Decision-Directed Fine Synchronization for Coded OFDM systems

Kai Shi, Erchin Serpedin, and Philippe Ciblat

Abstract

A new decision-directed (DD) synchronization scheme is proposed for joint estimation of carrier frequency offset (CFO) and sampling clock frequency offset (SFO) in coded orthogonal frequency division multiplexing (OFDM) systems. By exploiting the decisions provided by a Viterbi decoder and the information available on all the modulated subcarriers, we report accurate estimators of residual CFO and small SFO without relying on pilots. The performance analysis and simulation results indicate that the proposed novel DD scheme achieves much better performance than the conventional pilot-based schemes in both AWGN and frequency-selective channels.

Index Terms

OFDM, carrier frequency offset, sampling clock frequency offset, synchronization, FFT, Decision-Directed

K. Shi and Dr. E. Serpedin are with the Dept. of Electrical Engineering, Texas A&M University, College Station, TX 77843-3128, USA. Phone: (979) 458 2287, Fax: (979) 862 3128, email: serpedin@ee.tamu.edu. Dr. P. Ciblat is with ENST, 46 rue Barrault, 75013 Paris, France.

The corresponding author is Dr. Serpedin. This work was supported by the NSF Award No. CCR-0092901 and TITF Program-Texas A&M University. This work was submitted in part for presentation in ICASSP 2004 Conference.
I. INTRODUCTION

The orthogonal frequency division multiplexing (OFDM) system is well fit for high speed transmission in highly frequency-selective (F-S) channels. However, OFDM is sensitive to synchronization errors. Numerous papers [1]-[4] deal with coarse frame (timing) and carrier frequency offset (CFO) synchronization while quite a few ones discuss the estimation of residual CFO and possible sampling clock frequency offset (SFO), the so called fine synchronization. For coarse frame and CFO synchronization, most schemes [2]-[4] proposed some estimators before the fast Fourier transform (FFT). Since no reliable data decision is available before FFT, data-aided (DA) or blind estimators can be used for pre-FFT synchronization. However, after coarse synchronization, there might still be present a residual CFO and uncompensated SFO, which will introduce large phase rotations even for shorts packet [5]-[6].

To remove the effect of CFO and SFO, some authors proposed pilot based post-FFT synchronizers [7]-[10]. Although the estimator [7] appears to work under general channel conditions, no analytical result has been reported to assess its unbiasedness in F-S channels. Also, the alternative estimator [11] appears to be biased in F-S channels. In this paper, we propose a new decision-directed (DD) post-FFT CFO and SFO synchronization scheme without using pilots. By utilizing the conjugate product of two consecutive OFDM symbols and the reliable decisions provided by a Viterbi decoder, we report first a one-shot DD joint estimation scheme of CFO and SFO. To obtain highly-accurate CFO and SFO estimates in F-S channels, the one-shot estimates are further passed through first-order tracking loop filters that used to control the interpolator and frequency corrector blocks. It is shown that the proposed CFO and SFO estimators are unbiased in both AWGN as well as F-S channels. Analytical closed-form expressions of the mean-square error (MSE) of proposed estimators are also reported for AWGN channels.

The outline of paper is as follows. In Section II, signal and channel models are presented. Section III presents a short overview on the previous results regarding data-aided CFO and SFO estimation schemes. Novel decision-directed post-FFT CFO and SFO synchronization schemes together with performance analysis results for both open-loop and closed-loop performance are presented in Section IV. Computer simulations are presented in Section V in order to corroborate the theoretical performance analysis. The paper is concluded in Section VI.
II. SIGNAL MODELS

In the transmitter of a coded OFDM system, the binary source data is passed through a convolutional encoder and block interleaver and Gray mapped into $a_{l,k}$, which denotes the complex data modulated on the $f_k = k/T_u$ subcarrier frequency of the $l$th OFDM symbol, which is assumed of unit variance $E\{|a_{l,k}|^2\} = 1$. The transmitted complex baseband signal can be described by

$$s(t) = \frac{1}{\sqrt{T_u}} \sum_{l=0}^{\infty} \sum_{k=0}^{K/2} a_{l,k} e^{j2\pi(k/T_u)(t-T_0-lT)} g(t-lT) ,$$

where $g(t)$ is a rectangular pulse with unit amplitude during $0 \leq t < T$. To avoid intersymbol interference (ISI) in frequency-selective channels, each symbol is preceded by a guard interval (cyclic prefix) of length $T_g$, which represents a repetition of the last portion of symbol. To simplify the transmitter, we assume a discrete-time implementation (with the sampling period $T_s = T_u/N$) of $s(t)$, and that can be easily generated by means of an $N$-point inverse fast Fourier transform (IFFT). In addition, $K$ is chosen to be less than $N$ to avoid spectrum aliasing. Therefore, the symbol period is $T = T_g + T_u$, which corresponds to $M = N + N_g$ samples.

If the transmitted signal is passed through the wide-sense stationary uncorrelated scattering (WSSUS) channel with channel transfer function (CTF)

$$h(\tau, t) = \sum_i h_i(t) \delta(\tau - \tau_i) ,$$

the received signal sampled with the period $T_s'$, in the presence of carrier frequency offset (CFO) $f_c$, timing offset $n_c T_s'$ and small sampling clock frequency offset (SFO) $\epsilon = (T_s' - T_s)/T_s$, is given by

$$r(nT_s') = e^{j2\pi f_c nT_s'} \sum_i h_i(nT_s') s(nT_s' - \tau_i - n_c T_s') + w(nT_s') ,$$

where each path $h_i(nT_s')$ presents a Rayleigh distributed amplitude and a uniformly distributed phase. The channel energy is assumed normalized to unity $\sum_i E[|h_i(nT_s')|^2] = 1$. To avoid ISI, the normalized maximum delay spread $\tau_{\text{max}}$ (normalized by $T_s'$) is assumed less than the length of guard interval $N_g$. In addition, $w(nT_s')$ denotes complex additive white Gaussian noise (AWGN) with variance $\sigma_w^2 = E\{|w(nT_s')|^2\}$. The average signal-to-noise ratio (SNR) for data subcarriers is defined as $E_s/N_0 = 1/\sigma_w^2$. 
After the coarse timing estimation as [2]-[3], \( \hat{n}_e \) is used by the FFT window controller as shown in Fig. 1. Therefore, the FFT window can be assumed to start from the ISI-free area \([\tau_{\text{max}} + 1 + lM, N_g + lM]T_s^\prime \). There might be a small timing offset after coarse timing synchronization, which can be absorbed in CTF [6]. To reduce possible intercarrier interference (ICI), a coarse CFO estimation \( \hat{f}_e \) is used by the frequency corrector block. The output of the \( N \)-point FFT block can be expressed as:

\[
z_{l,k} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} r_{l,n} e^{-j2\pi kn/N},
\]

where \( r_{l,n} = r((n + N_g + lM)T_s^\prime) \). Taking into account the small SFO \( \epsilon \) and residual CFO \( f_N = f_eT_u \) (normalized by subcarrier spacing), after some manipulations similar to ones in [6], the \( l_{th} \) symbol of the \( N \)-point FFT block takes the expression:

\[
z_{l,k} \approx a_{l,k} H_{l,k} \left( \frac{\sin(\pi \theta_k)}{N \sin(\pi \theta_k/N)} \right) e^{j\pi \theta_k(N-1)/N} e^{j2\pi \theta_k(N_g+lm)/N} + \text{ICI} + n_{l,k},
\]

where \( n_{l,k} \) is AWGN noise with the same variance as \( w_n \) and

\[
\theta_k = f_N(1 + \epsilon) + \epsilon k \approx f_N + \epsilon k.
\]

In the approximate expression (5), the ICI caused by small CFO and SFO can be omitted since its power is very small with respect to the additive noise \( \left(n_{l,k}\right) \) power [5], [6]. We also omit the effect of slow drifts of the FFT window, caused by small SFO \( \epsilon \). However, the FFT window shift can be large enough to induce ISI if \( \epsilon \) is not compensated. We also assume the channel to be constant during on OFDM symbol duration, and its taps are denoted by \( h_{l,n} = h_n(lMT_s^\prime) \). Let \( H_{l,k} \) denote the Fourier transform of the CTF

\[
H_{l,k} = \sum_{n=0}^{N_{\text{max}}} h_{l,n} e^{-j2\pi kn/N}.
\]

In (5), the residual CFO and SFO introduce a time variant phase rotation \( \phi_{l,k} = 2\pi \theta_k(N_g + lM + (N - 1)/2)/N \), which depends on both the time index \( l \) and the subcarrier index \( k \). Thus, for a certain value of \( l \), the accumulated phase rotation \( \phi_{l,k} \) can be large enough to introduce data decision errors. To avoid possible decision errors, we need to estimate and compensate \( \phi_{l,k} \).

Alternatively, we can estimate CFO \( f_N \) and SFO \( \epsilon \) separately. For a feedback synchronization scheme, as we plan to utilize later, it is undesirable if the estimation parameter depends on the
time index $l$. As suggested by [6], the effect of time index can be cancelled out by taking the conjugate product of two consecutive OFDM symbols

$$x_{l,k} = z_{l,k} \cdot z_{l-1,k}^* = e^{j2\pi \rho \theta_k} a_{l,k} a_{l-1,k}^* |H_{l,k}|^2 + \text{noise},$$

where $\rho = M/N$, $*$ denotes conjugate operation and the channel is assumed to be quasi-static $H_{l,k} \approx H_{l-1,k}$. To simplify subsequent discussions, we assume hereafter that $a_{l,k}$ is M-PSK modulated data or pilots.

III. Previous Results

A post-FFT data-aided (DA) CFO and SFO estimator was proposed in [7] and [8]

$$\hat{f}_N = \frac{1}{2\pi \rho} \frac{\varphi_{l,1} + \varphi_{l,2}}{2}, \quad \hat{\epsilon} = \frac{1}{2\pi \rho} \frac{\varphi_{l,2} - \varphi_{l,1}}{K/2},$$

where $\varphi_{l,1/2} = \arg \left[ \sum_{k \in P_{1/2}} x_{l,k} \right]$ and $P_{1/2}$ denote 1 for the first and 2 for the second half of pilots which are symmetrically and uniformly distributed around DC ($k = 0$). It is easy to prove that estimators (9) are unbiased in the presence of AWGN and flat fading channels. In [8], the mean-square error of (9) in AWGN channels are reported:

$$\text{MSE}(\hat{f}_N) = \frac{1}{4\pi^2 \rho^2 N_P \cdot E_s/N_0}, \quad \text{MSE}(\hat{\epsilon}) = \frac{4}{\pi^2 \rho^2 K^2 N_P \cdot E_s/N_0},$$

respectively, where $N_P$ stands for the number of pilots per symbol. From (10), we can infer that the performance of estimators (9) can be improved by increasing the number of pilots $N_P$. However, increasing the number of pilots will decrease the system throughput.

IV. Proposed Synchronization Scheme

A. Decision-Directed Estimator

We notice that if data decisions $\hat{a}_{l,k}$ are available, they can be used to improve the performance of CFO and SFO estimators. For broadcasting systems, such as the European DVB system, reliable data decisions are available after the receiver has acquired some channel information based on the inserted pilots. Therefore, decision-directed (DD) schemes can be utilized to improve the tracking performance of CFO and SFO synchronizers in broadcasting systems. For burst transmission systems, such as IEEE 802.11a standard, the synchronization preamble contains continuous pilots. Reliable data decisions $\hat{a}_{l,k}$ are available right after coarse synchronization.
and channel estimation. Thus, for burst OFDM transmission systems, we can use a DD scheme not only for tracking but also for acquisition. To make use of reliable decisions, the Viterbi decoder output is re-interleaved and re-mapped\(^1\) to the complex data decisions \(\hat{d}_{l,k}\). Therefore, we propose the following estimator

\[
\hat{f}_N = \frac{1}{2\pi} \frac{\varphi_{l,1} + \varphi_{l,2}}{2}, \quad \hat{e} = \frac{1}{2\pi} \frac{\varphi_{l,2} - \varphi_{l,1}}{K/2 + 1} \tag{11}
\]

where

\[
\varphi_{l,(1|2)} = \arg \left[ A_{l,(1|2)} \right], \quad A_{l,(1|2)} = \sum_{k \in C_{(1|2)}} z_{t,k} \hat{a}_{l,k}^* \hat{z}_{l-1,k} \hat{a}_{l-1,k}, \tag{12}
\]

and \(C_1 = [-K/2,-1], C_2 = [1,K/2]\) denote the first and second half of data subcarriers, respectively. From (11), we expect that the performance of proposed estimators might be much better than (9) since all the data subcarriers have been utilized. In Appendix-A, we show that the MSE of proposed estimator in AWGN channels is given by

\[
\text{MSE}(\hat{f}_N) = \frac{1}{4\pi^2 \rho^2 K \cdot E_s/N_0}, \quad \text{MSE}(\hat{e}) = \frac{4}{\pi^2 \rho^2 K(K + 1)^2 \cdot E_s/N_0}. \tag{13}
\]

In Appendix-B, we have also shown that estimators (11) are unbiased in F-S channels for small \(\epsilon\). A similar proof can be carried out for the estimator (9) using some slight modifications. However, calculation of a closed-form expression for the MSE of estimators (9) and (11) in F-S channels appears to be intractable, and therefore, we resort to computer simulations to evaluate the MSE-performance of these estimators.

From the simulation results presented in Fig. 2, we can observe that the one-shot estimation is not accurate enough for correction. One may expect that averaging over several symbols can improve the performance. This is true in AWGN channels as shown in Fig. 3. We have averaged the estimation over 10 symbols and obtained much better performance. Unfortunately, in F-S channels, an error floor is found for large SFO \(\epsilon\). As mentioned in Appendix-B, the estimators (9) and (11) are not anymore unbiased in the presence of large SFO. However, Fig. 3 illustrates that for small SFO \(\epsilon\) there is no error floor in F-S channels.

\(^1\)Relative to DA schemes, DD schemes assume a low implementation upgrade: only the two components in the dash-line frame of Fig. 1 are need.
B. Closed-Loop Scheme

The above results suggest us to utilize a closed-loop synchronization scheme in the presence of F-S channels. As long as the Viterbi decoder outputs reliable data decisions, the DD SFO and CFO estimator starts to work. The one-shot estimates ($\hat{\phi}$ and $\hat{f}_N$) are post-processed by their corresponding first-order tracking loop filters

$$\hat{f}_t = \hat{f}_{t-1} + \gamma_f \hat{f}_N, \quad \hat{\phi}_t = \hat{\phi}_{t-1} + \gamma_c \hat{\phi}.$$  \hspace{1cm} (14)

Symbol by symbol, the above loop filters update the control parameters of number-controlled oscillators in the interpolator and frequency corrector. Different from the conventional decision-directed phase estimation for single carrier systems [13], the Viterbi decoder in our scheme will not introduce additional loop delay or performance degradation because reliable decisions are available before the DD SFO and CFO synchronizers start working. For large SFO, we obtain first coarse estimates. After correction with the first estimated value, a smaller residual SFO is left, and then we may obtain a more accurate estimate (lower error floor for smaller SFO in Fig. 3). Finally, very accurate estimates can be expected in these feedback tracking schemes.

After convergence, the estimators exhibit small fluctuations about the stable equilibrium points. Considering a linearized equivalent model, we can derive the tracking performance of the closed-loop schemes as follows [12]

$$\text{MSE} = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} S(f) |H(f)|^2 df,$$  \hspace{1cm} (15)

where $S(f)$ is the power spectral density (PSD) of loop noise present in CFO and SFO estimators (derived in Appendix-A)

$$S_{\text{CFO}}(f) = \frac{\sigma_n^2 T [1 - \cos(2\pi f T)]}{4\pi^2 \rho^2 K}, \quad S_{\text{SFO}}(f) = \frac{4\sigma_n^2 T [1 - \cos(2\pi f T)]}{\pi^2 \rho^2 K (K + 2)^2},$$  \hspace{1cm} (16)

and $H(f)$ is the closed-loop transfer function given by

$$H(f) = \frac{\gamma}{e^{2\pi f T} - (1 - \gamma)}.$$  \hspace{1cm} (17)

From (16) and simulation results in Fig. 4, we find that the loop noise is colored and its PSD is not flat over the loop bandwidth with a zero at DC. As shown in Fig. 4, similar results have been found for loop noise in DA schemes. Substituting (16)-(17) into (15), we can easily find the MSE of closed-loop DD estimators

$$\text{MSE}(\hat{f}_t) = \frac{\gamma_f^2 (2 - \gamma_f)}{4\pi^2 \rho^2 K \cdot E_s / N_o}, \quad \text{MSE}(\hat{\phi}_t) = \frac{4\gamma_c^2 (2 - \gamma_c)}{\pi^2 \rho^2 K (K + 2)^2 \cdot E_s / N_o}.$$  \hspace{1cm} (18)
Similarly, we obtain the MSE of closed-loop DA estimators in [7] and [8]

\[
\text{MSE}(\hat{f}_l) = \frac{\gamma_f^2}{4\pi^2 \rho^2 N_P \cdot E_s/N_0}, \quad \text{MSE}(\hat{\epsilon}_l) = \frac{4\gamma_e^2}{\pi^2 \rho^2 K^2 N_P \cdot E_s/N_0}.
\] (19)

V. COMPUTER SIMULATIONS

In our simulations, we assume an OFDM system with \( N = 128 \) subcarriers and guard interval of 16. There are 10 pilot subcarriers inserted into 120 QPSK data modulated subcarriers in the DA scheme, while no pilot is inserted in the DD scheme. We assume a 12-path frequency-selective channel with exponentially decaying power profile and the root-mean square (RMS) delay spread is 1.25 \( T_s \). Further, we utilize a rate 1/2 convolutional encoder with generator polynomial (133,171) and a block (16x15) interleaver to combat additive noise and spectral nulls, respectively. The simulation results are obtained using 2,000 Monte-Carlo trials for each SNR value. The SFO \( \epsilon \) and CFO \( f_N \) are assumed equal to 40 ppm and 0.06, respectively.

To simplify the channel estimation, we assume the channel is static during the whole packet (150 symbols), which is a reasonable assumption for indoor environment. Thus, in this static channel, the first two symbols are used for one-shot DA channel estimation. However, we remark that the channel can be time-varying, and a one-shot decision-directed channel estimator such as [14] can be used after initial DA channel estimation.

In Fig. 2, we show the one-shot estimation performance for DD and DA estimators. Simulation results show a good agreement between the analytical (analysis) results and the experimental results, and the proposed DD estimator is about 10 dB better than the DA scheme in both AWGN channels as well as F-S channels.

Fig. 3 illustrates the simulation results for various SFOs in the presence of AWGN and F-S channels. For AWGN channels, relative to the one-shot estimation in Fig. 2, averaging over several symbols can improve the performance of SFO estimator for both large and small SFO. However, due to the inaccurate approximation (see Appendix-B), for large SFO in F-S channels, (11) is not unbiased anymore, and an error floor can be observed for DD SFO and CFO estimators (a fact which is not illustrated here). Similar error floors can be found for DA SFO/CFO estimators in the presence of large SFO and F-S channels.

The comparison between analytical and simulation results of the PSD of closed loop noise for DD and DA estimators is shown in Fig. 4, bearing a good agreement for \( fT > 0.1 \). Because
of the spectrum leakage introduced by the estimation window in PSD estimator, close to zero frequency (DC), there is a discrepancy between the analytical and simulation results.

Next, we resort to the closed-loop scheme as described in Section IV-B, and consider $\gamma_c = 0.069$ and $\gamma_f = 0.081$ for both DD and DA schemes. As shown in Fig. 5, the one-shot estimates are smoothed by loop filters and the residual errors in the proposed DD scheme converge to zero within 30 symbols. The closed-loop tracking performance of DD scheme in Fig. 6, which is much better than that of the conventional DA scheme for both CFO and SFO synchronization, corroborates the closed-loop analysis result presented in Section IV.

VI. CONCLUSIONS

We introduce a new decision-directed post-FFT joint estimator for carrier frequency offset and sampling clock frequency offset in coded OFDM systems. By performance analysis and computer simulations, we prove that our new scheme obtain much better performance compared to conventional data-aided scheme in both AWGN and frequency-selective channels. Since we save the pilots for synchronization, the throughput of system is increased. With very few additional hardware, this new synchronization scheme can be implemented in many wireless OFDM systems.

Appendix-A: Performance analysis of one shot estimation in AWGN channels

In AWGN channels ($|H_{l,k}| = 1$), assuming correct decisions and substituting (5) into (12), one can find

$$u_{l,k} = z_{l,k} \hat{a}_{l,k} \hat{a}_{l-1,k} \approx e^{j2\pi_f \rho_k} + u_{l,k},$$

where $u_{l,k} = n_{l,k} a_{l,k} e^{-j\phi_{l-1,k}} + n_{l-1,k} a_{l-1,k} e^{j\phi_{l-1,k}}$, and the products of two noise terms are neglected. From (11), denoting $\alpha = e^{-j2\pi_f f_N}$ and making use of $\tan(a \pm b) \approx \tan(a) \pm \tan(b)$ (if $\tan(a) \tan(b) \ll 1$), we obtain:

$$\tan[4\pi\rho(\hat{f}_N - f_N)] \approx \frac{\Im(A_{l1}\alpha)}{\Re(A_{l1}\alpha)} + \frac{\Im(A_{l2}\alpha)}{\Re(A_{l2}\alpha)},$$

where $\Re(x)$ and $\Im(x)$ denote real and imaginary part of $x$, respectively. If the absolute value of CFO estimation error $e_l = \hat{f}_N - f_N$ is small, $\tan[4\pi\rho(\hat{f}_N - f_N)]$ can be well approximated by
Substituting (20) into (12) and (21), from (21) it follows that
\[
e_t \approx \frac{1}{4\pi\rho} \left[ \frac{\Im(A_{t,1}^\alpha)}{\Re(A_{t,1}^\alpha)} + \frac{\Im(A_{t,2}^\alpha)}{\Re(A_{t,2}^\alpha)} \right]
\approx \frac{1}{4\pi\rho} \left[ \frac{\Im(A_{t,1}^\alpha)}{\sum_{k \in C_1} \cos(\pi\rho k)} + \frac{\Im(A_{t,2}^\alpha)}{\sum_{k \in C_2} \cos(\pi\rho k)} \right]
\approx \frac{1}{2\pi\rho K} \Im[\alpha \sum_{k \in C_1, C_2} v_{t,k}],
\] (22)
where in the second equation we made use of eq. (6), and in the last equation we assumed \(\cos(\pi\rho k) \approx 1\) and \(\sin(\pi\rho k) \approx 0\) for small SFO \(\epsilon\). The mean-square error of CFO estimator can be obtained after some straightforward calculations
\[
\text{MSE}(\hat{f}_N) = \frac{1}{4\pi^2\rho^2 K \cdot E_s / N_0}. \tag{23}
\]
Similarly, one can find the MSE of SFO estimator
\[
\text{MSE}(\epsilon) = \frac{4}{\pi^2\rho^2 K (K + 2)^2 \cdot E_s / N_0}. \tag{24}
\]
From (22), after some manipulations, one can find that the CFO estimation noise is colored noise with autocorrelation
\[
R_e(m) := E\{e_t e_{t+m}\} = \begin{cases} \frac{\sigma_n^2}{4\pi\rho_0 K}, & m = 0 \\ -\frac{\sigma_n^2}{8\pi\rho_0 K}, & m = \pm 1 \\ 0, & \text{otherwise}. \end{cases} \tag{25}
\]
Thus, the power spectral density (PSD) of CFO estimation noise is given by
\[
S_{\text{CFO}}(f) = T \sum_m R_e(m) e^{-j2\pi m f T} = \frac{\sigma_n^2 T [1 - \cos(2\pi f T)]}{4\pi^2\rho_0 K}. \tag{26}
\]
Similarly, the PSD of SFO estimation noise can be obtained
\[
S_{\text{SFO}}(f) = \frac{4\sigma_n^2 T [1 - \cos(2\pi f T)]}{\pi^2\rho_0 K (K + 2)^2}. \tag{27}
\]

**Appendix-B: Unbiasedness of estimator (11) in frequency-selective channels**

From (7), it is easy to find that the power of CTF corresponding to the first half of subcarriers can be expressed as
\[
|H_{t,k}|^2 = \sum_{n=0}^{\tau_{\text{max}}} |h_{t,n}|^2 + 2\Re\left\{ \sum_{n=0}^{\tau_{\text{max}}} \sum_{m=n+1}^{\tau_{\text{max}}} h_{t,n}^* h_{t,m} e^{-j2\pi k(n-m)/N} \right\}. \tag{28}
\]
Assuming correct decisions \( \hat{a}_{l,k} \), we obtain

\[
A_{l,1} = \sum_{k \in C_1} |H_{l,k}|^2 e^{j2\pi \rho \theta_k} + \sum_{k \in C_1} v_{l,k},
\]

(29)

where

\[
v_{l,k} = n_{l,k} a_{l,k}^* H_{l-1,k} e^{-j\phi_{l-1,k}} + n_{l-1,k} a_{l-1,k} H_{l,k} e^{j\phi_{l,k}}.
\]

(30)

The first part of right-hand side of \( A_{l,1} \) can be rewritten as:

\[
A_{l,1} = e^{j2\pi f_N} \left( e^{-j\pi K/2} S_i(\epsilon, K) \sum_{n=0}^{\tau_{\max}} |h_{l,n}|^2 + \sum_{n=0}^{\tau_{\max}} \sum_{m=n+1}^{\tau_{\max}} \sum_{k \in C_1} 2\Re\{\Gamma_{l,n,m,k}\} e^{j2\pi \rho \epsilon k}\right),
\]

(31)

where \( S_i(\epsilon, K) = \frac{\sin[\pi(\rho \epsilon + K)/2]}{\sin[\pi(\rho \epsilon)]} \approx \frac{K}{2} \) and \( \Gamma_{l,n,m,k} = h_{l,n} h_{l,m} e^{-j2\pi \rho (n-m)/N} \).

Let us now focus on the second term of \( A_{l,1} \) present inside the bracket. Defining \( \Delta = n - m \), we obtain

\[
\sum_{k \in C_1} 2\Re\{\Gamma_{l,n,m,k}\} e^{j2\pi \rho \epsilon k} = h_{l,n} h_{l,m}^{*} \sum_{k \in C_1} e^{j2\pi \rho (\epsilon - \Delta/N) k} + h_{l,n}^{*} h_{l,m} \sum_{k \in C_1} e^{j2\pi \rho (\epsilon + \Delta/N) k}
\]

\[
= h_{l,n} h_{l,m}^{*} e^{-j\pi(\rho \epsilon - \Delta/N)(K/2+1)} \frac{\sin[\pi K(\rho \epsilon - \Delta/N)/2]}{\sin[\pi (\rho \epsilon - \Delta/N)]} + h_{l,n}^{*} h_{l,m} e^{-j\pi(\rho \epsilon + \Delta/N)(K/2+1)} \frac{\sin[\pi K(\rho \epsilon + \Delta/N)/2]}{\sin[\pi (\rho \epsilon + \Delta/N)]}.
\]

(32)

Assuming small SFO, \( \epsilon \ll \Delta/N (\Delta > 1) \), we make the following approximation \(^2\) (\( \epsilon \ll 1/M \)):

\[
\frac{\sin[\pi K(\rho \epsilon \pm \Delta/N)/2]}{\sin[\pi (\rho \epsilon \pm \Delta/N)]} \approx \frac{\sin[\pi K/(2N)]}{\sin[\pi (\Delta/N)]}.
\]

(33)

Thus, for small SFO, we obtain

\[
\sum_{k \in C_1} 2\Re\{\Gamma_{l,n,m,k}\} e^{j2\pi \rho \epsilon k} \approx e^{-j\pi K/2} \sum_{k \in C_1} 2\Re\{\Gamma_{l,n,m,k}\};
\]

(34)

and we can express \( A_{l,1} \) as:

\[
A_{l,1} \approx e^{j2\pi f_N} e^{-j\pi K/2} \sum_{k \in C_1} |H_{l,k}|^2 + \sum_{k \in C_1} v_{l,k}.
\]

(35)

Similarly, we can make the following approximation:

\[
A_{l,2} \approx e^{j2\pi f_N} e^{j\pi K/2} \sum_{k \in C_2} |H_{l,k}|^2 + \sum_{k \in C_2} v_{l,k}.
\]

(36)

Substituting (35)-(36) into (11), it is easy to find that (11) is approximately unbiased for slow fading F-S channels.

\(^2\)The accuracy of approximation depends on the value of \( \epsilon \). For large \( \epsilon \), the approximation is not accurate anymore and we cannot prove that (11) is unbiased.
REFERENCES


Fig. 1. The receiver structure of coded OFDM systems

Fig. 2. The one-shot estimation performance of DD and DA estimators
Fig. 3. The effect of averaging DD SFO estimator over 10 symbols

Fig. 4. The comparison between analysis (solid line) and simulation (dash line) results of the PSD of loop noise
Fig. 5. Convergence of DD closed-loop synchronization

Fig. 6. The closed-loop performance of DD and DA schemes