A compressed sensing approach for robust time-frequency analysis of signals corrupted by strong heavy-tailed noise is proposed. When using traditional time-frequency distributions and the corresponding ambiguity functions, the strong and impulsive nature of the noise introduces spurious peaks and compromises the sparse time-frequency signal reconstruction. In order to provide accurate localization of the signal power and reduce false positives, compressed sensing is applied to the robust ambiguity function based on the L-estimation approach. This enhances the sparse time-frequency trajectories that correspond to the instantaneous frequencies of signal components. Simulation examples involving non-Gaussian noise and signals with different instantaneous frequency laws are provided to demonstrate the effectiveness of the proposed approach.

1. INTRODUCTION

Compressive Sensing (CS) is a framework for simultaneous sensing and compression. It has been used in various applications involving one-dimensional and two-dimensional signals [1]-[7]. These include sparse signal and antenna arrays, sparse indoor and SAR imaging, sparse communication channel estimation, sparse subsurface and remote sensing, and sparse MTI for urban sensing. In essence, CS is based on $\ell_1$ reconstruction, in lieu of $\ell_2$ and the least squares minimization.

In CS, the number of measurements required to characterize a time series or to provide a detailed description of an imaged scene with proper target locations can be reduced depending on the sparseness of the data in a certain domain. The unknown sparse signal or image vector to be recovered can be sparse in its own domain or upon linear transformation that might be based on DFT, DWT, DCT, or using any other orthogonal basis expansion. The proper expansion depends on the sensing environment as well as the sensing modality and waveforms.

In general, a signal which is $K$ sparse in a specific domain can be completely characterized by $M$ measurements ($M>K$) with $M<<N$, where $N$ is the number of samples dictated by the Nyquist theorem. This can be achieved through convex optimization that uses the sparsity as important a priori information.

In this paper, we deal with a vectorized two dimensional joint-variable signal representation and apply compressive sensing to achieve high resolution sparse signal characterization in the time-frequency domain. In particular, we consider nonstationary signals corrupted by heavy-tailed noise. The $\ell_1$ minimization underlying CS emphasizes a sparse vector solution and reduces spurious, false positives, and noisy components that may accompany other solutions or representations based on different norms [1],[2]. The signal sparsity property, whether it is present in a single variable or multi-variable domains, has extensively been used as fundamental premise for an efficient signal processing, statistical estimation, classification and compression algorithms.

Since such approach uses an incomplete set of measurements, computationally efficient algorithms have been proposed and employed [8],[9].

More recently, compressed sensing has been successfully applied in the area of time-frequency signal representation. It is recognized that a nonstationary signal, which is neither sparse in time or frequency, may become sparse and, as such, amenable to CS application in the joint-variable domain. A simple example is a chirp signal whose power concentration around its instantaneous frequency renders most of the time-frequency points of zero or close to zero values. This property can be generalized to a wide class of nonstationary signals which are uniquely characterized by
their respective instantaneous frequency laws. These laws allow only a few of time-frequency points, or atoms, to take non-zero values, whereas the vast majority of points remain of negligible values. Clearly, the time-frequency sparsity would be reduced when dealing with multi-component signals, as it increases domain occupancy and decreases the ratio of zero to non-zero values.

Unlike most cases, in which the signal is typically sparse in one-domain and non-sparse in others, joint variable description of many signals, including those of polynomial phase characterization, yields sparse representation in two equivalent domains, such as the time-frequency domain and the ambiguity domain. These two domains are related by two-dimensional Fourier transform. CS can, therefore, be applied in either domain to reconstruct the signal in the other domain. For instance, when performed on the observations in the ambiguity domain, CS leads to a sparse reconstruction of signal instantaneous frequency \[10\]. This reconstruction was shown to be of higher resolution than that obtained from just applying the inverse two-dimensional transformation. It is noted, however, that the CS approach, although it provides desirable reconstruction for certain types of signals in the presence of Gaussian noise, it can face difficulties when considering other types of noise. Specifically, if the signal is corrupted by impulsive noises, the standard distributions provide poor signal representation. As such, compressive sensing techniques acting on ambiguity-domain samples, may subsequently lead to inaccurate or incorrect signal power time-frequency localization. This is due, in major part, to failure to properly account for the nature of the additive noisy pulses in the sparse time-frequency representation.

In this paper, we propose a robust approach to compressed sensing based time-frequency analysis. Namely, we use robust time-frequency representations, in lieu of standard distributions. Among possible different robust time-frequency approaches, the L-estimate form of the Wigner distribution has successfully been used \[11\], \[12\]. This approach is most suitable for signals with mixed Gaussian and impulse noise. The robust distribution can be combined with compressed sensing to gain the benefits of both techniques and provide enhanced solutions, which is difficult to achieve by using only one approach individually. This is especially important in the case of strong noise and multi-component signals, when certain spurious peaks cannot be simply removed by the robust distribution. We use the robust ambiguity domain as the domain of observations. The underdetermined problem formulation is obtained by only considering the samples around the origin in the ambiguity domain. These reduced observations are used in \(\ell_1\) minimization reconstruction to yield the sparsest time-frequency representation. However, one should note that the achieved sparse time-frequency representation does not provide the energy of the spectrum over time neither satisfies marginal conditions, but can be used for the instantaneous frequency estimation.

In reducing the number of observations by more than 98%, we show that the proposed robust approach significantly eliminates the influence of noisy pulses, providing better sparse time-frequency representation compared to the standard approach. Simulation examples involving non-Gaussian noise and signals with different instantaneous frequency laws are provided to demonstrate the effectiveness of the proposed approach.

The paper is organized as follows. The application of compressed sensing in the standard time-frequency analysis is given in Section II. The robust approach is proposed in Section III. The simulation results are illustrated in Section IV, while the concluding remarks are given in Section V.

2. TIME-FREQUENCY REPRESENTATION AND COMPRESSED SENSING

Let us consider the Wigner distribution, as one of the commonly used time-frequency distributions \[13\], defined by:

\[
WD(t,\omega) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})^* (t-\frac{\tau}{2}) e^{-j\omega\tau} d\tau,
\]

(1)

The ambiguity domain counterpart of the Wigner distribution is the ambiguity function. They are related by the two-dimensional Fourier transform as follows:

\[
A(\tau,\theta) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})^* (t-\frac{\tau}{2}) e^{-j\omega\tau} dt = \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} WD(t,\omega) e^{-j(\theta\tau-\omega\tau)} dtd\omega.
\]

(2)

The advantage of using the ambiguity domain and its relation with the time-frequency domain has been widely used in the analysis of multicomponent signals \[14\]-\[17\]. For multicomponent signals, the Wigner distribution produces undesired components called cross-terms, which appear between signal’s auto-terms, at the position of their arithmetic mean. On the other hand, the cross-terms are usually dislocated from the origin in the ambiguity domain and can be suppressed, or significantly attenuated, by the use of low-pass filtering. This is achieved by applying the kernel \(c(\tau,\theta)\):

\[
A_f(\tau,\theta) = A(\tau,\theta)c(\tau,\theta).
\]

(3)

The resulting time-frequency distributions based on \(A_f(\tau,\theta)\) belong to the Cohen class, defined as:
from the measurements are the amplitude and the phase function of the \( f_1 \), which
and 
\[ t \]
matrices exhibit a very low coherence with the FT matrix.

- Typically chosen as a random matrix, since random
discussed in [6].

- Components (where 
consider the Wigner distribution and the ambiguity
acquisition, which is typically desired in CS, is not a main
emphasizing cross-terms. The other objective is fast data
improving time-frequency signal resolution, while de-
domain [10]. It is remarked that the objective is to
approach and exploiting sparsity in the time-frequency

- Improved time-frequency signal power localization
can be achieved by using the compressed sensing
approach and exploiting sparsity in the time-frequency
domain [10]. It is remarked that the objective is to
improve time-frequency signal resolution, while de-
emphasizing cross-terms. The other objective is fast data
acquisition, which is typically desired in CS, is not a main
drive behind this time-frequency CS application. If we
consider the Wigner distribution and the ambiguity
function in the form of \( N \times 1 \) vectors \( W_1 \) and \( A_1 \),
respectively, then in the sense of compressed sensing we
can write:

\[ A_1 = \Psi W_1 \], \hspace{1cm} (5)

where \( \Psi \) is of \( N \times N \) dimension and represents the two-
dimensional Fourier transform matrix. By using the
measurement matrix \( \Phi \) (of size \( M \times N \)), \( M \) linear
measurements of vector \( A_1 \) can be taken as follows:

\[ A_1^M = \Phi A_1 = \Phi \Psi W_1 \]. \hspace{1cm} (6)

The sensing, or sampling matrix \( \Phi \) and the 2D Fourier
transform matrix \( \Psi \) should represent a low coherence	pair. The combined matrix is referred to as the
representation dictionary. The sampling matrix \( \Phi \) is
typically chosen as a random matrix, since random
matrices exhibit a very low coherence with the FT matrix.
The merits of using different sampling structures are
discussed in [6]. In the underlying problem, a mask
applied in the ambiguity domain constitutes the sampling
matrix and defines the reduced measurements, as shown
below. In order to reconstruct \( W_1 \) from the measurements
\( A_1^M \), one should search for the sparsest vector \( W_1 \) which
is consistent with \( A_1^M \).

- The sparsity assumption is defined as follows: the
\( N \times N \) time-frequency representation of signal with \( K 
\) components (where \( K < N \)) should have at most \( K \times N 
non-zero points. Hence, the compressed sensing approach is
applied as follows:

- Collect a set of samples from the ambiguity
domain;
- Solve the \( \ell_1 \)-norm minimization problem to
obtain the sparsest time-frequency distribution.

Here, it is important to select a suitable set of ambiguity
domain samples, which is done by an appropriate
ambiguity function masking. The mask can be formed as a
small area around the origin of the ambiguity plane or on
the basis of some other a priori information about signal
support in this domain. It has been shown that this
approach reduces the cross-terms as well as a certain
amount of Gaussian noise. However, in the presence of
impulsive noise (which can be even mixed with the
Gaussian noise), the standard time-frequency distributions
and corresponding ambiguity functions are unsuitable.
Hence, in the next section, we propose the robust sparsity-
based time-frequency approach.

3. ROBUST TIME-FREQUENCY
REPRESENTATION AND COMPRESSED SENSING

An ideal time-frequency distribution can be represented in the form:

\[ D_{\text{ideal}} = \sum_{i=1}^{P} A_i^2 (\tau) \delta (f - \phi_i (\tau) / 2 \pi) \] \hspace{1cm} (7)

where \( P \) is the number of signal components, while \( A_i \) and
\( \phi_i (\tau) \) are the amplitude and the phase function of the \( i \)-th
component, respectively. Hence, in the case of ideal
representation at each time instant the time-frequency
representation is characterized by a single point.
According to the theory, the localized distribution with the
smallest possible number of non-zero coefficients can
be obtained as a solution of \( \ell_0 \)-norm minimization of the
time-frequency distribution. However, in practice we may
use the near-optimal solutions based on the \( \ell_1 \)-norm
minimization. Here, we consider the robust compressed
sensing approach and thus the desired robust time-
frequency distribution \( D_1 \) can be obtained as:

\[ D_1 = \arg \min_{\Psi} \| \Psi \|_1 \] \hspace{1cm} \( \Psi^{-1} \{ D \} - A_1 = 0 \) \hspace{1cm} (8)

where \( A_1 \) denotes the set of samples from the robust
ambiguity domain function in the region defined by the
mask \( (\theta, \tau) \in \Omega \), while \( D \) denotes the robust
time-frequency distribution.

In the presence of noise, instead of (8), we may use the
approximation:

\[ D_1 = \arg \min_{\Psi} \| \Psi \|_1 \] \hspace{1cm} \( \| \Psi^{-1} \{ D \} - A_1 \|_2 \leq \epsilon \) \hspace{1cm} (\theta, \tau) \in \Omega .

The optimization is based on a small number of robust
ambiguity function points taken from the center part of
the ambiguity domain [10]. The mask in the ambiguity domain from which we obtain the measurement vector $A_L$ is illustrated in Figure 1.

The robust ambiguity function can be efficiently realized in the L-estimate form which is not only suitable for impulsive noise, but also for a mixed Gaussian and impulsive noise. The L-estimate ambiguity function and the L-estimate Wigner distribution can be observed as a 2D Fourier transform pair:

$$A_L(\theta, \tau) = \mathbb{F}_{2D}(W_{LD}(t, \omega)). \quad (9)$$

The basic idea behind the L-estimate form can be described as follows [11]:

1) The elements $x(n+m)\bar{x}(n-m)e^{-j4\pi km/N_s}$ are sorted. Consequently, the samples corrupted by impulse noise are shifted to the left and right end of the sorted sequence;

2) An amount of samples from the ends of sequence are discarded, while the mean value is calculated for the remaining samples. Therefore, the L-estimate Wigner distribution can be written as [11],[12]:

$$W_{LD}(n,k) = \sum_{i=-N_s/2}^{N_s/2-1} a_i r_i(n,k),$$

$$r_i(n,k) \in R(n,k) = \left\{ \Re(x(n+m)\bar{x}(n-m)e^{-j4\pi km/N_s}) \right\},$$

$$m \in [-N_s/2, N_s/2),$$

where the elements $r_i(n,k)$ are sorted in non-decreasing order as: $r_i(n,k) \leq r_{i+1}(n,k)$, while the coefficients $a_i$ are defined by:

$$a_i = \begin{cases} 
\frac{1}{N_s(1-2\alpha)+4\alpha}, & \text{for } i \in [(N_s-2\alpha\alpha(2-N_s)+N_s-1] \\
0, & \text{elsewhere} \end{cases} \quad (11)$$

where $N_s$ is even, while the parameter $\alpha$ takes values within the range $[0,1/2]$. Higher value of $\alpha$ provides better reduction of heavy-tailed noise, while smaller value of $\alpha$ improves spectral characteristics. Thus, the value of parameter $\alpha$ should be chosen to provide good trade-off between these requirements.

4. SIMULATION RESULTS

4.1. Example 1

Consider the monocomponent signal in the form:

$$x_1 = e^{j(16/3 \cos(3/2 \pi x)+13 \cos(\pi x))} + u(t),$$

where $u(t) = 0.17(u_1^3(t) + jv_1^3(t))$, where $u_1(t)$ and $v_1(t)$ are Gaussian noises with variance equal to 1 (the overall noise is heavy-tailed). The value of parameter $\alpha$ in the realization of the $\alpha$-trimmed coefficients $a_i$ is $\alpha=4/5$.

In order to provide faster computations, we use, in this example, the time-frequency representations of size 60x60 (large size would require higher computational time and significant processing power). Hence, the total number of points in the time-frequency as well as in the ambiguity domain is 3600. The time-frequency mask is obtained by using the central region of size 7x7 in the ambiguity domain. In so doing, we use approximately 1.4% of the total number of points in the ambiguity domain.

**Figure 2.** a) Wigner distributions (standard left, robust right), b) ambiguity functions (standard left, robust right), c) resulting sparse time-frequency representation (the standard approach - left, the robust approach - right)
Figure 2 illustrates the results obtained by using the standard (left column) and the robust approach (right column). The standard and robust time-frequency distributions are given in Figure 2.a.

The corresponding ambiguity functions are given in Figure 2.b: the standard ambiguity function – left, the robust ambiguity function - right. The resulting sparse time-frequency representations are given in Figure 2.c (standard approach on the left side, and the robust approach on the right side). Note that the standard compressed sensing approach, due to the noise influence, produces significant errors, where several noisy peaks appear. On the other side, the robust approach provides improved results, as it reduces the noise influence significantly. The number of non-zero points in the resulting L-estimate sparse time-frequency representation is approximately between 45 and 50 (estimated from different experiments), which is again a very small percentage of the total number of points in the time-frequency domain.

It is also important that the proposed robust approach do not decrease the quality of results in the case of non-impulsive noise, such as the common Gaussian noise. Hence, we consider the case with pure Gaussian noise, as well. The same signal is used, with the Gaussian noise variance equal to 1. The standard Wigner distribution and the L-estimate Wigner distribution in the presence of Gaussian noise are shown in Figure 3.a (left and right, respectively). The sparse time-frequency representations achieved as the results of compressed sensing are shown in Figure 3.b (standard approach left and robust approach right). Note that, in the case of Gaussian noise, the L-estimate approach provides the same quality of signal power distribution as the standard approach, since the L-estimate robust form also involves the mean value calculations.

### 4.2. Example 2

Next we consider the noisy multicomponent signal which consists of a chirp and a sine frequency modulated component:

\[ x_2 = e^{j(16/5 \cos(3/2 \pi t) + 6 \cos(\pi t) + 12 \pi t)} + e^{-j(5\pi t^2 + 20\pi t)} + v(t). \]

The noise parameters are the same as in the previous example. The time-frequency representations are given in Figure 4.a: the standard form on the left side, the robust form on the right side. The corresponding ambiguity functions are shown in Figure 4.b, while the sparse time-frequency representation are shown in Figure 4.c. Additionally, in Figure 4.d, we present one additional example for another noise realization (noise is generated in the same random way in each example).
5. CONCLUSION

The robust compressed sensing based approach for localized time-frequency representation of signals corrupted by heavy-tailed noise is proposed. Due to the pulsed non-Gaussian nature of the noise, the classical time-frequency analysis provides unsatisfactory performance of IF estimation. This poor performance motivated the consideration of alternative methods, such as the robust compressed sensing based approach. Using few samples from the L-estimate ambiguity function, the proposed approach succeeded in reducing the noise influence and providing proper sparse time-frequency representation. This improved performance was demonstrated by two simulation examples involving nonstationary signals with different frequency laws.

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