

Light Front Quantisation as an Initial-Boundary Value Problem

T. Heinzl¹ and E. Werner²
Institut für Theoretische Physik
Universität Regensburg
Universitätsstraße 31
93053 Regensburg, FRG

Abstract:

In the light front quantisation scheme initial conditions are usually provided on a single lightlike hyperplane. This, however, is insufficient to yield a unique solution of the field equations. We investigate under which additional conditions the problem of solving the field equations becomes well posed. The consequences for quantisation are studied within a Hamiltonian formulation by using the method of Faddeev and Jackiw for dealing with first-order Lagrangians. For the prototype field theory of massive scalar fields in 1+1 dimensions, we find that initial conditions for fixed light cone time *and* boundary conditions in the spatial variable are sufficient to yield a consistent commutator algebra. Data on a second lightlike hyperplane are not necessary. Hamiltonian and Euler-Lagrange equations of motion become equivalent; the description of the dynamics remains canonical and simple. In this way we justify the approach of discretised light cone quantisation.

PACS: 03.50 Kk; 11.10.Ef, Qr

¹e-mail: heinzl@vax1.rz.uni-regensburg.d400.de

²e-mail: werner@vax1.rz.uni-regensburg.d400.de

1. Introduction

In 1949 Dirac pointed out that in a relativistic quantum theory the choice of the time variable is not unique^[1]. In his "front form" of dynamics he suggested to use light fronts $x^+ \equiv x^0 + x^3 = \text{const}$ as surfaces of equal "time". This idea was revived about twenty years later in the context of current algebra^[2] as well as deep inelastic scattering and the parton model^[3,4] when people quantised field theories on lightlike hyperplanes and derived the corresponding Feynman rules^[5,6]. In the recent years interest has grown in using light front techniques to solve field theories non-perturbatively. This has been pioneered by Brodsky and Pauli within their program of discretised light cone (LC) quantisation^[7], which was later accompanied by light front Tamm-Dancoff methods^[8]. These works, however, did not address a long-standing problem of LC physics: how can effects related to a non-trivial vacuum structure such as spontaneous symmetry breaking arise from the LC vacuum which was believed to be trivial^[9-11]? This has only recently been clarified for some model field theories by demonstrating the important role of modes with LC momentum $k^+ = 0$ which carry the non-trivial vacuum aspects^[12,13]. In all of these approaches one makes explicit or implicit assumptions on the behaviour of the fields for large values of the spatial variable $x^- \equiv x^0 - x^3 \rightarrow \pm L \rightarrow \pm\infty$ in terms of asymptotic or boundary conditions (BC). It has never been checked whether these conditions which must hold for *all* LC times x^+ are consistent with the dynamics of the system. It has, however, been suggested that the system might so become over-determined. Already in the early seventies it was noted that quantising on a light front means quantising on the characteristic surfaces of the (classical) field equations^[14,15]. The conclusion was that one is dealing with a characteristic initial value problem (CIVP) when one wants to solve these equations. This implies that one has to specify data on both characteristics $x^+ = \text{const}$, $x^- = \text{const}$. Whether this is of any relevance for the quantum theory has not been studied before McCartor's work on LC quantisation of massless fields^[16]. He found that one should quantise free massless fermions in 1+1 dimensions on both characteristics, which essentially means that one has equal- x^+ and equal- x^- commutators. In a Hamiltonian picture, this amounts to having two different times. Accordingly, the Poincaré generators receive contributions from both characteristics. Though this method is surely legitimate, quantising on a cone-like surface instead of a single light front makes it difficult to establish a Hamiltonian formulation. Therefore, in this work we address the alternative question whether it is possible to choose a single light front $x^+ = \text{const}$ as the initial surface and develop a consistent Hamiltonian formalism which describes the dynamics completely in the parameter x^+ . We concentrate on the case of massive scalar fields in 1+1 dimensions, which turns out to be sufficiently general. Only a few remarks will be made concerning the massless case, which is generically different.

Our starting point is to clarify the relation between initial data for the Klein-Gordon (KG) equation and the specification of canonical commutators for the cases of both space-like and lightlike hypersurfaces. We find that the independent canonical commutators can be viewed as special initial data satisfying a consistency condition which serves as a testing ground for the quantisation prescription. These results are essentially derived by considering the Euler-Lagrange (EL) equation of motion. As a next step we perform an analogous investigation within a Hamiltonian picture. In former publications we have set up the Hamiltonian formulation^[13] by treating LC field theories as constrained systems in the sense of Dirac and Bergmann^[17,18]. The outcome of this procedure were Dirac brackets, i.e. modifications of the conventional Poisson brackets, which became the commutators after quantisation. This has been called a generalised correspondence principle. It has turned out recently, however, that there is a much more economic way of deriving the basic (or Dirac) brackets, namely the method of Hamiltonian reduction due to Faddeev and Jackiw^[19]. This method is also more intuitive; for the case at hand it simply reduces to demanding equivalence of EL and Hamiltonian equations of motion. This already determines the basic bracket. More technically, within the new method Dirac's primary constraints, stemming from the definition of the canonical momenta, are completely absent.

For our investigation the benefit of the Faddeev-Jackiw procedure is a very transparent analysis of the sensitivity of the basic field brackets (or commutators) on the chosen initial and/or BC. It turns out that characteristic data specified on the *cone* $x^\pm = const$ together with the canonical LC Hamiltonian do not lead to a consistent description of the dynamics. There is no canonical field commutator such that the Hamiltonian equation of motion matches the EL one (i.e. the KG equation). The only consistent way of quantising on a single light front is to replace the data on the second characteristic by appropriate BC that hold for all LC times. The dynamics is then well determined by the Hamiltonian and in accordance with the general (Riemann) solution of the KG equation. This we prove quite generally and check explicitly for the Pauli-Jordan^[20] or Schwinger^[21] function which is the free field commutator for arbitrary LC time differences. The essence of the proof is that the data on the second characteristic, $x^- = const$, are already determined by the data on $x^+ = const$ and the BC. The former, therefore, can no longer be specified independently, or, put in the quantum language: a single equal- x^+ commutator together with suitable BC in x^- is sufficient. In view of this result, which mathematically corresponds to an initial-boundary value problem, we regard the term "light front quantisation" as more appropriate than "light cone quantisation".

The paper is organised as follows. In Section 2 we review the relation between instantaneous quantisation (on a spacelike hypersurface) and the Cauchy problem for the KG equation. In Section 3 we compare quantisation on lightlike hyperplanes and the CIVP. The Hamiltonian formulation is set up in Section 4, whereas, in Section 5, we analyse the relation between (periodic) BC and the solution of the CIVP. Finally, in Section 6, we discuss some consequences of this work.

2. Instantaneous Quantisation as a Cauchy Problem

We start from the covariant action

$$S = \int_M d^2x \mathcal{L} = \int_M d^2x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right], \quad (2.1)$$

where M denotes Minkowski space. The principle of least action yields

$$\delta S = \int_M d^2x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right] \delta \phi + \int_{\partial M} d\sigma_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi = 0. \quad (2.2)$$

If we do not vary on the boundary ∂M of Minkowski space, $\delta \phi|_{\partial M} = 0$, the surface term in δS vanishes and we obtain the (massive) KG equation

$$(\square + m^2)\phi = 0. \quad (2.3)$$

This is a hyperbolic partial differential equation (PDE) of second order^[22,23]. The Cauchy problem for it can be formulated as follows: determine a solution $\phi(x^0, x^1)$ of (2.3) which satisfies the initial conditions

$$\phi(x^0 = 0, x^1) = f(x^1); \quad \dot{\phi}(x^0 = 0, x^1) = g(x^1). \quad (2.4)$$

The prescribed functions, f and g , which determine ϕ and its time derivative $\dot{\phi}$ on the surface $x^0 = 0$, are called Cauchy data. The Cauchy problem for the KG equation is well posed, i.e. it has a unique solution which we are going to derive now. As ϕ obeys (2.3) its Fourier representation must be of the form

$$\phi(x) = \int \frac{d^2p}{2\pi} \delta(p^2 - m^2) \chi(p) e^{-ip \cdot x}. \quad (2.5)$$

After performing the p^0 -integration and an inverse Fourier transform we obtain the mass-shell values of χ ,

$$\chi(p^0 = \pm \omega_p, p^1) = \int dx^1 e^{-ip^1 x^1} [\omega_p f(x^1) \pm g(x^1)] \equiv \omega_p f(p^1) \pm g(p^1), \quad (2.6)$$

where $\omega_p = (p_1^2 + m^2)^{1/2}$. On the mass-shell, therefore, χ is uniquely determined by the Cauchy data. After quantisation the amplitudes χ get replaced by creation and annihilation operators a and a^\dagger ; equation (2.6) then expresses the well known fact that these Fock operators are given in terms of the field *and* the velocity. By reinserting (2.6) into (2.5) the field ϕ can be expressed through its initial values, f and g ,

$$\phi(x) = \int dy^1 \left[f(y^1) \frac{\partial}{\partial y^0} \Delta(x - y) - g(y^1) \Delta(x - y) \right], \quad (2.7)$$

where we have introduced the Pauli-Jordan^[20] or Schwinger^[21] function (or better distribution)

$$\Delta(x) = -\frac{i}{2\pi} \int d^2p \delta(p^2 - m^2) \text{sgn}(p^0) e^{-ip \cdot x} = -\frac{1}{2} \text{sgn}(x^0) \theta(x^2) J_0(m\sqrt{x^2}) . \quad (2.8)$$

It is an invariant solution of the KG equation, which is antisymmetric, $\Delta(x) = -\Delta(-x)$, and vanishes outside the LC,

$$\Delta(x) = 0 , \quad \text{for } x^2 < 0 . \quad (2.9)$$

These properties, however, are just the requirements which are to be satisfied by the commutator of two free scalar fields so that one can identify

$$[\phi(x), \phi(0)] = i\Delta(x) , \quad (2.10)$$

where the factor i is necessary for hermiticity. An exhaustive discussion of these issues can be found in Schweber's textbook^[24]. With the help of (2.8) one infers the equal-time commutators

$$[\phi(x), \phi(0)]_{x^0=0} = i\Delta(x)|_{x^0=0} = 0 , \quad (2.11a)$$

$$[\dot{\phi}(x), \phi(0)]_{x^0=0} = i\dot{\Delta}(x)|_{x^0=0} = -i\delta(x^1) . \quad (2.11b)$$

This is completely consistent with the general formula (2.7). Replacing there ϕ , f and g by the commutators,

$$[\phi(x), \phi(0)] = \int dy^1 \left\{ [\phi(y), \phi(0)] \frac{\partial}{\partial y^0} \Delta(x-y) - [\dot{\phi}(y), \phi(0)] \Delta(x-y) \right\}_{y^0=0} , \quad (2.12)$$

and inserting (2.11) gives just the identity (2.10). Equation (2.12) is therefore a consistency condition which expresses the commutator for arbitrary time differences $x^0 > 0$ through its Cauchy data at time $x^0 = 0$. The latter are just the two independent canonical equal time commutators (2.11). To quantise consistently *both* of them have to be specified.

In the next sections we want to analyse the relation between data and commutators if one chooses a light front as the quantisation hypersurface. To this end we will derive the appropriate consistency condition analogous to (2.12).

3. Light Cone Quantisation as a Characteristic Initial Value Problem

Let us rewrite the KG equation (2.3) in terms of LC variables x^\pm , with $\partial^\pm = 2\partial/\partial x^\mp$,

$$(\partial^+ \partial^- + m^2)\phi = 0. \quad (3.1)$$

The lightlike hyperplanes $x^\pm = \text{const}$ are called the characteristics of this hyperbolic PDE (cf. any textbook on PDEs, e.g. Ref.s [22,23]). It has been shown explicitly in a recent preprint^[25] that specifying the field ϕ and an arbitrary number of derivatives on one characteristic only does not lead to a unique solution of (3.1). This is not the case, however, for the CIVP which is defined as follows: Determine a solution $\phi(x^+, x^-)$ which satisfies the initial conditions

$$\phi(x^+, x_0^-) = f(x^+), \quad (3.2a)$$

$$\phi(x_0^+, x^-) = g(x^-), \quad (3.2b)$$

and the continuity condition

$$\phi(x_0^+, x_0^-) = f(x_0^+) = g(x_0^-). \quad (3.3)$$

The functions f and g which specify ϕ on *both* characteristics are called characteristic data. According to Neville and Rohrlich^[14] the solution of the CIVP can be obtained from the solution (2.7) of the Cauchy problem via Gauss' theorem. To this end one introduces the quantity^[14,24]

$$F_\mu(x, y) = \Delta(x - y) \frac{\partial}{\partial y^\mu} \phi(y) - \phi(y) \frac{\partial}{\partial y^\mu} \Delta(x - y), \quad (3.4)$$

which is divergence free,

$$\frac{\partial}{\partial y^\mu} F^\mu(x, y) = 0, \quad (3.5)$$

since both ϕ and Δ satisfy the KG equation. One integrates (3.5) over the volume bounded by the simplex ABC (Fig.1), where on AB : $x^0 = 0$, on BC : $x^- = x_0^-$ and on CA : $x^+ = x_0^+$. Gauss' theorem yields

$$0 = \left(\int_{AB} + \int_{BC} + \int_{CA} \right) d\sigma(y) n_\mu F^\mu(x, y), \quad (3.6)$$

with $d\sigma$ and n_μ the appropriate surface elements and normal unit vectors. Explicitly one has

$$\begin{aligned}
& \int_{AB} dy^1 \left[\phi(y) \frac{\partial}{\partial y^0} \Delta(x-y) - \Delta(x-y) \frac{\partial}{\partial y^0} \phi(y) \right] = \\
& = \int_{CB} dy^+ \left[\phi(y) \frac{\partial}{\partial y^+} \Delta(x-y) - \Delta(x-y) \frac{\partial}{\partial y^+} \phi(y) \right] + \\
& + \int_{CA} dy^- \left[\phi(y) \frac{\partial}{\partial y^-} \Delta(x-y) - \Delta(x-y) \frac{\partial}{\partial y^-} \phi(y) \right].
\end{aligned} \tag{3.7}$$

According to (2.9), $\Delta(x-y)$ vanishes for spacelike argument, $(x-y)^2 < 0$, and one can extend the integration on the l.h.s. to infinity. Comparison with (2.7) then reveals the l.h.s. as the solution of the Cauchy problem so that one finally has with the data (3.2)

$$\begin{aligned}
\phi(x^+, x^-) &= \int_{x_0^+}^{\infty} dy^+ \left[f(y^+) \frac{\partial}{\partial y^+} \Delta(x^+ - y^+, x^- - x_0^-) - \Delta(x^+ - y^+, x^- - x_0^-) \frac{\partial f}{\partial y^+} \right] + \\
&+ \int_{x_0^-}^{\infty} dy^- \left[g(y^-) \frac{\partial}{\partial y^-} \Delta(x^+ - x_0^+, x^- - y^-) - \Delta(x^+ - x_0^+, x^- - y^-) \frac{\partial g}{\partial y^-} \right].
\end{aligned} \tag{3.8}$$

This is the desired solution of the CIVP for the KG equation in terms of the initial data, f and g , and the Pauli-Jordan function Δ , which in LC coordinates can be written as

$$\Delta(x^+, x^-) = -\frac{1}{4} [\text{sgn}(x^+) + \text{sgn}(x^-)] J_0(m\sqrt{x^+x^-}). \tag{3.9}$$

The Bessel function J_0 is the Riemann function of the KG equation, and for $x^\pm \geq x_0^\pm$ equation (3.8) coincides with the classical Riemann solution for hyperbolic PDEs applied to the case at hand (often also called the telegraph equation)^[22,23],

$$\begin{aligned}
\phi(x^+, x^-) &= \frac{1}{2} [f(x_0^+) + g(x_0^-)] R(x^+ - x_0^+, x^- - x_0^-) + \\
&+ \frac{1}{2} \int_{x_0^-}^{x^-} dy^- R(x^+ - x_0^+, x^- - y^-) \partial_y^+ g(y^-) + \\
&+ \frac{1}{2} \int_{x_0^+}^{x^+} dy^+ R(x^+ - y^+, x^- - x_0^-) \partial_y^- f(y^+).
\end{aligned} \tag{3.10}$$

The Riemann function R is explicitly given by

$$R(x^+, x^-) = J_0(m\sqrt{x^+x^-}) = \sum_{n=0}^{\infty} \frac{1}{n!n!} \left(-\frac{1}{4}m^2x^+x^-\right)^n. \tag{3.11}$$

For practical calculations it is often more convenient to work with the Riemann solution because one is dealing with functions instead of distributions.

As a side remark we note that the solutions (3.8) and (3.10) behave smoothly in the limit of vanishing mass, which will, however, be discussed elsewhere.

We emphasize that, in order to obtain a unique solution, data on *both* characteristics and not only on a single light front have to be specified. For the quantum theory (in $d = 1 + 1$) this implies quantisation on the light cone: the two independent commutators that have to be specified are

$$[\phi(x), \phi(0)]_{x^\pm=0} = i\Delta(x)|_{x^\pm=0} = -\frac{i}{4}\text{sgn}(x^\mp). \quad (3.12)$$

This is obtained from (3.9) by setting

$$\text{sgn}(x^\pm = 0) = 0, \quad (3.13)$$

which can be verified from the Fourier representation of Δ ^[14]. Let us derive the consistency condition analogous to (2.12). To this end we set $x_0^\pm = 0$, $f(x^+) = -\frac{i}{4}\text{sgn}(x^+)$, $g(x^-) = -\frac{i}{4}\text{sgn}(x^-)$. Then, after a partial integration in (3.8), $[\phi(x), \phi(0)]$ should be given by

$$\begin{aligned} [\phi(x), \phi(0)] \stackrel{?}{=} & -\frac{i}{4}\text{sgn}(y^+)\Delta(x^+ - y^+, x^-)|_{y^+=0}^\infty + i \int_0^\infty dy^+ \delta(y^+) \Delta(x^+ - y^+, x^-) + \\ & + -\frac{i}{4}\text{sgn}(y^-)\Delta(x^+, x^- - y^-)|_{y^-=0}^\infty + i \int_0^\infty dy^- \delta(y^-) \Delta(x^+, x^- - y^-). \end{aligned} \quad (3.14)$$

The surface terms vanish due to (2.9) and (3.13); the integrals give $\frac{i}{2}\Delta(x)$ each which, shows that (3.14) is really an identity. The consistency condition is then

$$\begin{aligned} [\phi(x), \phi(0)] = & \int_0^\infty dy^+ \left\{ [\phi(y), \phi(0)] \frac{\partial}{\partial y^+} \Delta(x - y) - \Delta(x - y) \frac{\partial}{\partial y^+} [\phi(y), \phi(0)] \right\}_{y^-=0} + \\ & + \int_0^\infty dy^- \left\{ [\phi(y), \phi(0)] \frac{\partial}{\partial y^-} \Delta(x - y) - \Delta(x - y) \frac{\partial}{\partial y^-} [\phi(y), \phi(0)] \right\}_{y^+=0}. \end{aligned} \quad (3.15)$$

It expresses the commutator for all x^\pm in terms of its characteristic values (3.12). Again this tells us that the CIVP corresponds to quantisation on the cone. For the quantisation of massless scalar fields in $d = 1 + 1$ the commutators (3.12) are postulated in the textbook Ref. [26]. McCartor has used analogous expressions for massless fermions^[16]. In two recent preprints^[27] quantisation on both $x^+ = \text{const}$ and $x^- = \text{const}$ has been performed for Yukawa theory and abelian gauge fields (in $d = 3 + 1$). Accordingly, one has to introduce *two* time parameters, x^+ and x^- . In the rest of the literature on "light cone" or "light front" quantisation, commutators are prescribed on one characteristic (usually $x^+ = 0$) only. This is indispensable if one wants to have a Hamiltonian formulation with *one* single evolution parameter x^+ , but seems to be in contradiction with the results above.

Before we answer the question, whether quantisation on a single light front is possible at all, we want to analyse why the relevance of data on a second characteristic (even in

the massive case) has been overlooked so far. For this purpose let us consider the Fourier representation (2.5) of ϕ written in LC variables,

$$\phi(x^+, x^-) = \frac{1}{4\pi} \int dp^+ dp^- e^{-ip^+ x^- / 2 - ip^- x^+ / 2} \delta(p^+ p^- - m^2) \chi(p^+, p^-). \quad (3.16)$$

The (erroneous) claim in the literature is now that the Fock operators which are obtained from quantising the Fourier amplitude χ can be expressed through the field ϕ alone, specified on one characteristic^[6,9,11]. This should be contrasted with (2.6) where both the field and the velocity on the initial surface are needed. Mathematically, the above claim amounts to assuming the identity

$$\delta(p^+ p^- - m^2) = \frac{1}{|p^+|} \delta(p^- - m^2/p^+), \quad (3.17)$$

which leads to

$$\int dx^- e^{ip^+ x^- / 2} \phi(x^+, x^-) \equiv \phi(x^+, p^+) = \frac{1}{|p^+|} \chi(p^+, m^2/p^+) e^{-im^2 x^+ / 2p^+}, \quad (3.18)$$

and (if we set $x^+ = 0$)

$$\chi(p^+, m^2/p^+) = |p^+| \phi(x^+ = 0, p^+) \equiv |p^+| g(p^+). \quad (3.19)$$

So it really seems that the amplitude χ is totally determined by the field ϕ on the characteristic $x^+ = 0$, the Fourier transform of which is $g(p^+)$. The data on the second characteristic, which are represented by the function $f(p^-)$, do not enter at all. This surely contradicts the solutions (3.8) and (3.10) which depend on both data. Note however, that the equations (3.17) and (3.18) are ill-defined at $p^+ = 0$. Both are actually identities for distributions rather than functions, and $1/|p^+|$ is not defined as a distribution^[28], unless one specifies some further regulating prescription. One possibility is to think of χ as being a test function and demand^[29]

$$\lim_{p^+ \rightarrow 0} \frac{1}{p^+} \chi(p^+, m^2/p^+) = 0. \quad (3.20)$$

This can be translated into a condition on ϕ : performing the limit $p^+ \rightarrow 0$ in (3.18) and using the KG equation we obtain with (3.20)

$$0 = \int dx^- \phi = -\frac{1}{m^2} \int dx^- \partial^+ \partial^- \phi = -\frac{2}{m^2} [\partial^- \phi(x^+, \infty) - \partial^- \phi(x^+, -\infty)]. \quad (3.21)$$

We see that the postulate (3.20) is equivalent to demanding that $\partial^- \phi$ and therefore ϕ be *periodic* in x^- for all $x^+ \geq 0$. Clearly, this is a severe restriction on ϕ , and one has to verify, whether such a condition is consistent with the characteristic data (3.2) and the solutions

(3.8) and (3.10). This will be done in the next section when we study the Hamiltonian formulation.

The moral so far is that there is a subtle interplay between initial or boundary data with respect to x^- and the singularity in the conjugate variable, i.e. at $p^+ = 0$. Only with sufficient conditions guaranteeing a well-defined inverse of p^+ do the manipulations leading to (3.19) make sense, i.e. only if ϕ is periodic in x^- , the statement that the data $f(x^+)$ are not needed seems to be correct. This will be verified in more detail in Section 5. It will also become more evident in the following that inverting the variable or better distribution p^+ is *the* problem of light front quantisation.

4. The Hamiltonian Formulation

In this section we want to analyse the relation between the data for the KG equation (viewed as a Hamiltonian equation of motion) and the basic bracket between the field variables, which is to become the commutator upon quantisation. In former publications^[13] we obtained this bracket as a Dirac bracket resulting from the Dirac-Bergmann algorithm^[17,18] for dealing with constrained systems. As the kinetic term in the Lagrangian (2.1), $\partial^+\phi\partial^-\phi$, is linear in the LC velocity, $\partial^-\phi \equiv 2\dot{\phi}$, the Lagrangian is called singular^[18]. This observation was then the starting point for running the Dirac-Bergmann machinery. However, as has been noted by Faddeev and Jackiw^[19], the appearance of (primary) constraints for first order Lagrangians is an artifact of the method one uses rather than of physical significance. To avoid the complexity of the Dirac-Bergmann algorithm they developed a much more economic method which in its simplest form reduces to demanding equivalence of EL and Hamiltonian equations. This already determines the basic brackets¹.

As the Hamiltonian equations express the velocity $\dot{\phi}$ in terms of the fields, it is necessary to perform a first integral of the KG equation, which amounts to invert the derivative ∂^+ by choosing an appropriate Green function $G(x^- - y^-)$. In momentum space, this inverse is a properly defined distribution $G(p^+)$ replacing the naive expression $1/p^+$ as was mentioned above. One finds

$$\dot{\phi}(x^+, x^-) = -\frac{m^2}{4} \int dy^- G(x^-, y^-) \phi(x^+, y^-) + h(x^+). \quad (4.1)$$

Up to possible boundary or finite volume terms the Green function G has to obey

¹The method of Faddeev and Jackiw has recently been applied to the quantisation of Light Front QCD^[30]. There, the basic brackets have been obtained as the symplectic structure of the (infinite-dimensional) phase space. This essentially boils down to the way we do it^[19].

$$\frac{d}{dx^-}G(x^-, y^-) = \delta(x^- - y^-), \quad (4.2)$$

or in momentum space

$$\frac{1}{2}p^+G(p^+) = 1. \quad (4.3)$$

Clearly, (4.2) and (4.3) do not determine G uniquely which is partially reflected by the appearance of the function h in (4.1). Both G and h have to be determined by initial and/or BC, which will be done below. In order to demonstrate the way how the method of Faddeev and Jackiw works, however, the formal expression (4.1) is sufficient.

The canonical hamiltonian of the KG system is

$$H_c \equiv \frac{1}{2}P^- = \frac{1}{4} \int dx^- m^2 \phi^2, \quad (4.4)$$

so that the Hamiltonian equation of motion becomes

$$\begin{aligned} \dot{\phi}(x^+, x^-) &= \{\phi(x^+, x^-), H_c\}^* + \frac{\partial \phi}{\partial x^+}(x^+, x^-) = \\ &= \frac{m^2}{4} \int dy^- \{\phi(x^+, x^-), \phi^2(x^+, y^-)\}^* + \frac{\partial \phi}{\partial x^+}(x^+, x^-). \end{aligned} \quad (4.5)$$

Here $\{, \}^*$ denotes the basic bracket and $\partial\phi/\partial x^+$ a possible explicit time dependence of ϕ which is not determined by the Hamiltonian. As any reasonable bracket should obey a Leibniz rule, one gets

$$\dot{\phi}(x^+, x^-) = \frac{m^2}{2} \int dy^- \{\phi(x^+, x^-), \phi(x^+, y^-)\}^* \phi(x^+, y^-) + \frac{\partial \phi}{\partial x^+}(x^+, x^-). \quad (4.6)$$

This coincides with (4.1) if one identifies

$$G(x^-, y^-) \equiv -2 \{\phi(x^+, x^-), \phi(x^+, y^-)\}^*, \quad (4.7a)$$

$$h(x^+) \equiv \frac{\partial \phi}{\partial x^+}(x^+, x^-). \quad (4.7b)$$

One notices that the basic bracket of two fields ϕ is given entirely in terms of the Green function G , i.e. the inverse of the derivative ∂^+ . To emphasize it once more, this inverse is only well defined if one specifies sufficient initial and/or boundary data. If this has been done, quantisation is straightforward (up to possible operator ordering problems), but then, of course dependent on the prescribed data.

Two properties the Green function G should fulfill are immediately clear if it is to become a commutator, namely translation invariance and antisymmetry,

$$G(x^-, y^-) = G(x^- - y^-), \quad (4.8a)$$

$$G(x^- - y^-) = -G(y^- - x^-). \quad (4.8b)$$

In the following we will discuss three examples. In all of them we specify data on the light front $x^+ = 0$, but the conditions dependent on x^- will be different.

Let us start with the characteristic data (3.2) with $x_0^\pm = 0$. Using the following identity for the Riemann function R of (3.11),

$$\frac{\partial}{\partial x^+} R(x^+, x^-) = -\frac{m^2}{4} \int_0^{x^-} dy^- R(x^+, y^-), \quad (4.9)$$

one finds that the Riemann solution (3.10) obeys

$$\dot{\phi}(x^+, x^-) = -\frac{m^2}{4} \int_0^{x^-} dy^- \phi(x^+, y^-) + \dot{f}(x^+). \quad (4.10)$$

The dependence on the data $f(x^+)$ on the second characteristic is clearly exhibited. Equation (4.10) corresponds to the choice of the Green function

$$G_1(x^-, y^-) = \theta(x^- - y^-) - \theta(-y^-), \quad (4.11)$$

which is vanishing on the second characteristic, $x^- = 0$. Obviously, this Green function breaks translation invariance. This can be cured by setting^[14] $x_0^- \rightarrow -\infty$, so that G_1 becomes

$$G_1(x^- - y^-) = \theta(x^- - y^-), \quad (4.12)$$

but still one does not have antisymmetry. G_1 is therefore inappropriate as a bracket or a commutator and must be abandoned. The moral is that quantisation on a single light front using characteristic data and the canonical Hamiltonian (4.4) is impossible! This does not come as a surprise since the Hamiltonian should be an integral over the initial value surface and in the case at hand would imply also integration over $x^- = \text{const}$ ^[16]. Recently, a Hamiltonian formulation has been set up which uses both x^+ and x^- as evolution parameters and therefore requires two Hamiltonians generating the time development^[27].

What we want to pursue instead is to investigate whether there is any way to use the simple Hamiltonian (4.4), quantise on a single light front and have a consistent formulation for the dynamics. Hence, from now on, we will follow the philosophy of "discretised LC quantisation"^[7] and enclose the physical system in a finite spatial volume, $-L \leq x^- \leq L$. This serves as a regulator in the infrared and makes the momentum variable p^+ discrete and denumerable. In a finite volume, BC have to be specified and these will be of crucial importance for the following. First note that BC with respect to x^- have to hold for *all*

LC times x^+ and can therefore have an influence on the dynamics. In the last section we have already seen that BC can do a good job in making the inverse of p^+ well-defined. Let us investigate this now in greater detail. We begin with the case of antiperiodic BC,

$$\phi(x^+, -L) + \phi(x^+, L) = 0 . \quad (4.13)$$

These are usually used for fermions^[7] but sometimes also for bosonic fields^[30]. The corresponding Green function is

$$G_2(x^-, y^-) = \frac{1}{2} \text{sgn}(x^- - y^-) , \quad (4.14)$$

which itself is antiperiodic,

$$G_2(L, y^-) + G_2(-L, y^-) = 0 . \quad (4.15)$$

In momentum space this corresponds to the principal value

$$\frac{1}{2} G_2(p^+) = \mathcal{P} \frac{1}{p^+} , \quad (4.16)$$

which is the canonical regularisation of the function $1/p^+$ ^[28]. According to (4.1) the time derivative of ϕ is given by

$$\dot{\phi}(x^+, x^-) = -\frac{m^2}{4} \int_{-L}^L dy^- G_2(x^-, y^-) \phi(x^+, y^-) + h_2(x^+) . \quad (4.17)$$

Antiperiodicity requires

$$h_2(x^+) = 0 . \quad (4.18)$$

As G_2 is the unique inverse of ∂^+ in this case and fulfills all necessary conditions we finally obtain from (4.7a) the consistent bracket

$$\{\phi(x^+, x^-), \phi(x^+, y^-)\}^* = -\frac{1}{4} \text{sgn}(x^- - y^-) . \quad (4.19)$$

After quantisation this coincides with the equal- x^+ -commutator of (3.12). This form is commonly used in the literature^[4,6,31]; the connection with the chosen BC, however, has only been discussed by a few authors^[32] and not been entirely clarified.

The discussion above can be still slightly generalised: Let us consider the BC

$$\phi(x^+, L) + \phi(x^+, -L) = 2b(x^+) , \quad (4.20)$$

with an arbitrary function b . This yields

$$\dot{\phi}(x^+, x^-) = -\frac{m^2}{4} \int_{-L}^L dy^- \frac{1}{2} \text{sgn}(x^- - y^-) \phi(x^+, y^-) + \dot{b}(x^+) . \quad (4.21)$$

The field ϕ can be decomposed,

$$\phi(x^+, x^-) = \varphi(x^+, x^-) + b(x^+) , \quad (4.22)$$

such that φ is antiperiodic. Again, in this case, a consistent quantisation is possible but now requires an explicit time dependence of ϕ , $\partial\phi/\partial x^+ = \dot{b}(x^+)$. The use of the BC (4.20) offers the possibility of introducing a spatially constant background field (represented by b) and could therefore be an alternative to the method of Ref. [13], where periodic BC have been used.

Periodic BC, however, also turn out to be consistent, as we are now going to show. They read explicitly

$$\phi(x^+, L) - \phi(x^+, -L) = 0 . \quad (4.23)$$

The appropriate Green function is the periodic sign function

$$G_3(x^-, y^-) = \frac{1}{2} \text{sgn}(x^- - y^-) - \frac{x^- - y^-}{2L} , \quad (4.24)$$

which is antisymmetric, translation invariant and obeys the conditions

$$\frac{\partial}{\partial x^-} G_3(x^- - y^-) = \delta(x^- - y^-) - \frac{1}{2L} , \quad (4.25a)$$

$$G_3(L, y^-) - G_3(-L, y^-) = 0 . \quad (4.25b)$$

This gives

$$\dot{\phi}(x^+, x^-) = -\frac{m^2}{4} \int_{-L}^L dy^- G_3(x^-, y^-) \phi(x^+, y^-) + h_3(x^+) . \quad (4.26)$$

In contradistinction to the antiperiodic case the BC (4.25b) obeyed by the Green function do *not* determine the function h_3 . However, G_3 obeys still another condition: it has a vanishing zero mode,

$$\int_{-L}^L dx^- G_3(x^-, y^-) = 0 . \quad (4.27)$$

Therefore, integrating (4.26) gives rise to

$$\frac{1}{2L} \int_{-L}^L dx^- \dot{\phi}(x^+, x^-) = h_3(x^+) . \quad (4.28)$$

By the same argument as in (3.21) we know that the integral over ϕ and therefore (by differentiation) over $\dot{\phi}$ vanishes. Thus h_3 is determined to be zero. Again one finds a consistent bracket

$$\{\phi(x^+, x^-), \phi(x^+, y^-)\}^* = \frac{1}{2} \left[\frac{1}{2} \text{sgn}(x^- - y^-) - \frac{x^- - y^-}{2L} \right]. \quad (4.29)$$

This bracket has earlier been obtained via the Dirac-Bergmann algorithm^[13] but much more effort was necessary.

With regard to the theory of PDEs, demanding BC alters the problem which is to be solved. One does no longer have a CIVP but rather an initial value problem in the variable x^+ and a boundary value problem with respect to x^- . This is an initial-boundary value^[23] or "mixed"^[22] problem.

5. Periodic Solutions of the KG Equation

In this section we explicitly want to show the consistency of BC with the general solution of the KG equation. We concentrate on the case of periodic BC.

A. Explicit Solution via Fourier-Laplace Transformation

Our starting point is the Hamiltonian version of the KG equation

$$\dot{\phi}(x^+, x^-) = -\frac{m^2}{4} \int_{-L}^L dy^- \left[\frac{1}{2} \text{sgn}(x^- - y^-) - \frac{x^- - y^-}{2L} \right] \phi(x^+, y^-), \quad (5.1)$$

where ϕ is periodic in x^- and $\phi(x_0^+ = 0, x^-) = g(x^-)$. Equation (5.1) is an integral differential equation for ϕ . To solve it we introduce the Laplace transform

$$\Phi(p, x^-) = \int_0^\infty dx^+ e^{-px^+} \phi(x^+, x^-); \quad \text{Re}(p) > 0. \quad (5.2)$$

Laplace transforming (5.1) yields

$$\Phi(p, x^-) = \frac{1}{p} g(x^-) - \frac{m^2}{4p} \int_{-L}^L dy^- \left[\frac{1}{2} \text{sgn}(x^- - y^-) - \frac{x^- - y^-}{2L} \right] \Phi(p, y^-). \quad (5.3)$$

This is an inhomogeneous Fredholm equation of the second kind^[33] for $\Phi(p, x^-)$ in the variable x^- , which can be solved by Fourier transformation with respect to x^- . To this end we expand

$$\phi(x^+, x^-) = \sum_n \phi_n(x^+) e^{-in\pi x^-/L}, \quad (5.4a)$$

$$g(x^-) = \sum_n g_n e^{-in\pi x^-/L}, \quad (5.4b)$$

$$\Phi(p, x^-) = \int_0^\infty dx^+ e^{-px^+} \sum_n \phi_n(x^+) e^{-in\pi x^-/L} =: \sum_n \Phi_n(p) e^{-in\pi x^-/L}, \quad (5.4c)$$

where we have introduced the discretised momenta^[7]

$$k_n^+ = 2\pi n/L. \quad (5.5)$$

The Green function has the Fourier series representation

$$\frac{1}{2} \text{sgn}(x^- - y^-) - \frac{x^- - y^-}{2L} = \frac{i}{L} \sum_{n \neq 0} \frac{1}{k_n^+} e^{-in\pi(x^- - y^-)/L}, \quad (5.6)$$

and it is obvious that its zero mode vanishes. Inserting (5.4a) into the KG equation immediately yields

$$\phi_{n=0}(x^+) = 0, \quad (5.7)$$

which is nothing but the statement that the zero mode of a free scalar field vanishes. The sums in (5.4) therefore run over $n \neq 0$ only. Fourier transforming (5.3) then yields the solution

$$\Phi_n(p) \equiv \int_0^\infty dx^+ e^{-px^+} \phi_n(x^+) = \frac{g_n}{p + im^2 L/4\pi n}; \quad n \neq 0. \quad (5.8)$$

Performing an inverse Laplace transformation gives via the residue technique

$$\phi_n(x^+) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px^+} \Phi_n(p) = g_n \exp(-i\hat{k}_n^- x^+/2), \quad (5.9)$$

where we have introduced the on-shell momentum

$$\hat{k}_n^- = m^2/k_n^+ = \frac{m^2 L}{2\pi n}. \quad (5.10)$$

Finally, the periodic solution of the KG equation is

$$\phi(x^+, x^-) = \sum_{n \neq 0} g_n e^{-ik_n^+ x^-/2 - i\hat{k}_n^- x^+/2} \equiv \sum_{n \neq 0} g_n e^{-i\hat{k}_n \cdot x}. \quad (5.11)$$

Again, as discussed at the end of Section 3, equation (5.11) suggests that ϕ is uniquely determined by data on a single light front $x^+ = 0$, given in terms of $g(x^-)$ and periodic BC holding for all x^+ . In the next subsection we will prove that this is indeed the case.

B. Relation between the Characteristic Data

If the last statement is true, the data $g(x^-)$ and $f(x^+)$ can no longer be independent. f should rather be expressible in terms of g . To verify this we consider the solution (3.10) of the characteristic problem and set $x_0^+ = 0$, $x_0^- = -L$. Demanding periodic BC results in a Fredholm equation of the first kind^[33] for f ,

$$\int_0^{x^+} dy^+ k_L(x^+ - y^+) f(y^+) - h_L(x^+) = 0, \quad (5.12)$$

where we have introduced the abbreviations

$$k_L(x^+ - y^+) = \frac{1}{2(x^+ - y^+)} m \sqrt{2L(x^+ - y^+)} J_1\left(m \sqrt{2L(x^+ - y^+)}\right), \quad (5.13a)$$

$$h_L(x^+) = \frac{1}{2} \int_{-L}^L dy^- J_0\left(m \sqrt{x^+(L - y^+)}\right) \partial_y^+ g(y^-). \quad (5.13b)$$

Again, this can be solved via Laplace transformation. Introducing the transforms

$$K_L(p) = \int dx^+ e^{-px^+} k_L(x^+), \quad (5.14a)$$

$$F(p) = \int dx^+ e^{-px^+} f(x^+), \quad (5.14b)$$

$$H_L(p) = \int dx^+ e^{-px^+} h_L(x^+), \quad (5.14c)$$

one obtains the simple expression $F = H_L/K_L$ or

$$f(x^+) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{px^+} \frac{H_L(p)}{K_L(p)}; \quad x^+ > 0. \quad (5.15)$$

K_L is given by the integral

$$K_L(p) = \int_0^\infty dz e^{-pz^2/2Lm^2} J_1(z), \quad (5.16)$$

which can be evaluated^[34] to give

$$K_L(p) = 1 - \exp(-m^2 L/2p), \quad (5.17)$$

so that finally f is determined as

$$f(x^+) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{px^+} \frac{H_L(p)}{1 - \exp(-m^2 L/2p)}; \quad x^+ > 0. \quad (5.18)$$

As H_L is a known functional of g , we have shown that the data f on $x^- = -L$ are given in terms of the data on $x^+ = 0$ and cannot be specified independently.

We note that the method breaks down in the limit of vanishing mass, $m \rightarrow 0$. In this case k_L is zero and (5.12) reduces to $h_L(x^+) = 0$, which from (5.13b) is nothing but the trivial statement that g is periodic in x^- . This breakdown can also be seen by directly studying the Riemann solution (3.10) in the massless case. Periodic BC do not lead to a relation between the data. Antiperiodic BC, however, are more restrictive: they eliminate right moving waves, $f(x^+) = 0$, so that $\phi = g(x^-)$ only.

C. The Periodic Pauli-Jordan Function

In this subsection we want to check our formula (5.18) for an explicit solution of the KG equation with known values on both characteristics, namely the (periodic) Pauli-Jordan function. We start from the Fourier representation (2.8) of the Pauli-Jordan function and replace

$$\int dk^+ \rightarrow \frac{2\pi}{L} \sum_{n \in \mathbb{Z}} . \quad (5.19)$$

This yields the periodic Pauli-Jordan function Δ_L in a finite spatial volume,

$$\begin{aligned} \Delta_L(x^+, x^-) &= -\frac{i}{2L} \sum_n \int dp^- \delta(p_n^+ p^- - m^2) \text{sgn}(p_n^+ + p^-) e^{-ip_n^+ x^- / 2 - ip^- x^+ / 2} \\ &\equiv \sum_n \Delta_n(x^+) e^{-in\pi x^- / L} . \end{aligned} \quad (5.20)$$

As Δ_L is a periodic solution of the KG equation its zero mode must vanish, according to (5.7), i.e.

$$\Delta_0(x^+) = 0 . \quad (5.21)$$

The sum in (5.20) therefore only runs over $n \neq 0$. Thus we can unambiguously perform the p^- integration (cf. (3.17)),

$$\begin{aligned} \Delta_{n \neq 0}(x^+) &= -\frac{i}{2L} \int dp^- \frac{1}{|p_n^+|} \delta(p^- - m^2 / p_n^+) \text{sgn}(p_n^+ + p^-) e^{-ip^- x^+ / 2} = \\ &= -\frac{i}{4\pi n} \exp(-i\hat{p}_n^- x^+ / 2) . \end{aligned} \quad (5.22)$$

Note that the expression $1/|p_n^+|$ is always well defined because $p_n^+ \neq 0$. Finally we obtain

$$\Delta_L(x^+, x^-) = - \sum_{n \neq 0} \frac{i}{4\pi n} e^{-i\hat{p}_n^- x^+ / 2 - in\pi x^- / L} . \quad (5.23)$$

This coincides with formula (5.11) applied to the special case of the Pauli-Jordan function. On the characteristics, Δ_L has the values

$$\Delta_L(x^+ = 0, x^-) = - \sum_{n \neq 0} \frac{i}{4\pi n} e^{-in\pi x^- / L} = - \left[\frac{1}{4} \text{sgn}(x^-) - \frac{x^-}{4L} \right] \equiv g(x^-) , \quad (5.24a)$$

$$\Delta_L(x^+, x^- = -L) = - \frac{i}{4\pi} \sum_{n \neq 0} \frac{(-1)^n}{n} \exp\left(-i \frac{m^2 L}{4\pi n} x^+\right) \equiv f(x^+) . \quad (5.24b)$$

As a cross-check we note that $\Delta_L(x^+ = 0, x^-)$ coincides (up to a factor of -2) with the periodic Green function G_3 of (4.24). The commutators corresponding to (5.24) are

$$[\phi(x), \phi(0)]_{x^+ = 0} = ig(x^-) , \quad (5.25a)$$

$$[\phi(x), \phi(0)]_{x^- = -L} = if(x^+) . \quad (5.25b)$$

Our objective is to obtain (5.24b) from (5.24a) by using formula (5.18) so that the commutator (5.25b) on $x^- = -L$ can be expressed in terms of the commutator (5.25a) on $x^+ = 0$. To this end we calculate (using the notation of Subsection B)

$$h_L(x^+) = -\frac{1}{2} J_0\left(m\sqrt{x^+L}\right) + \frac{J_1\left(m\sqrt{2Lx^+}\right)}{m\sqrt{2Lx^+}} , \quad (5.26)$$

which has the Laplace transform

$$H_L(p) = -\frac{1}{2p} \exp\left(-\frac{m^2 L}{4p}\right) - \frac{1}{m^2 L} \exp\left(-\frac{m^2 L}{2p}\right) + \frac{1}{m^2 L} . \quad (5.27)$$

Inserting this into (5.18) one finds that f should be

$$f(x^+) \stackrel{?}{=} - \frac{1}{8\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{px^+} \left[\frac{1}{p \sinh(m^2 L / 4p)} - \frac{1}{m^2 L} \right] . \quad (5.28)$$

For $x^+ > 0$ the second term does not contribute. The first term has single poles at

$$p_n := -i \frac{m^2 L}{4n\pi} ; \quad n = \pm 1, \pm 2, \dots . \quad (5.29)$$

Note that there is no pole at $n = 0$. A residue evaluation of (5.28) yields

$$-\frac{1}{8\pi i} \int_{c-i\infty}^{c+i\infty} dp e^{px^+} \left[\frac{1}{p \sinh(m^2 L/4p)} - \frac{1}{m^2 L} \right] = -\frac{i}{4\pi} \sum_{n \neq 0} \frac{(-1)^n}{n} \exp\left(-i \frac{m^2 L}{4\pi n} x^+\right) \equiv f(x^+). \quad (5.30)$$

This coincides exactly with (5.24b) which is what we wanted to show. As a final step we want to derive a consistency condition analogous to (2.12) and (3.15), expressing the commutator $[\phi(x), \phi(0)]$ in terms of its values on $x^+ = 0$ only. This information is implicitly provided by (5.23) from which we obtain

$$[\phi(x), \phi(0)] = i\Delta_L(x) = \sum_{n \neq 0} i\Delta_n(x^+ = 0) e^{-i\hat{p}_n \cdot x}, \quad (5.31)$$

with Δ_n given in terms of the commutator on $x^+ = 0$,

$$i\Delta_n(x^+ = 0) = \frac{1}{2L} \int_{-L}^L dx^- e^{in\pi x^-/L} [\phi(x), \phi(0)]_{x^+=0}. \quad (5.32)$$

Inserting this into (5.31) finally yields

$$\begin{aligned} [\phi(x), \phi(0)] &= \frac{1}{2L} \int_{-L}^L dy^- [\phi(y), \phi(0)]_{y^+=0} \sum_{n \neq 0} e^{-in\pi(x^- - y^-)/L - i\hat{p}_n^- x^+/2} = \\ &= \int_{-L}^L dy^- [\phi(y), \phi(0)] \left. \frac{\partial}{\partial y^-} \Delta_L(x - y) \right|_{y^+=0}. \end{aligned} \quad (5.33)$$

This is the desired consistency condition for the initial-boundary value problem. An equal- x^- commutator as in (3.15) does not appear.

6. Conclusions

We have shown that a single-time formalism with the canonical LC Hamiltonian and data/commutators specified on *two* lightlike hyperplanes $x^\pm = \text{const}$ fails to describe the dynamics in accordance with the Euler-Lagrange equation of motions. However, quantisation on a *single* light front (characteristic) $x^+ = \text{const}$ is consistent if (i) the field under consideration is massive and (ii) additional boundary conditions are specified in the x^- direction. Hamiltonian and Euler-Lagrange description of the dynamics are then completely equivalent. The LC Hamiltonian formulation is canonical and simple. Technically, we have found that the phase space reduction method of Faddeev and Jackiw is a very useful tool for these investigations as well as for light front quantisation in general. As a result we

can say that (under the above mentioned conditions) the attempts of quantising LC field theories in finite volumes are on a safe basis.

One might argue that our conclusions are valid only for the free theory. However, as has been emphasized by Jackiw^[4], the equal- x^+ -commutators are also initial data for the interacting theory, as long as there is no dynamics evolving within the initial surface. Due to causality, this is obviously the case for spacelike surfaces but also for lightlike ones, if there is a mass gap. Again the massless case seems to be peculiar.

There is no such straightforward argument concerning the relevance of boundary conditions for interacting theories, but there are at least some indications. For theories with constant background fields, caused e.g. by non-vanishing vacuum expectation values, periodic boundary conditions seem to be a natural choice^[13]. Quite generally one would guess that boundary conditions are less important in the perturbative than in the non-perturbative regime. This has been verified for LC- ϕ^4 -theory where perturbative contributions to the self energy caused by boundary effects were shown to vanish in the infinite volume limit^[35]. On the other hand, if non-perturbative physics is due to non-trivial topological properties, boundary conditions play a crucial role. This is well known from the attempts of quantising gauge theories in finite volumes where boundary conditions define additional topological quantum numbers^[36]. In the context of LC quantum field theory this has been discussed for the Schwinger model^[37,38]. For LC gauge theories with massless gauge bosons (no Higgs phenomenon), however, one has to clarify the issue of massless particles propagating within the quantisation surface.

This implies that one should extend the analysis of this paper to fields of higher spin (fermions, gauge bosons,...) and also to higher dimensions. To some extent this has been done in Ref.[15] (without discussion of boundary conditions) where it is claimed that the characteristic initial value problem "is undetermined for field theories containing massive fields of spin higher than 1/2". If true there might also be a problem for the LC formulation of massive gauge bosons acquiring their masses via the Higgs mechanism. This raises the general issue of a light front description of mass generation: The conclusions of this paper only hold for massive fields; the massless case is generically different. There is probably no smooth transition between the massless and the massive case with respect to the quantisation prescription. This can be seen already in the Schwinger model (QED in 1+1 dimensions). There, massless fermions, which have to be quantised on the light cone^[16], transform into a massive boson, which can be quantised on a single front with periodic boundary conditions^[37]. It is therefore very difficult to set up a direct bosonisation formula because the latter has to relate fields quantised on different hypersurfaces.

In higher dimensions it is clear that there is a much greater variety of initial and/or boundary conditions and solutions^[25,39]. For massive fields a formula analogous to (3.8) still holds with just additional integrations over the additional dimensions^[14]. The discussion of sections 3 to 5 should therefore (with minor modifications) still be valid.

Acknowledgements

We are indebted to M.J. Lavelle, G. Fischer and S. Simbürger for reading (versions of) the manuscript.

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Figure Captions

Figure 1: The integration contour used in the derivation of (3.8). To know the field ϕ at the point P one has to specify either ϕ and $\dot{\phi}$ on AB (Cauchy problem) or ϕ on AC and CB (CIVP). Data outside the characteristic cone of P need not be specified^[14,23].