)

where \( \gamma = \max_i |\gamma_i| \) and \( \varepsilon = \max_j |\varepsilon_j| \). The above model can be further rewritten as

Minimize \( J = x \cdot \gamma + \beta \cdot \varepsilon, \)

\[
\begin{align*}
-BZ + (n - 1)Z & \leq \gamma \cdot e, \\
(A - mI)W + \varepsilon \cdot e & \geq 0, \\
e^T W & = 1, \\
W, \gamma, \varepsilon & \geq 0,
\end{align*}
\]

(36)

which is an LP model and can be solved using LINDO software package or Microsoft EXCEL solver. Let \( W^* = (w^*_1, \ldots, w^*_m)^T \) be the optimal weight vector generated by (36), based on which the overall weighted assessment value of each alternative can be computed and the decision can be made. Such an approach is referred to as the WMDN approach.

3.5. The extension to group decision-making

In group decision-making, different DMs may be able to provide different preference information because of their differences in domain knowledge and educational backgrounds. Consider a group decision-making problem where \( k_1 \) DMs provide the fuzzy preference relations \( P^{(k)}(k = 1, \ldots, k_1) \) on decision alternatives and \( k_2 \) DMs give the multiplicative preference relations \( A^{(l)}(l = 1, \ldots, k_2) \) on the relative importance weights of decision attributes. If the fuzzy preference relations \( P^{(k)}(k = 1, \ldots, k_1) \) and the multiplicative preference relations \( A^{(l)}(l = 1, \ldots, k_2) \) can be aggregated into a collective fuzzy preference relation \( P = (p_{ij})_{n \times n} \) and a collective multiplicative preference relation \( A = (a_{ij})_{m \times m} \), respectively, then the general model (19) can be directly used to
perform a group decision analysis; otherwise, the following extended optimization model should be utilized:

\[
\text{Minimize } J = \left\{ \sum_{k=1}^{k_1} \left( z_k \sum_{i=1}^{n} |\gamma_i^{(k)}|^p \right) + \sum_{l=1}^{k_2} \left( \beta_l \sum_{j=1}^{m} |\epsilon_j^{(l)}|^p \right) \right\}^{1/p},
\]

\[
\begin{align*}
\text{s.t.} & \quad [B^{(k)} Z - (n - 1) Z] W - \Gamma^{(k)} = 0, \quad k = 1, \ldots, k_1, \\
& \quad (A^{(l)} - m I) W - E^{(l)} = 0, \quad l = 1, \ldots, k_2, \\
& \quad e^T W = 1, \\
& \quad W \succ 0,
\end{align*}
\]

where

\[
B^{(k)} = \begin{bmatrix}
\sum_{j=2}^{n} p_{1j}^{(k)} & p_{12}^{(k)} & \cdots & p_{1n}^{(k)} \\
p_{21}^{(k)} & \sum_{j=1, j \neq 2}^{n} p_{2j}^{(k)} & \cdots & p_{2n}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1}^{(k)} & p_{n2}^{(k)} & \cdots & \sum_{j=1}^{n-1} p_{nj}^{(k)}
\end{bmatrix}, \quad k = 1, \ldots, k_1,
\]

\[
\Gamma^{(k)} = (\gamma_1^{(k)}, \ldots, \gamma_n^{(k)})^T, \quad E^{(l)} = (\epsilon_1^{(l)}, \ldots, \epsilon_m^{(l)})^T, \quad l = 1, \ldots, k_2,
\]

3.6. The choice of parameters

Both models (19) and (37) involve several parameters such as $p$, $\alpha$ and $\beta$. Theoretically, $p$ can take any positive number and an optimum value can also be determined by minimizing the objective function when the weight parameters $\alpha$ and $\beta$ are given. However, this will significantly increase the computational complexity of the models. On the other hand, the optimal $p$ value determined in this way is only applicable to the case considered and lacks generality. Therefore, a very practical way is to set $p = 1$ or $2$ or $\infty$. For $p = 1$ or $\infty$, models (19) and (37) are simplified as LP models. For $p = 2$, the models are simplified as quadratic programming (QP) models. It is known from the theory of LP that for an LP model, we can always conduct a sensitivity analysis to find an interval for each objective function coefficient and when the objective function coefficient varies within this interval, the optimum solution of the LP model remains unchanged. So, when $p$ is set as $1$ or $\infty$, the corresponding LP model will be relatively robust to the changes of $\alpha$ and $\beta$. In other words, if $p$ is set as $p = 2$, the corresponding QP model will be more sensitive to the changes of $\alpha$ and $\beta$. Therefore, if a DM expects a robust decision result, he/she should choose $p = 1$ or $\infty$; otherwise, he/she can choose $p = 2$. If they feel difficult in choosing an appropriate $p$ value, it is suggested that $p = 1, 2$ and $\infty$ be all set for decision-making so that the final decision can be made on a majority basis.

$\alpha$ and $\beta$ are weight parameters and are designed to increase the flexibility of the models so that they can reflect well the importance that a DM attaches to different preference relations. If the DM is unwilling or unable to provide a multiplicative preference relation on the weights of attributes or does not want to consider it, then he can set $\alpha = 1$ and $\beta = 0$. In this case, model (19) becomes

\[
\text{Minimize } J = \left\{ \sum_{i=1}^{n} |\gamma_i|^p \right\}^{1/p},
\]

\[
\begin{align*}
\text{s.t.} & \quad [B Z - (n - 1) Z] W - \Gamma = 0, \\
& \quad e^T W = 1, \\
& \quad W \succ 0,
\end{align*}
\]

(39)
which is the generalization of the LDM-1 for integrating subjective fuzzy preference relation and objective decision matrix information [18]. On the contrary, if the DM is unable to provide a fuzzy preference relation on decision alternatives or prefers not to consider it, then he can set \( \alpha = 0 \) and \( \beta = 1 \). Accordingly, model (19) can be written as

\[
\text{Minimize } J = \left( \sum_{i=1}^{n} |\varepsilon_i|^p \right)^{1/p},
\]

\[
\text{s.t.}\begin{cases}
(A - mI)W - E = 0, \\
e^TW = 1, \\
W \geq 0,
\end{cases}
\]

which is a generalized new model for deriving priorities from a pairwise comparison matrix. For \( p = 1 \), it becomes a GP model for generating priorities from a pairwise comparison matrix:

\[
\text{Minimize } J = \sum_{i=1}^{n} |\varepsilon_i|,
\]

\[
\text{s.t.}\begin{cases}
(A - mI)W - E = 0, \\
e^TW = 1, \\
W \geq 0.
\end{cases}
\]

For \( p = 2 \), (40) becomes the following QP model for generating a priority vector from a pairwise comparison matrix:

\[
\text{Minimize } J = W^T[(A - mI)^T(A - mI)]W,
\]

\[
\text{s.t.}\begin{cases}
e^TW = 1, \\
W \geq 0.
\end{cases}
\]

For \( p = \infty \), (40) becomes a minimax LP model for generating a priority vector from a pairwise comparison matrix:

\[
\text{Minimize } J = \delta,
\]

\[
\text{s.t.}\begin{cases}
-\delta \cdot e \leq (A - mI)W \leq \delta \cdot e, \\
e^TW = 1, \\
W, \delta \geq 0,
\end{cases}
\]

where \( \delta = \max_j |\varepsilon_j| \).

Usually, the DM may want to consider both the fuzzy preference relation on alternatives and the multiplicative preference relation on the weights of attributes. In this case, he can attach a different relative importance to each of the relations. For example, he can use the well-known AHP scale 1–9 to help determine the values of \( \alpha \) and \( \beta \). If the fuzzy preference relation is thought to be equally important as the multiplicative preference relation, then \( \alpha = \beta = 1/2 \). If the former is thought to be moderately important than the latter, then he may set \( \alpha = 3\beta \), i.e. \( \alpha = 3/4 \) and \( \beta = 1/4 \). If the fuzzy preference relation is considered as strongly important than the multiplicative preference relation, then \( \alpha = 5\beta \) can be set, namely, \( \alpha = 5/6 \) and \( \beta = 1/6 \). If the former is regarded as very strongly important than the latter, then the DM may set \( \alpha = 7\beta \), which leads to \( \alpha = 7/8 \) and \( \beta = 1/8 \). If the former is extremely important than the latter, then the DM can set \( \alpha = 9\beta \), i.e., \( \alpha = 9/10 \) and \( \beta = 1/10 \). It is suggested that the DM should test more than one set of values to conduct a sensitivity analysis so that a credible decision result can be obtained.

For the parameters \( \alpha_k \) and \( \beta_l \) in model (37), they are designed to reflect the relative importance of different DMs in a group decision-making. They can be determined in terms of factors such as the DMs’ domain knowledge, positions in the group decision-making and experience. If there is no significant evidence to show the inequality of the DMs in a group decision-making, \( \alpha_k \) and \( \beta_l \) can be set to be equal.

4. A numerical example

In this section, we provide a numerical example to illustrate how a DM’s fuzzy preference relation on alternatives and multiplicative preference relation on attributes can be integrated with the objective decision matrix information to reach a decision. The example is based on the published work by Fan et al. [5].
A potential buyer, the DM, intends to buy a house. He has four alternatives to choose from: \(A_1, \ldots, A_4\). The attributes (factors) he considers include price in \$ (\(G_1\)), size in m\(^2\) (\(G_2\)), distance to work in kms (\(G_3\)) and environmental characteristics (\(G_4\)). Among the four attributes, \(G_2\) and \(G_4\) are benefit attributes, which receive high values for desirable alternatives, and \(G_1\) and \(G_3\) are cost attributes, which receive low values for desirable alternatives. The decision matrix for this MADM problem is

\[
X = \begin{bmatrix}
3.0 & 100 & 10 & 7 \\
2.5 & 80 & 8 & 5 \\
1.8 & 50 & 20 & 11 \\
2.2 & 70 & 12 & 9 
\end{bmatrix}
\]

Both the fuzzy and multiplicative preference relations of the DM are given as

\[
P = \begin{bmatrix}
- & 0.44 & 0.64 & 0.54 \\
0.56 & - & 0.69 & 0.60 \\
0.36 & 0.31 & - & 0.40 \\
0.46 & 0.40 & 0.60 & - 
\end{bmatrix}
\]

\[
an d \quad A = \begin{bmatrix}
1 & 1/2 & 2 & 3 \\
2 & 1 & 3 & 4 \\
1/2 & 1/3 & 1 & 2 \\
1/3 & 1/4 & 1/2 & 1 
\end{bmatrix}
\]

To find the best choice for the DM, we first normalize the original decision matrix \(X\) by Eqs. (1)–(2). The normalized decision matrix \(Z\) turns out to be

\[
Z = \begin{bmatrix}
0 & 1 & 5/6 & 1/3 \\
5/12 & 3/5 & 1 & 0 \\
1 & 0 & 0 & 1 \\
2/3 & 2/5 & 2/3 & 2/3 
\end{bmatrix}
\]

From the given fuzzy preference relation \(P\) and Eq. (15), we have

\[
B = \begin{bmatrix}
1.62 & 0.44 & 0.64 & 0.54 \\
0.56 & 1.85 & 0.69 & 0.60 \\
0.36 & 0.31 & 1.07 & 0.40 \\
0.46 & 0.40 & 0.60 & 1.46. 
\end{bmatrix}
\]

Its maximum eigenvalue is \(\lambda_{\max} = 3\) and the corresponding normalized principal right eigenvector is derived from Eq. (14) as \(D = (d_1, d_2, d_3, d_4)^T = (0.2713, 0.3446, 0.1537, 0.2304)^T\), which ranks the four alternatives as \(A_2 > A_1 > A_4 > A_3\), where the symbol “\(>\)” means “is preferred or superior to”. This is the DM’s subjective preference ranking and \(A_2\) is his subjectively best choice. However, such a subjective preference ranking may be unable to be realized because both the decision matrix information and the weight preference relation have not been considered yet.

For the multiplicative preference relation \(A\), the following priority vectors can be derived from model (40) or its equivalent models (41)–(43) for \(p = 1, 2, \infty\):

\[
W = (0.2778, 0.4692, 0.1603, 0.0928)^T \text{ for } p = 1,
\]

\[
W = (0.2775, 0.4691, 0.1594, 0.0936)^T \text{ for } p = 2,
\]

\[
W = (0.2772, 0.4686, 0.1595, 0.0946)^T \text{ for } p = \infty,
\]

which are all quite close to the eigenvector weights \(W = (0.2772, 0.4673, 0.1601, 0.0954)^T\) with \(\lambda_{\max} = 4.0310\) and \(CR = 0.011 < 0.1\) derived by Saaty’s EM method. This shows that the generalized model (40) is a good option to derive priorities from a pairwise comparison matrix. The above four priority vectors on the attributes lead to the four sets of overall weighted assessment values for the four alternatives by Eqs. (4) or (5), which are shown as follows:

\[
D = (0.6336, 0.5575, 0.3706, 0.5416)^T \text{ by the } W \text{ for } p = 1,
\]

\[
D = (0.6333, 0.5566, 0.3714, 0.5416)^T \text{ by the } W \text{ for } p = 2,
\]

\[
D = (0.6331, 0.5562, 0.3718, 0.5417)^T \text{ by the } W \text{ for } p = \infty,
\]

\[
D = (0.6325, 0.5560, 0.3726, 0.5421)^T \text{ by the EM weights.}
\]
The four sets of assessment values all rank the four alternatives as $A_1 > A_2 > A_4 > A_3$. This is the ranking derived from the multiplicative preference relation and the objective decision matrix information, but the DM’s fuzzy preference relation has not been considered yet. So, this ranking may still be not the final ranking.

To determine the final ranking of the four alternatives, both the fuzzy and multiplicative preference relations and the decision matrix information have to be considered at the same time. This can be done by solving model (19) or its equivalent models (24), (25) and (35) for $p = 1$, $2$ or $\infty$. For different combinations of subjective preferences and objective information, Tables 1–6 show the estimated optimal relative weights of the four attributes and the overall weighted assessment values of the four alternatives as well as their final rankings obtained by different approaches for $p = 1$, $2$ and $\infty$.

Since $(\alpha, \beta) = (0, 1)$ and $(\alpha, \beta) = (1, 0)$ stand for two extreme combinations, which either take no account of fuzzy preference relation or consider no multiplicative preference relation, they should be avoided when the two preference relations have to be considered simultaneously. In this situation, no matter how $\alpha$ and $\beta$ are combined, the three approaches, WLDN, WLSDN and WMDN, all produce the same final ranking: $A_1 > A_2 > A_4 > A_3$. So, $A_1$ is the best choice for this DM.

It is also observed that the weights obtained by the WLSDN approach are more sensitive to the changes of $\alpha$ and $\beta$ than those obtained by the WLDN and the WMDN approaches. This observation confirms our analysis in Section 3.6 that model (19) with $p = 1$ or $\infty$ is more robust to changes of $\alpha$ and $\beta$ than with $p = 2$. 

---

**Table 1**
Optimal relative weights of the four attributes obtained by the WLDN approach

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.3413</td>
<td>0.3152</td>
<td>0.3435</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.2723</td>
<td>0.4643</td>
<td>0.2051</td>
<td>0.0582</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.2728</td>
<td>0.4692</td>
<td>0.1603</td>
<td>0.0928</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.2728</td>
<td>0.4692</td>
<td>0.1603</td>
<td>0.0928</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.2728</td>
<td>0.4692</td>
<td>0.1603</td>
<td>0.0928</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.2728</td>
<td>0.4692</td>
<td>0.1603</td>
<td>0.0928</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.2778</td>
<td>0.4692</td>
<td>0.1603</td>
<td>0.0928</td>
</tr>
</tbody>
</table>

**Table 2**
Overall weighted assessment values of the four alternatives and their rankings by the WLDN approach

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.6014</td>
<td>0.6748</td>
<td>0.3413</td>
<td>0.5826</td>
<td>$A_2 &gt; A_1 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.6547</td>
<td>0.5972</td>
<td>0.3305</td>
<td>0.5429</td>
<td>$A_1 &gt; A_2 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.6336</td>
<td>0.5575</td>
<td>0.3706</td>
<td>0.5416</td>
<td>$A_1 &gt; A_2 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.6336</td>
<td>0.5575</td>
<td>0.3706</td>
<td>0.5416</td>
<td>$A_1 &gt; A_2 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.6336</td>
<td>0.5575</td>
<td>0.3706</td>
<td>0.5416</td>
<td>$A_1 &gt; A_2 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.6336</td>
<td>0.5575</td>
<td>0.3706</td>
<td>0.5416</td>
<td>$A_1 &gt; A_2 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.6336</td>
<td>0.5575</td>
<td>0.3706</td>
<td>0.5416</td>
<td>$A_1 &gt; A_2 &gt; A_4 &gt; A_3$</td>
</tr>
</tbody>
</table>

**Table 3**
Optimal relative weights of the four attributes obtained by the WLSDN approach

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.3263</td>
<td>0.2965</td>
<td>0.3773</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.2820</td>
<td>0.4630</td>
<td>0.1955</td>
<td>0.0594</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.2751</td>
<td>0.4692</td>
<td>0.1782</td>
<td>0.0775</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.2747</td>
<td>0.4699</td>
<td>0.1732</td>
<td>0.0821</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.2750</td>
<td>0.4701</td>
<td>0.1694</td>
<td>0.0856</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.2761</td>
<td>0.4697</td>
<td>0.1636</td>
<td>0.0905</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.2775</td>
<td>0.4691</td>
<td>0.1594</td>
<td>0.0939</td>
</tr>
</tbody>
</table>
Finally, we point out that the example examined in this section is also applicable to a couple who want to buy a house or something else when one provides a fuzzy preference relation and the other gives a multiplicative preference relation. The final decision will be based on both preference relations and objective decision matrix information.

5. Concluding remarks

In this paper, we developed a general MADM approach for integrating subjective preferences and objective decision matrix information so that decision can be made on an integrated basis. Three special cases of the approach were investigated. The extension of the approach to group decision-making and the choice of decision parameters were also discussed. A numerical example was examined to illustrate the application of the proposed MADM approaches. Since the approaches consider both the fuzzy and the multiplicative preference relations simultaneously, any approach considering only one of them can be seen as a special case of our approaches.

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References