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Density of hyperbolicity in dimension one

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1. Introduction

In this paper we will solve one of the central problems in dynamical systems:

Theorem 1 (Density of hyperbolicity for real polynomials). *Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.*

Here we say that a real polynomial is *hyperbolic* or *Axiom* A, if the real line is the union of a repelling hyperbolic set, the basin of hyperbolic attracting periodic points and the basin of infinity. We call a C^1 endomorphism of the compact interval (or the circle) *hyperbolic* if it has finitely many hyperbolic attracting periodic points and the complement of the basin of attraction of these points is a hyperbolic set. By a theorem of Mañé for C^2 maps, this is equivalent to the following conditions: all periodic points are hyperbolic and all critical points converge to periodic attractors. Note that the space of hyperbolic maps is an open subset in the space of real polynomials of fixed degree, and that every hyperbolic map satisfying the mild “no-cycle” condition (which states that orbits of critical points are disjoint) is structurally stable; see [dMvS93]. Theorem 1 solves the 2nd part of Smale's eleventh problem for the 21st century [Sma00]:

Theorem 2 (Density of hyperbolicity in the C^k topology). *Hyperbolic* (*i.e. Axiom* A) *maps are dense in the space of C^k maps of the compact interval or the circle*, $k = 1, 2, \ldots, \infty, \omega$.

This theorem follows from the previous one. Indeed, one can approximate any smooth (or analytic) map on the interval by polynomial maps, and therefore by Theorem 1 by hyperbolic polynomials. Similarly, one can approximate any map of the circle by trigonometric polynomials. If a circle map does not have periodic points, it is semi-conjugate to the rotation and it can be approximated by an Axiom A map (this is a classical result). If a circle map does have a periodic point, then using this periodic point we can construct a piecewise smooth map of an interval conjugate to the circle map.

1.1. *History of the hyperbolicity problem.* The problem of density of hyperbolicity goes back in some form to Fatou; see [Fat20, p. 73] and [McM94, §4.1]. Smale gave this problem 'naively' as a thesis problem in the 1960's; see [Sma98]. Back then some people even believed that hyperbolic systems are dense in all dimensions, but this was shown to be false in the late 1960's for diffeomorphsms on manifolds of dimension ≥ 2. The problem whether hyperbolicity is dense in dimension one was studied by many people, and it was solved in the C^1 topology by [Jak71], a partial solution was given in the C^2 topology by [BM00] and C^2 density was finally proved in [She04].

From the 1980's spectacular progress was made in the study of quadratic polynomials. In part, this work was motivated by the survey papers of May (in *Science* and *Nature*) on connections of the quadratic maps $f_a(x) = ax(1-x)$ with population dynamics, and also by popular interest in computer pictures of Julia sets and the Mandelbrot set. Mathematically, the realization that quasi-conformal mappings and the measurable Riemann mapping theorem were natural ingredients, enabled Douady, Hubbard, Sullivan and Shishikura to go far beyond the work of the pioneers Julia and Fatou. Using these quasiconformal rigidity methods, Douady, Hubbard, Milnor, Sullivan and Thurston proved in the early 1980's that bifurcations appear monotonically within the family $f_a\colon [0,1] \to [0,1]$, $a \in [0,4]$. In the early 1990's, as a byproduct of his proof on the Feigenbaum conjectures, Sullivan proved that hyperbolicity of the quadratic family can be reduced to proving that any two topologically conjugate nonhyperbolic quadratic polynomials are quasi-conformally conjugate. In the early 1990's McMullen was able to prove a slightly weaker statement: each real quadratic map can be perturbed to a (possibly complex) hyperbolic quadratic map. A major step was made when, in 1997, Graczyk and Światek (see [GŚ97] and [GŚ98]), and Lyubich (see [Lyu97]) proved independently that hyperbolic maps are dense in the space of real quadratic maps. Both proofs require complex bounds and growth of moduli of certain annuli. The latter part was inspired by Yoccoz's proof that the Mandelbrot set is locally connected at nonrenormalizable parameters, but is heavily based on the fact that $z^2 + c$ has only one quadratic critical point (the statement is otherwise wrong). Using their result, Kozlovski was able to prove hyperbolic maps are dense within the space of smooth unimodal maps in [Koz03].

In 2003, the authors were able to prove density of hyperbolicity for real polynomials with real critical points, see [SKvS]. The main step in that proof was to obtain estimates for Yoccoz puzzle pieces both from above and below. In the present paper, we solve the original density of hyperbolicity questions completely for real, one-dimensional, dynamical systems.

1.2. *Strategy of the proof and some remarks.* The main ingredient for the proof of Theorem 1 is the rigidity result [SKvS].

The first step in proving Theorem 1 is to prove complex bounds for real maps in full generality. This was done previously in [LvS98], [LY97] and [GŚ96] in the real unimodal case, and in the (real) multimodal minimal case in [She04]. The proof of the remaining case (multimodal nonminimal) will be given in Section 3. As in [SKvS] one has quasi-conformal rigidity for the box mappings we construct; see Theorem 4.

Next we show (roughly speaking) that if a real analytic family of real analytic maps f_λ has nonconstant kneading type, then either f_0 is hyperbolic or f_λ displays a critical relation for λ arbitrarily close to 0. This will be done in Section 4, by a strategy which is similar to the unimodal situation dealt with in [Koz03], but we use the additional combinatorial complexity in the multimodal case and the existence of box mappings and their quasi-conformal rigidity.

With this in mind, it is is fairly easy to construct families of polynomial maps f_λ, so that f_λ has more critical relations than f_0 for (some) parameters λ arbitrarily close to 0: approximate an artificial family of C^3 maps by a family of polynomials (of much higher degree). In this way one can approximate the original polynomial by polynomials (of higher degree) so that each critical point either is contained in the basin of attracting periodic points or satisfies a critical relation, i.e., is eventually periodic. From this, and the Straightening Theorem, the main theorem will immediately follow.

Of course it is natural to ask about the Lebesgue measure of parameters for which f_λ is 'good'. At this moment, we are not able to prove the general version of Lyubich's results [Lyu02] that for almost every $c \in \mathbb{R}$, the quadratic map $z \mapsto z^2+c$ is either hyperbolic or stochastic. (This result was strengthened by Avila and Moreira [AM], who proved that for almost all real parameters the quadratic map has nonzero Lyapounov exponents.) This would prove the famous Palis conjecture in the real one-dimensional case; see [Pal00]. See, however, [BSvS04].

2. Notation and terminology

Let Z be a topological space and $x \in Z$. The connected component of Z containing x will be denoted as $\mathrm{Comp}_x Z$, or, if it is not misleading, as $Z(x)$. Similar notation applies to a connected subset of Z.

Let $I = (a, b)$ be an interval on the real line. For any $\alpha \in (0, \pi)$ we use $D_\alpha(I)$ to denote the set of points $z \in \mathbb{C}$ such that the angle $\angle azb$ is greater than α. $D_\alpha(I)$ is a Poincaré disc: it is equal to the set of points $z \in \mathbb{C}$ with $d_P(z, I) < d(\alpha)$ where d_P is the Poincaré metric on $\mathbb{C} \setminus (\mathbb{R} \setminus I)$, and $d(\alpha) > 0$ is a constant depending only on α.

Let f be a real C^1 map of a closed interval $X = [0, 1]$ with a finite number of critical points which are not of inflection type (so each critical point of f is either a local maximum or minimum) and are contained in $\mathrm{int}(X)$. The set of critical points of f will be denoted as $\mathrm{Crit}(f)$.

Denote the critical points of f by $c_1 < c_2 < \cdots < c_b$. These critical points divide the interval $[0,1]$ into a partition $\mathcal{P}$ which consists of elements $\{[0,c_1), c_1, (c_1,c_2), c_2, \dots, (c_b,1]\}$.

For every point $x \in [0,1]$ we can define a sequence $\nu_f(x) = \{i_k\}_{k=0}^\infty$ such that $i_k = j$ if $f^k(x)$ belongs to the j-th element of the partition $\mathcal{P}$, $0 \le j \le 2b$. This sequence is called *the itinerary* of x.

We say that $f, \tilde{f}$ are *combinatorially equivalent* if there exists an order-preserving bijection h from the postcritical set (i.e., the iterates of the critical points) of f onto the corresponding set for $\tilde{f}$ which conjugates f and $\tilde{f}$. Obviously, the itineraries of the corresponding critical points of f and $\tilde{f}$ are the same.

In many cases we want to control only critical points which do not converge to periodic attractors and for this purpose we introduce the following notion. Two maps f and $\tilde{f}$ are called *essentially combinatorially equivalent* if there exists an order preserving bijection $h : \cup_c \mathrm{orb}_f(c) \to \cup_{\tilde{c}} \mathrm{orb}_{\tilde{f}}(\tilde{c})$ which conjugates f and $\tilde{f}$, where the union is taken over the set of critical points whose iterates do not converge to a periodic attractor.

For a critical point c of f, let $\mathrm{Forw}(c)$ denote the set of all critical points contained in the closure of the orbit $\{f^n(c)\}_{n=0}^\infty$, and let $\mathrm{Back}(c)$ be the set of all critical points c' with $\mathrm{Forw}(c') \ni c$. So if $c' \in \mathrm{Forw}(c)$, then either $f^n(c) = c'$ for some $n \ge 0$ or $\omega(c) \ni c'$. Let $[c] = \mathrm{Forw}(c) \cap \mathrm{Back}(c)$. Now, $[c]$ is equal to $\{c\}$ if c is nonrecurrent and equal to the collection of critical points $c' \in \omega(c)$ with $\omega(c) = \omega(c')$ otherwise.

An open set $I \subset X$ is called *nice* if for any $x \in \partial I$ and any $n \ge 1$, $f^n(x) \notin I$. Let Ω be a subset of $\mathrm{Crit}(f)$. An *admissible neighborhood* of Ω is a nice open set I with the following property:

- I has exactly $\#\Omega$ components each of which contains a critical point in Ω;
- for each connected component J of the domain of definition of the first return map to I, either J is a component of I or J is compactly contained in I.

Given an admissible neighborhood I of Ω, $\mathrm{Dom}(I)$ will denote the union of the components of the domain of the first entry map to I which intersect the orbit of c for some $c \in \Omega$, $\mathrm{Dom}'(I) = \mathrm{Dom}(I) \cup I$, and $\mathbf{D}(I) = \mathrm{Dom}(I) \cap I$. We use $R_I : \mathbf{D}(I) \to I$ to denote the first entry map E_I to I restricted to $\mathbf{D}(I)$. For each admissible neighborhood I of Ω, let

$$C_1(I) = \Omega \setminus \mathrm{Dom}(I) \text{ and } \mathrm{C}_2(I) = \{c' \in \Omega : I(c') \subset \mathrm{Dom}(I)\}.$$

3. Induced holomorphic box mappings

In this section we will prove the existence of complex bounds, i.e., the existence of box mappings. There are several definitions of box mappings. Here we will use a definition which is slightly more general than the one given in [SKvS].

Definition 1 (Complex box mappings). A holomorphic map

$$F : U \to V \tag{1}$$

between open sets in $\mathbb{C}$ is a *complex box mapping* if the following hold:

- V is a union of finitely many pairwise disjoint Jordan disks;
- Every connected component V' of V is either a connected component of U or the intersection of V' and U is a union of Jordan disks with pairwise disjoint closures which are compactly contained in V',
- For each component U' of U, $F(U')$ is a component of V and $F|U'$ is a proper map with at most one critical point;
- Each connected component of V contains at most one critical point of F.

The *filled Julia set* of F is defined to be

$$K(F) = \{z \in U : F^n(z) \in U \text{ for any } n \in \mathbb{N}\};$$

and the *Julia set* is $J(F) = \partial K(F)$.

Such a *complex box mapping* is called *real-symmetric* if F is real, all its critical points are real, and the domains U and V are symmetric with respect to $\mathbb{R}$.

A *real box mapping* is defined similarly: replace "Jordan disks" by "intervals", and "holomorphic" by "real analytic".

We say that a box mapping F is *induced* by a map f if any branch of F is some iterate of a complex extension of the map $f\colon X \to X$.

This type of box mapping naturally arises in the following setting: let $f : \Delta \to \mathbb{C}$ be a holomorphic map, $f(X) \subset X$, where Δ is some complex neighborhood of X. Fix some critical points of f and an appropriate neighborhood V of these critical points, consider the first entry map $R : U \to V$ of f to V. We will see that if the domain V is carefully chosen, then the map $R : U \to V$ is a complex box mapping.

Let us define a graph $\mathbf{Cr}$=$\mathbf{Cr}(f)$ as follows: the vertices of $\mathbf{Cr}$ are the critical points of f, and there is an edge between two distinct critical points c_1, c_2 if and only if $c_1 \in \mathrm{Forw}(c_2)$ or $c_2 \in \mathrm{Forw}(c_1)$. A subset Ω of $\mathrm{Crit}(f)$ is called *connected* if it is connected with respect to the graph.

A subset Ω of $\mathrm{Crit}(f)$ is called a *block* if it is contained in a connected component of $\mathbf{Cr}(f)$ and if $\mathrm{Back}(c) \subset \Omega$ holds for all $c \in \Omega$. Clearly, a connected component of $\mathbf{Cr}(f)$ is a block, and it is maximal in the sense that it is not properly contained in another block. A block is called *nontrivial* if it is disjoint from the basin of periodic attractors and there exists $c \in \Omega$ with an infinite orbit.

THEOREM 3 (Existence of complex box mappings). *Let $f : X \to X$ be a real analytic map with nondegenerate critical points.*

I. *Let c_0 be a nonperiodic recurrent critical point of f. Then there exists an admissible neighborhood I of $[c_0]$ such that $R_I \colon \mathbf{D}(I) \to I$ extends to a real-symmetric complex box mapping $F : U \to V$ with* $\mathrm{Crit}(F) = [c_0]$, *and F carries no invariant line field on its filled Julia set.*

II. *Assume that Ω is a nontrivial block of critical points such that*

- *each recurrent critical point $c \in \Omega$ has a nonminimal ω-limit set*;
- *if Ω' is the component of the graph $\boldsymbol{Cr}(f)$ which contains Ω, then f is not infinitely renormalizable at any $c' \in \Omega'$.*

Then, for any $K > 0$ there exists an admissible neighborhood I of Ω, such that $R_I : \mathbf{D}(I) \to I$ extends to a complex box mapping $F : U \to V$ with the following properties:

- *For each $c \in \Omega$, $V(c)$ is contained in $D_{\theta_0}(I(c))$, where $\theta_0 \in (0, \pi)$ is a universal constant*;
- *There exists a universal constant $\theta_1 > 0$ such that any connected component U' of U satisfies*
$$fU' \subset D_{\theta_1}(fU' \cap \mathbb{R});$$
- *Let Q be the closure of $\partial(U \cap \mathbb{R}) \cup \partial(V \cap \mathbb{R})$. Then there exists a constant $C > 0$ such that*
$$\mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U', \partial V') > C \text{ and } \mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U', \partial U'') > C$$
where $\mathrm{dist}_{\mathbb{C}\setminus Q}$ is the hyperbolic distance in $\mathbb{C} \setminus Q$, V' is a connected component of V and $U' \neq U''$ are connected components of U;
- *The filled Julia set of F has Lebesgue measure zero*;
- *If U' is a connected component of U and compactly contained in a component V' of V, then* $\mathrm{mod}(V' \setminus U') \geq K$;
- *For each $c \in U \cap \Omega$, $|f(\mathrm{Comp}_c(V) \cap \mathbb{R})| > K|f(\mathrm{Comp}_c(U)) \cap \mathbb{R}|$.*

In the case of minimal $\omega(c_0)$ the existence of the box mapping is proved in [She04], and the absence of an invariant line field follows from the same argument in Sections 6 and 7 of [She03]. So we only have to prove the nonminimal case. The proof of this theorem will occupy the next two subsections.

3.1. *Complex bounds from real bounds.* Let Ω be as in Theorem 3. Our goal of this subsection is to prove that for an appropriate choice of an admissible neighborhood I of Ω, the real box mapping R_I extends to a complex box mapping with the desired properties. To this end, it is convenient to introduce geometric parameters $\mathrm{Space}(I)$, $\mathrm{Gap}(I)$ and $\mathrm{Cen}(I)$ as follows.

For any intervals $j \subset t$, where the components of $t \setminus j$ are denoted by l, r, define

$$\mathrm{Gap}(l, r) = \frac{1}{\mathrm{Space}(t, j)} := \frac{|t||j|}{|l||r|}.$$

So if $\mathrm{Gap}(l, r)$ is large, then the gap interval j is at least larger than one of the intervals l or r. At the same time, if $\mathrm{Space}(t, j)$ is large, then there is a large space around the interval j inside t. The parameter $\mathrm{Gap}(I)$ is defined as

$$\mathrm{Gap}(I) = \inf_{(J_1,J_2)} \mathrm{Gap}(J_1, J_2),$$

where (J_1, J_2) runs over all distinct pairs of components of $\mathrm{Dom}'(I)$.

To introduce the parameter $\mathrm{Space}(I)$, let

$$I^* = \bigcup_{c' \in \mathrm{C}_2(I)} I(c'), \quad I^\sharp = I - I^*. \tag{2}$$

The parameter $\mathrm{Space}(I)$ is defined to be

$$\mathrm{Space}(I) = \inf_J \mathrm{Space}(\mathrm{Comp}_J\, I, J),$$

where the infimum is taken over all components J of the domain of R_I which are contained in $I^\sharp$. In the following construction we shall be unable to guarantee that all components of $\mathbf{D}(I)$ are compactly contained in I.

Furthermore, for any $c \in \Omega$, let $\hat{J}(c)$ be the component of $\mathrm{Dom}'(I)$ which contains $f(c)$ (so $\hat{J}(c) = \emptyset$ if $f(c) \notin \mathrm{Dom}'(I)$), and define

$$\mathrm{Cen}_1(I) = \max_{c \in \Omega \setminus \mathrm{C}_2(I)} \frac{|\hat{J}(c)|}{|f(I(c))|},$$

$$\mathrm{Cen}_2(I) = \max_{c \in \mathrm{C}_2(I)} \left(\left| \frac{|\hat{J}(c)|}{|f(I(c))|} - 2 \right| \right),$$

and $\mathrm{Cen}(I) = \max(\mathrm{Cen}_1(I), \mathrm{Cen}_2(I))$.

Proposition 1. *There exists $\varepsilon_0 > 0$, $C_0 > 0$ and $\theta_0 \in (0, \pi)$ (depending only on $\#\Omega$) with the following properties. Let I be an admissible neighborhood of Ω such that* $\mathrm{Cen}(I) < \varepsilon_0$, $\mathrm{Space}(I) > C_0$ *and* $\mathrm{Gap}(I) > C_0$. *Assume also* *that* $\max_{c'\in\Omega}|I(c')|$ *is sufficiently small. Then there exists a real-symmetric complex box mapping* $F : U \to V$ *whose real trace is real box mapping* R_I. *Moreover, the map* F *satisfies the properties specified in Theorem* 3.

To prove this proposition we need a few lemmas. Let $\mathcal{U} \subset \mathbb{C}$ be a small neighborhood of X so that $f\colon X \to X$ extends to a holomorphic function $f : \mathcal{U} \to \mathbb{C}$ which has only critical points in X. Here, as before, $X = [0, 1]$.

Fact 1 (Lemma 3.3 in [dFdM99]). For every small $a > 0$, there exists $\theta(a) > 0$ satisfying $\theta(a) \to 0$ and $a/\theta(a) \to 0$ as $a \to 0$ such that the following holds. Let $F : \mathbb{D} \to \mathbb{C}$ be univalent and real-symmetric, and assume that $F(0) = 0$ and $F(a) = a$. Then for all $\theta \geq \theta(a)$, we have $F(D_\theta((0, a))) \subset D_{(1-a^{1+\tau})\theta}((0, a))$, where $0 \leq \tau < 1$ is a universal constant.

Lemma 1. *For any* $\theta > 0$ *there exists* $\eta_1 = \eta_1(f, \theta) > 0$ *such that if* J *is an interval which does not contain a critical point and* $|J| < \eta_1$, *then* $f : J \to fJ$ *extends to a conformal map* $f : U \to D_\theta(fJ)$ *such that* $U \subset D_{(1-M|fJ|^{1+\tau})\theta}(J)$, *where* $M > 0$ *is a constant depending on* f.

◁ Taking two small neighborhoods $N \Subset N'$ of $\mathrm{Crit}(f)$, assuming $|J|$ is small enough, we have either $J \cap N = \emptyset$ or $J \subset N'$. In the former case, f defines a conformal map onto a definite complex neighborhood of fJ, and the lemma follows by applying Fact 1 to the inverse of this conformal map. In the latter case, f can written as $Q \circ \phi$, where ϕ is a conformal map onto a definite neighborhood of fJ and Q is a real quadratic polynomial. The lemma follows from Fact 1 applied to ϕ^{-1} and the Schwarz lemma. ▷

Let us say that a sequence of open intervals $\{G_i\}_{i=0}^{s}$ is a *chain* if G_i is a component of $f^{-1}(G_{i+1})$ for each $i = 0, \ldots, s-1$. The *order* of this chain is the number of G_i's which contain a critical point, $0 \leq i < s$.

Lemma 2. *For any* $\theta \in (0, \pi)$ *there exists* $\eta = \eta(f, \theta) > 0$ *and* $\theta' \in (0, \pi)$ *such that the following holds. Let* I *be an admissible neighborhood of* Ω *with* $|I| < \eta$ *and* $\mathrm{Cen}_2(I) < 1$. *Let* J *be a component of* $\mathrm{Dom}'(I)$, *let* $s \geq 0$ *be minimal with* $f^s(J) \subset I^\sharp$, *and let* K *be the component of* $I^\sharp$ *containing* $f^s(J)$. *Then there exists a Jordan disk* U *with* $J \subset U \subset D_{\theta'}(J)$ *such that* $f^s : U \to D_\theta(K)$ *is a well-defined proper map.*

◁ Let $\{G_j\}_{j=0}^{s}$ be the chain with $G_s = K$ and $G_0 = J$. Since f has no wandering interval, $\max_{j=1}^{s}|G_j|$ is small provided that $|K| \leq |I|$ is sufficiently small. Moreover since $f^s : J \to K$ is a first return map, the intervals G_j, $1 \leq j \leq s$ are pairwise disjoint; thus

$$\sum_{j=1}^{s}|G_j| \leq |X| = 1.$$

Therefore, assuming $|I|$ is sufficiently small, we obtain that $\sum_{j=1}^{s}|G_j|^{1+\tau}$ is as small as we want.

First consider the case that $f^s|J$ is a diffeomorphism. Let η_1 and M be as in Lemma 1. Then provided that $\max_{j=1}^{s}|G_j| < \eta_1$ and $\sum_{j=1}^{s}|G_j|^{1+\tau} < 1/(2M)$, that lemma implies that there is a sequence of Jordan disks U_j with $U_j \subset D_{\theta/2}(G_j)$, $0 \le j \le s$, such that $U_s = D_\theta(K)$ and $f : U_j \to U_{j+1}$ is a conformal map for all $0 \le j < s$. The lemma follows when $U = U_0$.

Now assume that $f^s|J$ is not diffeomorphic, and let $s_1 < s$ be maximal such that G_{s_1} contains a critical point c. Then as above, we obtain Jordan disks U_j for all $s_1 < j \le s$ such that $U_s = D_\theta(K)$, such that

- for all $s_1 < j < s$, $f : U_j \to U_{j+1}$ is a conformal map;
- $U_j \subset D_{\theta/2}(G_j)$.

Provided that I is sufficiently small, we have $c \in \bigcup_{c' \in \Omega} \mathrm{Back}(c') = \Omega$. By the minimality of s we have $c \in \mathrm{C}_2(I)$ and so by the assumption on $\mathrm{Cen}_2(I)$, $|f(G_{s_1})|/|G_{s_1+1}| = |f(I(c))|/|\hat{J}(c)|$ is bounded away from zero. Therefore, provided that $|G_{s_1+1}|$ is sufficiently small, we have a Jordan disk U_{s_1} with $G_{s_1} \subset U_{s_1} \subset D_{\theta_1}(G_{s_1})$ such that $f : U_{s_1} \to U_{s_1+1}$ is a 2-to-1 proper map, where $\theta_1 \in (0,\pi)$ is a constant depending only on θ. Repeat the argument for the shorter chain $\{G_j\}_{j=0}^{s_1}$ and so on. Since the order of the chain $\{G_j\}_{j=0}^{s}$ is bounded from above by $\#\Omega$, the procedure stops within $\#\Omega$ steps, completing the proof. $\triangleright$

Proof of Proposition 1. Assume that $|I|$ and $\mathrm{Cen}_2(I)$ are both very small. For each $c \in \Omega \setminus \mathrm{C}_2(I)$, define $V_c = D_{\pi/2}(I(c))$. By Lemma 2, there exists a constant $\theta_0 \in (0,\pi)$ and for each component J of $\mathrm{Dom}'(I)$, there exists a Jordan disk $U(J)$ with $J \subset U(J) \subset D_{\theta_0}(J)$ such that if $r = r(J)$ is the minimal nonnegative integer with $f^r(J) \subset I(c)$ for some $c \in \Omega \setminus \mathrm{C}_2(I)$, then $f^r : U(J) \to V_c$ is a well-defined proper map.

For $c \in \mathrm{C}_2(I)$, define $V_c = U(I(c))$. For each component J of $\mathrm{Dom}(I) \cap I^\sharp$, let $\hat{J}$ be the component of $\mathrm{Dom}'(I)$ which contains $f(J)$, and let $U(J)$ be the component of $f^{-1}(U(\hat{J}))$ which contains J. Then $U(J)$ is a Jordan disk with $U(J) \cap \mathbb{R} = J$, and $f : U(J) \to U(\hat{J})$ is a well-defined proper map.

Clearly, for each component J of the domain of R_I, if $c \in \Omega$ is such that $R_I(J) \subset I(c)$, and if $R_I|J = f^s|J$, then $f^s : U(J) \to V_c$ is a well-defined proper map.

Assume now that $\mathrm{Space}(I)$ is very big and and $\mathrm{Cen}_1(I)$ is very small. Then for each $c \in \Omega \setminus \mathrm{C}_2(I)$ and for each component J of $\mathrm{Dom}(I) \cap I(c)$, $\mathrm{mod}(V_c \setminus \overline{U}_J)$ is bounded from below by a large constant. In fact, if $J \not\ni c$ then by Lemma 1, $U(J) \subset D_{\theta_0/2}(J)$, which implies that $\mathrm{mod}(V_c \setminus \overline{U(J)}) \ge \mathrm{mod}(D_{\pi/2}(I(c)) \setminus \overline{D_{\theta_0/2}(J)})$ is large since $\mathrm{Space}(I, J)$ is large. If $J \ni c$, then by assumption, $|\hat{J}|/|f(I(c))|$ is small, so that $U(J)$ is contained in a round disk centered at c with radius much smaller than $|I(c)|$; hence $\mathrm{mod}(V_c \setminus \overline{U(J)})$ is again big. Note that provided that $\mathrm{Space}(I)$ is large enough,

$$\bigcup_{J \subset I(c)} U(J) \subset B\left(c, \frac{|I(c)|}{4}\right) \cup D_\alpha(I(c)), \tag{3}$$

where $\alpha \in (0, \pi)$ is a constant close to π.

Next let us assume that $\mathrm{Gap}(I)$ is large and show that there exists $\delta > 0$ such that for any components J_1 and J_2 of $\mathrm{Dom}(I)$,

$$\mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U(J_1), \partial U(J_2)) > \delta. \tag{4}$$

To this end, we may assume that J_1 and J_2 are contained in $I(c)$ for some $c \in \Omega \setminus \mathrm{C}_2(I)$, and that $|\hat{J}_1| \leq |\hat{J}_2|$. Recall that

$$f(U(J_i)) = U(\hat{J}_i) \subset D_{\theta_0}(\hat{J}_i), \ i = 1, 2. \tag{5}$$

In particular, provided that $\mathrm{Gap}(\hat{J}_1, \hat{J}_2)$ is larger than some number which only depends on θ_0,

$$\overline{U(J_1)} \cap \overline{U(J_2)} = \emptyset. \tag{6}$$

Let us consider the following two cases:

Case 1. $J_1 \ni c'$. Since there exist only finitely many components of $\mathrm{Dom}'(I)$ with length not smaller than $|J_1|$, there are only finitely many pairs (J_1, J_2) satisfying the property, and thus (4) follows from (6).

Case 2. $J_1 \not\ni c'$. In this case, (5) implies that $d(\partial U(J_1), \partial U(J_2))/|J_1|$ is big, provided that $\mathrm{Gap}(\hat{J}_1, \hat{J}_2)$ is big enough. Moreover, Lemma 1 implies that $U(J_1) \subset D_{\theta_0/2}(J_1)$. All these imply that the distance between $\mathrm{dist}_{\mathbb{C}\setminus\partial J_1}(\partial U(J_1), \partial U(J_2))$ is large, where $\mathrm{dist}_{\mathbb{C}\setminus\partial J_1}$ denotes the hyperbolic distance in $\mathbb{C} \setminus \partial J_1$. As $\mathrm{dist}_{\partial J_1} \leq \mathrm{dist}_{\mathbb{C}\setminus Q}$, (4) follows.

Now we define a complex box mapping $F\colon U \to V$ by setting $U = \bigcup_J U(J)$, $V = \bigcup_{c\in\Omega} V(c')$ and by defining F so that its real trace is R_I.

To complete the proof, it remains to show that the filled Julia set of F has measure zero. In fact, the property (3) implies that for a.e. $z \in K(F)$, $\omega(z)$ contains a critical point. For the set of points with this last property, one argues as in Proposition 2.2 and Theorem 5.1 of [She04] to show that this set has Lebesgue measure zero. □

For later use, let us include the following easy proposition to end this section.

PROPOSITION 2. *For any $\rho > 0$, there exists $\eta = \eta(f, \rho)$ with the following property. Let I be a nice interval, and let $\{J_i\}$ be a collection of components of the domain of the first return map R_I such that*

- *$R_I|J_i$ is monotone*;
- $\mathrm{Space}(I, J_i) > \rho$;
- *these J_i have pairwise disjoint closures.*

Assume that $|I| < \eta$ and I is disjoint from the immediate basin of periodic attractors. Then for any $\theta \in (\pi/2, \pi)$, $R_I : \bigcup J_i \to I$ extends to a complex box mapping $\phi : \bigcup U_i \to V$ such that $V = D_\theta(I)$ and $U_i \subset D_{\theta'}(J_i)$. Moreover, $\theta' \to \theta$ as $|I| \to 0$.

Proof. For each J_i, let s_i denote the return time of J_i to I. Then $\sum_{j=1}^{s} |f^j J_i| \le 1$. Provided that I is a small interval which is disjoint from the basin of periodic attractors, we have that $\sup_{j=1}^{s} |f^j J_i|$ is small, so that $\prod_{j=1}^{s}(1 - M|f^j J_i|)^{1+\tau}$ is close to 1. By Lemma 1, it follows that $f^{s_i} : J_i \to I$ extends to a conformal map $F : U_i \to V$ with $V = D_\theta(I)$ and $U_i \subset D_{\theta'}(J_i)$, with

$$\theta' = \prod_{j=1}^{s} \left(1 - M|f^j J_i|^{1+\tau}\right) \theta.$$

Since θ'/θ is close to 1 and J_i is well-inside I, $\mathrm{mod}(V \setminus U_i)$ is greater than a positive constant. Since J_i's have pairwise disjoint intervals, these U_i's have pairwise disjoint closure, so that $F : \bigcup U_i \to V$ defines a complex box mapping. $\square$

3.2. *Choice of an admissible neighborhood.* We shall prove here:

PROPOSITION 3. *Let Ω be a subset of* $\mathrm{Crit}(f)$ *as in Theorem* 3. *For any $\varepsilon > 0$ and $C > 0$ there exists an arbitrarily small admissible neighborhood I of Ω such that such that* $\mathrm{Gap}(I) > C$, $\mathrm{Space}(I) > C$, *and* $\mathrm{Cen}(I) < \varepsilon$.

First we observe that there exists a forward invariant finite set Z which is disjoint from the forward orbits of the critical points, such that for all $c \in \Omega$, the length of the component of $X \setminus f^{-n}(Z)$ which contains c tends to 0 as $n \to \infty$. In fact, this has been shown in Section 6.1 of [SKvS]. More precisely, this last property is equivalent to the fact that c is an accumulation point of $\bigcup_{n=0}^{\infty} f^{-n}(Z)$; thus if it holds for some c, then it holds for $c' \in \mathrm{Forw}(c) \cup$ *Back*(c). By Fact 6.1 and Lemma 6.1 of [SKvS], this property holds for $c \in \Omega$ which has an infinite orbit; thus it holds for all $c \in \Omega$.

Clearly, a component of $X \setminus f^{-n}(Z)$ is a nice interval. Throughout this subsection, all nice intervals are of this form.

Let us say that a nice interval I is *C-nice* if for each return domain J of I, we have $\mathrm{Space}(I, J) > C$.

The first step to prove Proposition 3 is the following:

Proposition 4. *Let* Ω *be a subset of* $\mathrm{Crit}(f)$ *as in Theorem* 3. *For any* $C > 0$ *there exists an arbitrarily small admissible neighborhood of* Ω *such that* $\mathrm{Space}(T) > C$.

Proof. Let us first prove the following:

Claim. For any $C > 0$ and any $c \in \Omega$, there exists an arbitrarily small C-nice interval $I \ni c$.

In fact, this claim was proved in Theorem 3.4 of [She04] in the case that c is recurrent. So let us assume that c is nonrecurrent. Let $K_1 \Supset K_2 \Supset \dots$ be a sequence of nice intervals containing c such that for each $i \geq 1$, $\mathrm{Space}(K_i, K_{i+1}) > 1$ and such that c does not return to K_1. Taking a large n, we show that K_n is C-nice. To this end, let $x \in K_n$ be a point which returns to K_n under iterates of f, and let L_j denote the entry domain of K_j containing x for all $1 \leq j \leq n$. Then by Theorem A of [vSV04], there exists a constant $\xi > 0$ such that $\mathrm{Space}(L_j, L_{j+1}) > \xi$ for all $1 \leq j < n$; thus $\mathrm{Space}(L_1, L_n) > C$ provided that n is large enough. As c is nonrecurrent, we have $L_1 \subset K_n$, for otherwise we would have $L_1 \supset K_n \ni 0$, which implies that 0 returns to K_1. Therefore $\mathrm{Space}(K_n, L_n) > C$.

To complete the proof, let $\Omega_1 = \{c_1, c_2, \dots, c_n\}$ be a minimal subset of Ω with the following property: for each $c \in \Omega \setminus \Omega_1$, $\mathrm{Forw}(c) \cap \Omega_1 \neq \emptyset$. By the minimality of Ω_1, we have that $\mathrm{Forw}(c_i) \not\ni c_j$ for any $i \neq j$. For each $1 \leq j \leq n$, let I_j be a small C-nice interval containing c_j. Then $I^\sharp = \bigcup_{j=1}^n I_j$ is a nice open set and

$$I := I^\sharp \cup \bigcup_{c \in \Omega \setminus \Omega_1} \mathrm{Comp}_c(\mathrm{Dom}'(I^\sharp))$$

is an admissible neighborhood of Ω. Let us show that $\mathrm{Space}(I) > C$ (provided that $\sup |I_j|$ is small enough).

To this end, let J be a component of $\mathrm{Dom}(I)$ which is compactly contained in I. Since $I \setminus I^\sharp \subset \mathrm{Dom}(I)$, we have $J \subset I_j$ for some $1 \leq j \leq n$, and that J is also a return domain of I_k for some $1 \leq k \leq n$. If $j = k$ then $\mathrm{Space}(I_j, J) > C$ since I_j is a C-nice interval. If $j \neq k$, then c_j does not enter a fixed nice interval $T_k \ni c_k$. Let J' be the entry domain of T_k which contains J. Arguing as in the proof of the claim above, provided that I_k is small enough, $\mathrm{Space}(J', J) > C$. On the other hand, $J' \not\ni c_j$, which implies that $J' \subset I_j$. Thus $\mathrm{Space}(I_j, J) > C$. □

Now let us fix for the moment a small constant $\rho > 0$. Given an admissible neighborhood T of Ω, we define a new admissible neighborhood $\mathcal{A}(T) = \mathcal{A}_\rho(T)$ with the following properties:

- for $c \in C_1(T)$, $\mathcal{A}(T)(c)$ is the maximal admissible neighborhood of c such that $|\mathcal{A}(T)(c)|/|T(c)| < \rho$;

- for $c \in \Omega \setminus (C_1(T) \cup C_2(T))$, $\mathcal{A}(T)(c)$ is the component of $\mathrm{Dom}(T)$ which contains c;
- for any $c \in C_2(T)$, when $k(c)$ is the minimal positive integer such that $R_T^{k(c)}(c) \in T(c')$ for some $c' \in \Omega \setminus C_2(T)$, then $\mathcal{A}(T)(c)$ is the maximal interval containing c such that $R_T^{k(c)}(\mathcal{A}(T)(c)) \subset \mathcal{A}(T)(c')$.

Clearly $\mathrm{C}_2(\mathcal{A}(T)) \subset \mathrm{C}_2(T)$. For $c \in \Omega \setminus C_2(T)$, let $k(c) = 0$. Note that for any $0 \le k \le k(c)$, $R_T^k(\mathcal{A}(T)(c)) \subset \mathcal{A}(T)$.

Lemma 3. *Assume that* $\mathrm{C}_2(\mathcal{A}(T)) = \mathrm{C}_2(T)$. *Then*

1. *For each* $c \in \mathrm{C}_2(T)$, $R_T(\mathcal{A}(T)(c)) \subset \mathcal{A}(T)$;
2. *For each* $c \in \Omega$ *and* $x \in T(c) \setminus \mathcal{A}(T)(c)$, *there exists an interval* $J(x)$ *with* $x \in J(x) \subset T(c) \setminus \mathcal{A}(T)(c)$ *such that* $R_T^{k(c)+1}$ *maps* $J(x)$ *onto a component of* T *diffeomorphically*;
3. *For each landing domain* J *of* $\mathcal{A}(T)$, *there exists an interval* $\hat{J}$ *with* $J \subset \hat{J} \subset \mathrm{Dom}'(T)$ *such that if* s *is the landing time of* J *into* $\mathcal{A}(T)$, *then* f^s *maps* $\hat{J}$ *diffeomorphically onto a component of* T.

$\lhd$ Let us prove the first statement by contradiction. It is enough to prove that $R_T(c) \in \mathcal{A}(T)$, so assume that this is not the case. For $0 \le i \le k(c)$, let $c_i \in \Omega$ be such that $R_T^i(c) \in T(c_i)$. Let $m \le k(c) - 1$ be maximal so that $R_T(c_m) \notin \mathcal{A}(T)(c_{m+1})$. Let $p \in \mathbb{N}$ be minimal such that $R_T^p(c_m) \in \mathcal{A}(T)$. By the maximality of m, we obtain $R_T(\mathcal{A}(T)(c_i)) \subset \mathcal{A}(T)(c_{i+1})$ for $i = m+1, \ldots, k-1$. Hence $R_T^i(c_m) \notin \mathcal{A}(T)$ for all $1 \le i \le k-m$, and so $p > k-m$. But $R_T^{k-m}(\partial\mathcal{A}(T)(c_m))$ is contained in $\partial\,\mathrm{Dom}(T)$, which implies that $R_T^p(\partial\mathcal{A}(T)(c_m)) \notin T$. Since $\mathcal{A}(T) \Subset T$, the minimality of p gives that $c_m \notin \mathrm{C}_2(\mathcal{A}(T))$. However, since $c_m \in \mathrm{C}_2(T) = \mathrm{C}_2(\mathcal{A}(T))$ this gives a contradiction.

Let us now pass to the proof of the second statement. By the first statement, for each $c \in \Omega$, $R_T^{k(c)}$ maps each component of $T(c) \setminus \mathcal{A}(T)(c)$ onto a component of $T \setminus \mathcal{A}(T)$ in a diffeomorphic way. For each $x \in T(c) \setminus \mathcal{A}(T)(c)$, we take $J(x)$ to be the maximal interval so that $R_T^{k(c)}(J(x))$ is contained in (a component of) $\mathrm{Dom}(T)$. Clearly, $R_T^{k(c)+1}$ maps $J(x)$ onto a component of T in a diffeomorphic way.

The third statement follows from the observation that any branch of the first landing map to $\mathcal{A}(T)$ can be written as the composition of the first landing map to T with finitely many maps of the form $R_T^{k(c)+1}|J(x)$, $x \in T(c)\setminus\mathcal{A}(T)(c)$. $\rhd$

Lemma 4. *Let* $c_1, c_2 \in \Omega$, *let* $x \in \mathcal{A}(T)(c_1) \cap \mathrm{Dom}(\mathcal{A}(T))$ *be such that* $R_{\mathcal{A}(T)}(x) \in \mathcal{A}(T)(c_2)$, *and let* s *be such that* $R_{\mathcal{A}(T)} = f^s$ *near* x. *Consider the* *chain* $\{G_i\}_{i=0}^s$ *with* $G_s = T(c_2)$ *and* $G_0 \ni x$. *Then the order of the chain is not greater than* $\#\Omega$. *Moreover, if* $c_1 \notin \mathrm{C}_2(\mathcal{A}(T))$, *then* $G_0 \subset \mathcal{A}(T)(c_1)$.

$\lhd$ First observe that $\mathcal{A}(T)(c) \supset \mathrm{Comp}_c \mathrm{Dom}(T(c))$ for all $c \in \Omega$. It follows that for each $c \in \Omega$, there can be at most one i with $0 < i \leq s$ such that $G_i \ni c$. Thus the order of the chain $\{G_i\}_{i=0}^s$ is at most $\#\Omega$.

Now let us assume that $c_1 \notin \mathrm{C}_2(\mathcal{A}(T))$ and show that $G_0 \subset \mathcal{A}(T)(c_1)$. Let k be the minimal positive integer such that $R_T^k = f^s$ near x. Since $G_0 \subset \mathrm{Comp}_x \mathrm{Dom}(T)$, we may assume that $c_1 \in \mathrm{C}_2(T)$. If $G_0 \Subset T(c_1)$, then $k > k(c_1)$, and $R_T^{k(c_1)}(G_0)$ is contained in a component of $\mathrm{Dom}(T)$ so that $G_0 \subset \mathcal{A}(T)(c_1)$. Therefore we may assume that $G_0 = T(c_1)$. Then $k \leq k(c_1)$, so $f^i(x) \notin T^\sharp$ for all $1 \leq i \leq s-1$. It follows that $R_T^k(\mathcal{A}(T)(c_1)) \subset \mathcal{A}(T)(c_2)$, which implies that $c_1 \in \mathrm{C}_2(\mathcal{A}(T))$. The contradiction completes the proof. $\rhd$

The following lemma is usually referred to as the *Koebe principle*. See [vSV04] for a proof.

Lemma 5. *Assume that* $\{G_i\}_{i=0}^s$ *is a chain such that* G_s *is contained in a small neighborhood of a nonperiodic and recurrent critical point.*

1. *For each* $N > 0$ *and* $C > 0$ *there exists* $C' > 0$ *such that if the order of the chain* $\{G_i\}_{i=0}^s$ *is at most* N *and* $\{J_i\}_{i=0}^s$ *is a chain with* $J_i \subset G_i$, $i = 0, \ldots, s$ *then if* $\mathrm{Space}(G_s, J_s) \geq C'$ *then* $\mathrm{Space}(G_0, J_0) > C$.
2. *For each* $C > 0$ *there exists* $K > 1$ *such that if* $f^s|G_0$ *is a diffeomorphism, and* $\mathrm{Space}(G_s, J_s) \geq C$ *then* $|Df^s(x)|/|Df^s(y)| \leq K$ *for each* $x, y \in J_0$. *Moreover,* $K \to 1$ *as* $C \to \infty$.

In the next two lemmas we will make a convenient choice for $\rho > 0$, but still write $\mathcal{A}(T) = \mathcal{A}_\rho(T)$.

Lemma 6. *For any* $\varepsilon > 0$ *and* $C > 0$ *there exists* $\rho_0 \in (0,1)$ *such that if* $\mathrm{Space}(T) > 1/\rho_0^{-1}$ *and* $\rho < \rho_0$, *then* $\mathrm{Space}(\mathcal{A}(T)) > C$ *and* $\mathrm{Cen}_1(\mathcal{A}(T)) < \varepsilon$.

$\lhd$ By Lemma 4 and the above Koebe principle, it suffices to show that $|\mathcal{A}(T)(c)|/|T(c)|$ is small for every $c \in \Omega$, provided that $\mathrm{Space}(T)$ and ρ^{-1} are sufficiently large. To this end, let s be such that $R_T^{k(c)} = f^s$ on $T(c)$, and consider the chain $\{G_i\}_{i=0}^s$ with $G_s = T(f^s(c))$ and $G_0 = T(c)$. The order of this chain is bounded from above by $k(c) \leq \#\Omega$. Since $\mathrm{Space}(T(f^s(c)), f^s(\mathcal{A}(T)(c)))$ is large, again by the above Koebe principle, we obtain the desired estimate. $\rhd$

Lemma 7. *For any* $\varepsilon > 0$ *and* $C > 0$ *there exists* $\rho_1 \in (0,1)$ *with the following property. Assume that* $\mathrm{C}_2(\mathcal{A}(T)) = \mathrm{C}_2(T)$, $\mathrm{Space}(T) > \rho_1^{-1}$ *and*

$\rho < \rho_1$. *Then* $\mathrm{Gap}(\mathcal{A}(T)) > C$. *Moreover, if* $\mathrm{C}_2(\mathcal{A}^2(T)) = \mathrm{C}_2(T)$, *then* $\mathrm{Cen}(\mathcal{A}(T)) < \varepsilon$.

$\lhd$ Assume that $\mathrm{Space}(T)$ is large and that ρ is small. Then by Lemma 6, for each $c \in \Omega$, $\mathcal{A}(T)(c)$ is deep inside $T(c)$.

Let us first show that $\mathrm{Gap}(\mathcal{A}(T))$ is big. To this end, let J_1 and J_2 be distinct components of $\mathrm{Dom}'(\mathcal{A}(T))$ and let s_1, s_2 be their landing times to $\mathcal{A}(T)$. Without loss of generality, assume $s_1 \le s_2$. It is enough to show that the gap between J_1 and J_2 is much bigger than J_2. Let $\hat{J}_i$, $i = 1, 2$, be as in Lemma 3 (3). By the Koebe principle, J_i is deep inside $\hat{J}_i$, so it suffices to show that $J_1 \cap \hat{J}_2 = \emptyset$. Let us prove this by contradiction. Assume that $J_1 \cap \hat{J}_2 \neq \emptyset$. Since both J_1 and $\hat{J}_2$ are pull backs of the nice set T, so either $J_1 \supset \hat{J}_2$ or $J_1 \subset \hat{J}_2$. Since $J_1 \cap J_2 = \emptyset$, the first alternative cannot happen. Therefore, $J_1 \subset \hat{J}_2$. It follows that for all $0 \le i \le s_2$, $f^i(J_1) \subset f^i(\hat{J}_2) - f^i(J_2)$; hence $f^i(J_1) \cap \Omega = \emptyset$. But $f^{s_1}(J_1)$ is a component of $\mathcal{A}(T)$, a contradiction.

Now let us also assume that $\mathrm{C}_2(\mathcal{A}^2(T)) = \mathrm{C}_2(\mathcal{A}(T))$, and show that $\mathrm{Cen}(\mathcal{A}(T))$ is small. In Lemma 6, we have already shown that that $\mathrm{Cen}_1(\mathcal{A}(T))$ is small. So it remains to show that $\mathrm{Cen}_2(T)$ is small. To this end, take $c \in \mathrm{C}_2(T)$ and let $c' \in \Omega$ be such that $R_T(c) \in T(c')$. By assumption we have $R_T(c) \in \mathcal{A}^2(T)(c')$. Since $|\mathcal{A}^2(T)(c')|/|\mathcal{A}(T)(c')|$ is small, the components of $\mathcal{A}(T)(c') - \{R_T(c)\}$ have almost the same length. If $J \ni f(c)$ is the landing domain to $\mathcal{A}(T)$ and if s is the landing time, then $f^s : J \to f^s(J)$ extends to a diffeomorphism onto $T(c')$ which implies by the Koebe principle that $f^s|J$ is almost linear. Thus the components of $J - \{f(c)\}$ have almost the same length. $\rhd$

Proof of Proposition 3. By Proposition 4, for any $C > 0$ there exists an admissible neighborhood T_0 of Ω with $\mathrm{Space}(T_0) > C$. Let ρ be a small constant. For $n \ge 0$, define inductively $T_{n+1} = \mathcal{A}(T_n)$. Then, since $\mathrm{C}_2(T_n) \supset \mathrm{C}_2(T_{n+1})$ there exists $N \le 2\#\Omega$ such that

$$\mathrm{C}_2(T_{N-1}) = \mathrm{C}_2(T_N) = \mathrm{C}_2(T_{N+1}).$$

By Lemmas 6 and 7, defining $I = T_N$ completes the proof. $\square$

The following proposition will also be needed in Section 4.

Proposition 5. *Let f be a real–analytic interval map with nondegenerate critical points, and let Ω be a connected component of the graph $\boldsymbol{Cr}(f)$ such that f is not infinitely renormalizable at any $c \in \Omega$. Let Ω_1 be the subset of Ω consisting of all points c such that $\omega(c) \ni c$ is minimal, and let $\Omega_2 = \Omega \setminus \Omega_1$. Then given a sufficiently small admissible neighborhood I of Ω_2 there exist a universal constant $C > 0$ and an arbitrarily small admissible neighborhood Y of Ω_1 with the following properties*:

- *each component of Y is a C-nice interval*;
- $R_Y\colon \mathbf{D}(Y) \to Y$ *extends to a complex box mapping*;
- $I \cup Y$ *is an admissible neighborhood of* Ω;
- *define* $\hat{\mathbf{D}}(Y)$ *to be the union of all return domains of Y which intersect* $\bigcup_{c\in\Omega} \mathrm{orb}(c)$, *which are disjoint from* $\mathbf{D}(Y)$ *and which return to Y before entering I, (so if J is a connected component of* $\hat{\mathbf{D}}(Y)$ *then* $f_0^n(J) \cap I = \emptyset$ *for $n = 0, \dots, s$, where s is the return time to Y of J). Then*

$$\inf \mathrm{Gap}(J_1, J_2) > 0,$$

where the infimum is taken over all distinct components J_1, J_2 of $\hat{\mathbf{D}}(Y)$.

Proof. As proved in [She04], for each $c \in \Omega_1$ there exists an arbitrarily small admissible neighborhood $K([c])$ of $[c]$ so that $R_{K([c])} : \mathbf{D}(K([c])) \to K([c])$ extends to a complex box mapping. Moreover, as the proof shows, we can choose $K([c])$ so that each component K' of $K([c])$ is C-nice, where $C > 0$ is a universal constant.

Let us write Ω_1 as a disjoint union of $[c_i]$'s. For each i, let K_i be a small admissible neighborhoods of $[c_i]$ as above (all taken from the same puzzle partition). Then $K = \bigcup_i K_i$ is an admissible neighborhood of $\bigcup_i [c_i]$ and $\mathbf{D}(K) = \bigcup_i \mathbf{D}(K_i)$. Consequently, there exists an arbitrarily small admissible neighborhood K of Ω_1 such that $R_K : \mathbf{D}(K) \to K$ extends to a complex box mapping and such that each component of K is a C-nice interval.

Now let us take a small admissible neighborhood P of Ω such that $P(c) = K(c)$ for all $c \in \Omega_1$ so that $\mathrm{Space}(P) \geq C$. This can be done as in the proof of Proposition 4. Provided that $|K|$ is small enough, we may choose P such that $P(c) \subset I$ for all $c \in \Omega_2$.

Arguing as above, we obtain a new admissible neighborhood $T \subset P$ of Ω such that $\mathrm{Space}(T)$ and $\mathrm{Gap}(T)$ are both bounded away from zero. Note that the components of $Y = \bigcup_{c\in\Omega_1} T(c)$ are obtained by pull–back of K if we use the first return map $R_K : \mathbf{D}(K) \to K$ (since iterates of points in Ω_1 never enter $P \setminus K$). So $R_Y : \mathbf{D}(Y) \to Y$ extends to a complex box mapping.

Let J_1 and J_2 be two distinct components of $\hat{\mathbf{D}}(Y)$. Since $T \setminus Y \subset I$, the first entry of J_i to Y is the same as the first entry to T, $i = 1, 2$. It follows that $\mathrm{Gap}(J_1, J_2)$ is bounded from below by a positive constant.

Finally let us assume that I is so small that the forward orbit of any $c \in \Omega_1$ does not enter I and show that $I \cup Y$ is nice. Otherwise, there exists a component U of $I \cup Y$ and $z \in \partial U$ such that $f^n(z) \in I \cup Y$ for some $n \geq 1$. If $U \subset I$, then $f^n(z) \in Y$ (since I is nice); thus U is properly contained in the domain of the first entry map to Y, which implies that T is not nice, a contradiction. Similarly, if $U \subset Y$, then U is contained in the domain of the first entry map to I, which implies that the forward orbit of $c \in U \cap \Omega_1$ enters I, contradicting the assumption on I. $\square$

The following theorem is the direct analogue of the Rigidity Theorem in [SKvS] for the box mappings defined in the previous section. The proof is the same.

THEOREM 4 (Rigidity Theorem for box mappings). *Let $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ be two combinatorially equivalent real-symmetric complex box mappings without neutral cycle or escaping critical point. Moreover, suppose that there exists a* q.c. *homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that h conjugates f and $\tilde{f}$ on the boundaries of their domains of definition.*

Then there exists a q.c. *homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ which conjugates f and $\tilde{f}$ on their domains and such that $\phi = h$ outside U.*

4. Instantaneous change of combinatorics in analytic families

In this section we shall use the two theorems from the previous section to prove that under certain conditions the only structurally stable maps within analytic families of analytic maps are hyperbolic maps. The main condition we put on such families is that all the maps in the family are regular (see the definition below). This condition was introduced in [Koz03] in a similar context. It seems conceivable that this condition is superfluous; however we do not know how to prove the theorem below without it.

Definition 2. A C^2 interval map $f : X \to X$ is *nondegenerate* if each critical point c of f is nondegenerate and contained in $\mathrm{int}(X)$. A nondegenerate interval map is called regular if each of its neutral periodic orbits contains a critical point with an infinite orbit in the interior of its attracting basin.

Definition 3. A critical point c of an interval map $f : X \to X$ is called *controlled* if either it is contained in the basin of a hyperbolic attracting cycle, or it is *precritical*, i.e., there exists $n \geq 1$ such that $f^n(c) \in \mathrm{Crit}(f)$.

We shall often consider a real analytic family $f_\lambda : X \to X$ of nondegenerate interval maps parametrized by a parameter λ from an open interval $\Lambda \ni 0$. Here by saying this family is *real analytic*, we mean (as usual) that the map $(\lambda, x) \mapsto f_\lambda(x)$ is real analytic. Note that each of the critical points of f_λ depends real-analytically on λ (in particular, the number of critical points of f_λ is the same for all λ). If c is a critical point of f_{λ_0} for some $\lambda_0 \in \Lambda$, then we use $c(\lambda)$ to denote the corresponding critical point of f_λ.

Let us say that such a family f_λ, $\lambda \in (-1, 1)$, is a *special* family passing through f_0, if the following hold:

1. f_0 has no neutral cycle;

2. Hyperbolic attracting cycles of f_0 do not bifurcate as $\lambda \in (-1, 1)$ varies;

3. If $f_0^k(c) = c'$ holds for some $k \geq 1$, and $c, c' \in \mathrm{Crit}(f_0)$, then $f_\lambda^k(c(\lambda)) = c'(\lambda)$ holds for all $\lambda \in (-1, 1)$;

4. If c is a critical point of f_0 such that $f_0^n(c)$ converges to a hyperbolic attracting cycle of f_0 as $n \to \infty$, then for all λ, $f_\lambda^n(c(\lambda))$ converges to a hyperbolic attracting cycle of f_λ;

5. For any $c \in \mathrm{Crit}(f_0)$ and any $n \geq 1$, $f_0^n(c)$ is not contained in the boundary of the immediate basin of a hyperbolic attracting cycle.

THEOREM 5. *Let f_λ, $\lambda \in (-1, 1)$, be a special family of nondegenerate real analytic interval maps. Assume that the map f_0 has a critical point c_0 which is not controlled such that the itinerary $\nu_{f_\lambda}(c_0(\lambda))$ is nonconstant as λ varies in $[0, 1)$ and that either of the following holds*:

1. *f_λ is regular for all $\lambda \in (-1, 1)$,*

2. *For any noncontrolled critical point c of f_0, $\omega_{f_0}(c) \ni c$ is minimal.*

Then there is an arbitrarily small $\lambda \in (0, 1)$ such that the number of controlled critical points of f_λ is greater than that for f_0.

In the unimodal case this theorem was proved previously in [Koz03]. The proof of the above theorem follows the same strategy, except that we need to deal with the possibility of more general types of critical relations (compared to the unimodal case). Moreover, we use a method of [LAdM03] to construct a holomorphic motion of the boundary of the box mappings (although one could also proceed as in [Koz03] or [LvS00]).

One can extend the above theorem to multi-parameter families easily (see [Koz03]).

Before proving the above theorem, we prove a simple proposition.

PROPOSITION 6. *Consider a special family f_λ, $\lambda \in (-1, 1)$ of nondegenerate interval maps. Let $\mathcal{C}$ be the set of critical points of f_0 which are contained in the basin of hyperbolic attracting cycles. Let $\lambda_0 \in (0, 1)$. Assume that for all $c \in \mathrm{Crit}(f_0) \setminus \mathcal{C}$, and all $\lambda \in [0, \lambda_0)$, the following hold*:

- $\nu_{f_\lambda}(c(\lambda)) = \nu_{f_0}(c)$,
- *$c(\lambda)$ is not contained in the basin of a hyperbolic attracting cycle.*

Then for all $c \notin \mathrm{Crit}(f_0) \setminus \mathcal{C}$, $\nu_{f_{\lambda_0}}(c(\lambda_0)) = \nu_{f_0}(c)$. Moreover, if c is a recurrent critical point of f_0 with a minimal ω-limit set, then $c(\lambda_0)$ is a recurrent critical point of f_{λ_0} with a minimal ω-limit set.

$\triangleleft$ Let $\mathcal{C}_1$ be the subset of $\mathrm{Crit}(f_0) \setminus \mathcal{C}$ consisting of all critical points for which the conclusion of the lemma does not hold. To prove the first statement of the proposition, we need to show that $\mathcal{C}_1 = \emptyset$.

Arguing by contradiction, assume that there exists $c \in \mathcal{C}_1$. Let $\nu_{f_0}(c) = \{i_k\}_{k=0}^{\infty}$ and $\nu_{f_{\lambda_0}}(c(\lambda_0)) = \{j_k\}_{k=0}^{\infty}$. By continuity it is easy to see that $i_k \neq j_k$ if and only if $f_{\lambda_0}^k(c(\lambda_0)) \in \mathrm{Crit}(f_{\lambda_0})$ and $f_0^k(c) \notin \mathrm{Crit}(f_0)$. Clearly the number of k's with $i_k \neq j_k$ is finite. Indeed, otherwise, the orbit $\{f_{\lambda_0}^k(c(\lambda_0))\}_{k=1}^{\infty}$ would hit $\mathrm{Crit}(f_{\lambda_0})$ infinitely many times, and so the map f_{λ_0} would have a super attractive critical periodic point and some iterate of $c(\lambda_0)$ would be mapped to this point by f_{λ_0}, which implies that for all λ sufficiently close to λ_0, $c(\lambda)$ would be in the basin of a hyperbolic attracting cycle, contradicting the assumption of the proposition. Therefore, there exists a maximal positive integer k such that $i_k \neq j_k$. Let $c'(\lambda_0) := f_{\lambda_0}^k(c(\lambda_0))$. Note that $c' \in \mathrm{Crit}(f_0) \setminus \mathcal{C}$.

By the maximality of k, $c' \notin \mathcal{C}_1$. Moreover,

$$\nu_{f_{\lambda_0}}(f_{\lambda_0}(c')) = \nu_{f_{\lambda_0}}(f_{\lambda_0}^{k+1}(c(\lambda_0)) = \nu_{f_0}(f_0^{k+1}(c)).$$

Thus, $\nu_{f_0}(f_0^{k+1}(c)) = \nu_{f_0}(f_0(c'))$, i.e., $[c', f_0^k(c)]$ is a homterval of f_0. Since f_0 has no wandering interval, it follows that $f_0^k(c)$ is contained in the closure of the immediate basin of a periodic attractor. By the definition of special family, this implies that c is contained in the basin of a hyperbolic attracting cycle of f_0, which contradicts the assumption of this proposition. This proves that $\mathcal{C}_1 = \emptyset$.

To prove the second statement, we observe that the property that a critical point is recurrent with a minimal ω-limit set can be characterized by the itinerary of the point. Since $\nu_{f_{\lambda_0}}(c(\lambda_0)) = \nu_{f_0}(c)$, the statement follows. $\triangleright$

Remark 4.1. A C^3 special family of interval maps is defined as in the real-analytic case except that we only require the maps to be C^3. The above proposition holds for C^3 families of interval maps.

The proof of Theorem 5 will follow from the next proposition.

PROPOSITION 7. *Let f_λ, $\lambda \in (-1,1)$ be a special family of real analytic nondegenerate interval maps. Assume that the number of controlled critical points of f_λ is the same for all $\lambda \in (-1,0]$. Then there exists $\varepsilon > 0$ such that for every critical point c of f_0 which is not contained in the basin of periodic attractors, and for all $\lambda \in [0,\varepsilon)$, we have $\nu_{f_\lambda}(c(\lambda)) = \nu_{f_0}(c)$.*

Proposition 7 *implies Theorem* 5. Let $\mathcal{C}$ denote the set of critical points of f_0 which are contained in the basin of hyperbolic attracting cycles, and let $\mathcal{C}' = \mathrm{Crit}(f_0) \setminus \mathcal{C}$.

Suppose that the assertion of the theorem does not hold. Then there exists a maximal $\lambda_0 > 0$ such that the number of controlled critical points of

f_λ is constant for all $\lambda \in [0, \lambda_0)$. Then for all $c \in \mathcal{C}'$, $\nu_{f_\lambda}(c_i(\lambda))$ is the same for all $\lambda \in [0, \lambda_0)$. By Proposition 6 we obtain that

$$\nu_{f_{\lambda_0}}(c(\lambda_0)) = \nu_{f_0}(c) \text{ for all } c \in \mathcal{C}'.$$

Claim. If f_{λ_0} is regular, then f_{λ_0} has no neutral cycle.

Arguing by contradiction, assume that that f_{λ_0} has a neutral periodic point p. Since f_{λ_0} is regular, there exists a non pre-periodic critical point c of f_{λ_0} in the interior of the attracting basin of the orbit of p. For all $\lambda \in (-1, 1)$, let $c(\lambda)$ denote the corresponding critical point of f_λ. Note that $c(0) \in \mathcal{C}'$; otherwise $c(\lambda_0)$ would be contained in the basin of a hyperbolic attracting cycle of f_{λ_0}. For all $\lambda \in [0, \lambda_0]$, $\nu_{f_\lambda}(c(\lambda)) = \nu_{f_{\lambda_0}}(c)$ is pre-periodic. Then there exists $k \geq 0$ and $n \in \mathbb{N}$ such that $\nu_{f_\lambda}(f^k(c(\lambda)))$ has period n for all $\lambda \in [0, \lambda_0]$. Therefore, either

$$f_\lambda^k(c(\lambda)) = f_\lambda^{k+n}(c(\lambda)), \tag{7}$$

or $c(\lambda)$ is contained in the basin of a periodic attractor of f_λ which has period $\leq 2n$. As both sides of (7) are real-analytic in λ and the equation is not satisfied for $\lambda = \lambda_0$, we conclude that (7) has only isolated roots. So there exists $\lambda_1 > 0$ such that (7) is not satisfied for all $\lambda \in (0, \lambda_1)$. Moreover, by an easy continuity argument, the assumption that f_0 has no neutral cycle implies that there exists $\lambda_2 > 0$ such that f_λ has no neutral cycle of period $\leq 2n$ for all $\lambda \in [0, \lambda_2)$. Therefore, for a small positive value of λ, $c(\lambda)$ is contained in the basin of a hyperbolic attracting cycle of f_λ. Since $c(0) \in \mathcal{C}'$, f_λ has more controlled critical points than f_0, a contradiction. This completes the proof of the claim.

The map f_{λ_0} satisfies the same assumptions as the map f_0. We rename f_{λ_0} by f_0. So for small negative values of λ, $\nu_{f_\lambda}(c_i(\lambda)) = \nu_{f_0}(c_i(0))$ for all $i \in \mathcal{I}_1$. Moreover, by assumption, there exists an arbitrarily small positive value of λ such that $\nu_{f_\lambda}(c(\lambda)) \neq \nu_{f_0}(c)$ for some $c \in \mathcal{C}'$. This contradicts Proposition 7. $\square$

4.1. *Holomorphic families of complex box mappings.* The proof of Proposition 7 involves holomorphic families of complex box mappings.

Let $\Lambda \subset \mathbb{C}$ be a topological disk, and $\lambda_0 \in \Lambda$. A *holomorphic motion* of a set $Z \in \mathbb{C}$ based on (Λ, λ_0) is a family of injections $H_\lambda : Z \to Z_\lambda \subset \mathbb{C}$ such that H_{λ_0} is the identity map and such that for any $z \in Z$, $\lambda \mapsto H_\lambda(z)$ is a holomorphic map. By Slodkowski's theorem (the λ-lemma) [Slo91], such a holomorphic motion can be extended to be a holomorphic motion of the whole complex plane based on the same Λ, and $H_\lambda : \mathbb{C} \to \mathbb{C}$ is a $K(r)$-qc map, where r is the hyperbolic distance between λ_0 and λ, and moreover, $K(r) \to 1$ as $r \to 0$.

We say that $F_\lambda : U_\lambda \to V_\lambda$, $\lambda \in \Lambda$ is a *holomorphic family of a complex box mapping based on* (Λ, λ_0) if the following hold:

- For each $\lambda \in \Lambda$, F_λ is a complex box mapping (with nondegenerate critical points);
- There exists a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ based on (Λ, λ_0) such that $H_\lambda(U_{\lambda_0}) = U_\lambda$, $H_\lambda(V_{\lambda_0}) = V_\lambda$ and $H_\lambda \circ F_{\lambda_0} = F_\lambda \circ H_\lambda$ holds on ∂U_{λ_0}.

We say that the family $F_\lambda, \lambda \in \Lambda$, is *real-symmetric* if Λ is symmetric with respect to the real line, $\lambda_0 \in \mathbb{R}$, and for all $\lambda \in \Lambda \cap \mathbb{R}$, F_λ and H_λ are real-symmetric. Each holomorphic family of complex box mappings appearing below is real-symmetric.

The following is a consequence of the Rigidity Theorem 4.

THEOREM 6. *Let $F_\lambda : U_\lambda \to V_\lambda$ be a real-symmetric holomorphic family of complex box mappings based on $(\mathbb{D}, 0)$. Suppose that*

- *F_0 has no escaping critical point*;
- *The map F_0 has no neutral or attracting cycles*;
- *There is a critical point c_0 of F_0 and its itinerary is not constant for all $\lambda \in (-1, 1)$*;
- *F_0 carries no invariant line field on its filled Julia set.*

Then there exist a critical point c and arbitrarily small $\lambda \in (0, 1)$ such that the itineraries $\nu_{F_0}(c)$ and $\nu_{F_\lambda}(c(\lambda))$ are different.

Proof. Let H_λ denote the holomorphic motion associated with F_λ and denote the Beltrami coefficient of H_λ by μ_λ. Define $\hat{\nu}_\lambda$ to be μ_λ outside of U_0 and zero on the filled Julia set of the map F_0, and everywhere else define it as the pullback of μ_λ by F_0. Obviously, μ_λ depends on λ holomorphically. By the Measurable Riemann Mapping Theorem [Ahl87], there exists a holomorphic motion $\tilde{H}_\lambda \colon \mathbb{C} \to \mathbb{C}$ such that $\tilde{H}_\lambda(x) = H_\lambda(x)$ for $x \in \{-1, 1\}$ and such that the Beltrami coefficient of H_λ is $\hat{\nu}_\lambda$. Since the map F_0 preserves the Beltrami coefficient $\hat{\nu}_\lambda$, the map

$$\tilde{F}_\lambda = \tilde{H}_\lambda \circ F_0 \circ \tilde{H}_\lambda^{-1} : \tilde{H}_\lambda(U) \to \tilde{H}_\lambda(V)$$

is a complex box mapping. It also depends holomorphically on λ. In order to complete the proof of Theorem 6 we will need the following lemma.

LEMMA 8. *Take $\lambda \in (-1, 1)$. Then the maps F_0 and F_λ are combinatorially equivalent if and only if $F_\lambda = \tilde{F}_\lambda$.*

$\triangleleft$ The "if" part is obvious. So assume that F_0 and F_λ are combinatorially equivalent. By the Rigidity Theorem for box mappings, there exists a q.c. homeomorphism $\phi_\lambda \colon \mathbb{C} \to \mathbb{C}$ which conjugates F_0 and F_λ and coincides with H_λ on $\mathbb{C} \setminus U_0$. The Beltrami coefficient of ϕ_λ is equal to $\hat{\nu}_\lambda$. Indeed, this is clear outside of the Julia set of $F_0 \colon U_0 \to V_0$, and follows from the absence of invariant line fields on the Julia set. Thus, $\phi_\lambda = H_\lambda$ and $F_\lambda = \tilde{F}_\lambda$. $\triangleright$

Let $\hat{\Lambda} = \{\lambda \in (-1,1) : \nu_{F_0}(c) = \nu_{F_\lambda}(c(\lambda))$ for all $c \in \mathrm{Crit}(F_0)\}$. The above lemma implies that $F_\lambda = \tilde{F}_\lambda$ for $\lambda \in \hat{\Lambda}$. If Λ has an accumulation point then by analytic continuation $F_\lambda = \tilde{F}_\lambda$ for all $\lambda \in \mathbb{D}$. Since this contradicts the third assumption of Theorem 6, it follows that $\hat{\Lambda}$ has no accumulation point in $(-1,1)$. This completes the proof of Theorem 6. $\square$

The proof actually gives the following version of the theorem, which is sometimes more convenient.

THEOREM 7. *Let $F_\lambda : U_\lambda \to V_\lambda$ be a real-symmetric holomorphic family of a complex box mapping based on $\mathbb{D}$. Assume that*

- *F_0 has no escaping critical point*;
- *The map F_0 has no neutral or attracting cycles*;
- *F_0 carries no invariant line field on its filled Julia set*;
- *The set $\{\lambda \in (-1,1) : F_\lambda$ is combinatorially equivalent to $F_0\}$ has an accumulation point in $(-1,1)$.*

Then there exists a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ which conjugates F_0 to F_λ for all $\lambda \in \mathbb{D}$.

Given a special family f_λ, $\lambda \in (-1,1)$ of interval maps, and a complex box mapping $F : U \to V$ induced by f_0, we say that this complex box mapping *persists in a neighborhood* Λ *of* 0 in $\mathbb{C}$, if for all $\lambda \in \Lambda$, there exists a complex box mapping $F_\lambda : U_\lambda \to V_\lambda$ induced by f_λ (for $\lambda \in \Lambda \cap (-1,1)$) such that $F_\lambda, \lambda \in \Lambda$, forms a holomorphic family.

The following lemma will be used several times to prove the persistence of certain complex box mappings.

LEMMA 9. *Consider a real-analytic family f_λ of interval maps. Let c be a critical point of f_0 which is not contained in the basin of periodic attractors, let U be a neighborhood of* $\mathrm{Crit}(f_0)$ *and the periodic attractors of f_0 and let $\theta_0 \in (0,\pi)$ be a constant. Let $I \ni c$ be a small nice interval. Let J_i, $i = 1, 2, \ldots$ be the domains of the first entry map to I with the property that $f_0^j(J_i) \cap U = \emptyset$ for all $0 \le j < s_i$, where s_i is denoted the first entry time of J_i. Let $V = D_{\theta_0}(I)$ and let U_i be the components of $f_0^{-s_i} V$ containing J_i. Then given* *any holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ based on Λ, there exists a neighborhood $\Lambda' \subset \Lambda$ of 0 in $\mathbb{C}$ and for each i there exists a holomorphic motion $H_{\lambda,i} : \mathbb{C} \to \mathbb{C}$ based on Λ' such that for all $\lambda \in \Lambda'$, $f_\lambda^{s_i} \circ H_{\lambda,i} = H_\lambda \circ f_0^{s_i}$ holds on U_i.*

Proof. Let Q be the set of all points $x \in X$ with the property that $f_0^k(x) \notin U$ for all $k \geq 0$. By [Man93], Q is a hyperbolic set, which persists under small perturbation. So, there exists a neighborhood $\Lambda_0 \subset \mathbb{C}$ of zero, a neighborhood $Z \subset \mathbb{C}$ of Q with $f_0(Z) \supset Z$, and a holomorphic motion $H_{\lambda,0} : \mathbb{C} \to \mathbb{C}$, $\lambda \in \Lambda_0$, such that $H_{\lambda,0} \circ f_0 = f_\lambda \circ H_{\lambda,0}$ on Z. Let $Z' \Subset Z$ be a smaller neighborhood of Q.

By Lemma 1, $f_0^j U_i \subset D_{\theta_0/2}(f_0^j J_i)$ for all $0 \leq j \leq s_i$. There exists a (large) integer N such that if $s_i \geq N$, then for all $0 \leq j \leq s_i - N$, $f_0^j(J_i)$ is contained in the small neighborhood Z' of Q. Let $\mathcal{I}$ be the set of all i's with $s_i \leq N$. Note that $\mathcal{I}$ has only finitely many elements, so we can define desired holomorphic motions $H_{\lambda,i}$ based on some open neighborhood Λ_1 of 0 for all $i \in \mathcal{I}$. Now take $i \notin \mathcal{I}$, and let $k(j)$ be such that $f_0^{s_i - j}(J_i) = J_{k(j)}$ for all $N \leq j \leq s_i$. Then $k(N) \in \mathcal{I}$ so that $H_{\lambda,k(N)}$ is well-defined for $\lambda \in \Lambda_1$. Since $U_{k(N)} \subset Z'$, by shrinking Λ_1 if necessary, we may assume that $H_{\lambda,k(N)}(U_{k(N)}) \subset H_{\lambda,0}(Z)$. Now for $\lambda \in \Lambda_0 \cap \Lambda_1$, define $H_{\lambda,k(N)+1} = f_\lambda^{-1} \circ H_{\lambda,k(N)} \circ f_0$, which gives us a well-defined holomorphic motion of $U_{k(N)+1}$ satisfying the required property. Moreover, $H_{\lambda,k(N)+1}(U_{k(N)+1}) \subset H_{\lambda,0}(Z)$, so that we may repeat the same procedure to define the desired holomorphic motions $H_{\lambda,k(j)}$ for all $N < j \leq s_i$. This completes the proof. $\square$

We shall also need the following lemma.

LEMMA 10. *For any $M > m > 0$ there exists $r \in (0,1)$ with the following property. Let $Q \subset \mathbb{C}$ be a closed set consisting of at least two points and let $d_{\mathbb{C}\setminus Q}$ denote the hyperbolic metric on $\mathbb{C} \setminus Q$. Let H_λ and H'_λ be holomorphic motions based on the unit disk such that $H_\lambda|Q = H'_\lambda|Q$. If Z, Z' are disjoint subsets of $\mathbb{C} \setminus Q$ so that $d_{\mathbb{C}\setminus Q}(Z, Z') > M$, then for all $\lambda \in \mathbb{D}_r$,*

$$d_{\mathbb{C}\setminus Q}(H_\lambda(Z), H_\lambda(Z')) > m.$$

In particular, $H_\lambda(Z) \cap H'_\lambda(Z') = \emptyset$.

The proof of this lemma uses the following fact:

Fact 2 (Lemma 2.3 in [LAdM03]). For any $M > m > 0$ there exists $\delta > 0$ with the following property. Let $S, \tilde{S} \subset \mathbb{C}$ be two hyperbolic Riemann surfaces and $h_1, h_2 : S \to \tilde{S}$ be $(1+\delta)$-q.c. maps homotopic rel boundary. Let Z and Z' be subsets of S. If $d_S(Z, Z') > M$, then $d_{\tilde{S}}(h_1(Z), h_2(Z')) > m$.

Proof of Lemma 10. Let δ be as in Fact 2. By the λ-lemma, there exists $r \in (0,1)$ such that the maximal dilation of H_λ and H'_λ are both bounded from above by $1+\delta$. Applying Fact 2 to $S = \mathbb{C} \setminus Q$ and $\tilde{S} = \mathbb{C} \setminus H_\lambda(Q)$ gives us the result. □

4.2. *Proof of Proposition* 7. Now let us consider a special family f_λ, $\lambda \in (-1, 1)$ of real analytic nondegenerate interval maps, with the property that for each $c \in \mathrm{Crit}(f)$ which is not in the basin of periodic attractors, we have for every $\lambda \in (-1, 0)$,

$$\nu_{f_\lambda}(c(\lambda)) = \nu_{f_0}(c). \tag{8}$$

Our goal is to prove that (8) holds for small positive values of λ.

First let us assume that c is a recurrent critical point of f_0 and that $\omega_{f_0}(c)$ is minimal. By Theorem 3, we can find a small admissible neighborhood I of $[c]$ such that $R_I\colon \mathbf{D}(I) \to I$ extends to a complex box mapping F which carries no invariant line filled on its Julia set. As this box mapping has only finitely many branches, there exists a neighborhood $\Lambda_c \subset \mathbb{C}$ of 0 such that we can find a holomorphic family of complex box mappings F_λ induced by f_λ, $\lambda \in \Lambda_c$, such that $F_0 = F$. By Theorem 7, there exists a holomorphic motion $H_\lambda\colon \mathbb{C} \to \mathbb{C}$ conjugating F to F_λ, for all $\lambda \in \Lambda$. In particular, for $\lambda \in \Lambda \cap \mathbb{R}$, F_λ is qc conjugate to F_0, from which it follows that (8) holds for $\lambda \in \Lambda_c \cap \mathbb{R}$.

If f_0 is infinitely renormalizable at c and $c' \in \mathrm{Back}(c)$, then we can take I to be a union of properly periodic intervals so that $\mathbf{D}(I) = I$. Letting s be the entry time of c' into I, we have that $f_0^s(c')$ is contained in the filled Julia set of F_0. As $H_\lambda(f_0^s(c')) = f_\lambda^s(c')$ holds for negative values of λ, we conclude that this equation holds for all λ; hence (8) holds for c' and small positive values of λ.

Let Ω be a maximal block of critical points, i.e., a connected component of the graph $\mathbf{Cr}(f)$. If each critical points in Ω has a finite orbit, then by analytic continuation it is easy to see that (8) holds for all $c \in \Omega$. If Ω contains a point c at which f is infinitely renormalizable, then the argument above shows that (8) holds for all $c \in \Omega$. So let us assume that Ω is a nontrivial block and f is not infinitely renormalizable at any $c \in \Omega$. Let Ω_1 denote the subset of Ω consisting of all recurrent critical points with minimal ω-limit set, and let $\Omega_2 = \Omega \setminus \Omega_1$. Using the holomorphic family of box mappings constructed above, we get again by Theorem 7 that (8) holds for all $c \in \Omega_1$. Let us show that it also holds for all $c \in \Omega_2$.

To deal with the critical points in Ω_2, we use a similar strategy, but the argument is more complicated for two reasons. Firstly, we have to consider complex box mappings with infinitely many branches, and secondly the complex box mappings we are able to construct do not contain all noncontrolled critical points in its filled Julia set.

Let us first choose a small admissible neighborhood I of Ω_2 such that $R_I\colon \mathbf{D}(I) \to I$ extends to a complex box mapping

$$F\colon U \to V \text{ with } V \supset I \supset \Omega_2$$

satisfying the properties specified in Theorem 3. We will assume that I is so small that the ω-limit sets of critical points in Ω_1 are disjoint from $\overline{I}$.

Next we choose an admissible neighborhood Y of Ω_1 according to Proposition 5. In particular, $R_Y\colon\ \mathbf{D}(Y) \to Y$ extends to a complex box mapping

$$G\colon A \to B \text{ with } B \supset Y \supset \Omega_1.$$

Let Y be so small that the iterates of components of $\mathbf{D}(Y)$ do not intersect I before returning to Y. Let $\alpha \in (0, \pi)$ be a constant close to π so that $B \supset \hat{B} := \bigcup_{c\in\Omega_1} D_\alpha(Y(c))$. (That α is close to π means that the components of $\hat{B}$ are close to the real line.) Let $\hat{\mathbf{D}} = \hat{\mathbf{D}}(Y)$ be as in that proposition, i.e., this is the union of all first return domains J of Y which enter Y before entering I (i.e. $f_0^n(J) \cap I = \emptyset$ for $n = 0, \ldots, s$, where s is the return time to Y of J), for which $J \cap \mathbf{D}(Y) = \emptyset$ and which intersect $\cup_{c\in\Omega}\mathrm{orb}(c)$. Note that the intervals J are subsets of Y, return to Y but do not contain iterates of any $c \in \Omega_1$ (see the definition of $\mathbf{D}(Y)$ in Section 2). Shrinking Y further we get that for each return domain J of Y, either J is a connected component of Y or $\mathrm{Space}(Y, J)$ is greater than a universal constant $\rho > 0$. By Proposition 2, $R_Y\colon \hat{\mathbf{D}} \to Y$ extends to a complex box mapping

$$\hat{G}\colon \hat{A} \to \hat{B} \text{ with } B \supset \hat{B} \supset Y \supset \Omega_1.$$

For each n, let A_n denote the union of the components of the domain of G^n which intersect the real line.

CLAIM. *The Julia set of $G\colon A \to B$ is a Cantor set. In other words, the maximal diameter of the components of A_n shrinks to zero as $n \to \infty$.*

Proof. From the definition of Ω_1 and $\mathbf{D}(Y)$, A (and therefore $G^{-n}(B)$) has finitely many components. As in Proposition 5 there exists a sequence of admissible neighborhoods $Y(k)$ of Ω_1 consisting of pullbacks of $Y = B \cap \mathbb{R}$, so that each component of $Y(k)$ is C-nice. Now, $R_{Y(k)}\colon\ \mathbf{D}(Y(k)) \to Y(k)$ extends to a complex box mapping $G(k)\colon A(k) \to B(k)$ with $B(k) \cap \mathbb{R} = Y(k)$ and so that the diameter of each component of $B(k)$ tends to zero as k tends to zero. Each component of $B(k)$ agrees on the real line with a component $B_i'(k)$ of $G^{-n}(B)$ for some n. Let $B'(k)$ be the union of such components $B_i'(k)$ and consider the first return map $R_{B'(k)}\colon A'(k) \to B'(k)$ to $B'(k)$ of $G\colon A \to B$. By Proposition 2.3 in [LvS98] there exists N so that $R_{B'(k)}^{-N}(B'(k)) \subset A(k)$. Then arguing as in the proof of Proposition 3.1 in [LvS98], and using the fact that components of $Y(k)$ are C-nice, we get that the maximal diameter of puzzle-pieces containing points z which eventually enter critical puzzles of G of every level, is small. By hyperbolicity, the remaining puzzle-pieces also tend to zero in diameter, completing the proof of the claim.

From this claim it follows that we can choose N so large that $A_N \Subset \hat{B}$. Let $\tilde{Y} \subset \cup_{c\in\Omega_1} A_N(c)$ be a small neighborhood of Ω_1 satisfying the following property: if U' is an iterate of a component of U such that $U' \cap \tilde{Y} \neq \emptyset$, then $U' \Subset A_N$ and, moreover, the Euclidian distance between the boundaries of U' and A_N is greater than some constant independent of U'. Such a neighborhood $\tilde{Y}$ exists because the iterates of critical points of Ω_1 never enter I and because the diameter of iterates of components of U are commensurable with their real traces.

Fix a small neighborhood $T \subset \mathbb{R}$ of the critical points $\mathrm{Crit}(f_0) \setminus \Omega$ and the attracting cycles, so that the orbits of points in Ω never enter T. Let Q be the set of all real points whose forward orbit never enters $T \cup I \cup \tilde{Y}$. As in the proof of Lemma 9, Q is hyperbolic and there exists a holomorphic motion $H_{\lambda,0} : \mathbb{C} \to \mathbb{C}$ based on a neighborhood Λ_0 of 0 in $\mathbb{C}$ such that $f_\lambda \circ H_{\lambda,0} = H_{\lambda,0} \circ f_0$ holds on Q.

Statement 1. *Let $c \in \Omega$. If $f_0^n(c) \in Q$ for some $n \geq 0$, then $f_\lambda^n(c(\lambda)) \in Q_\lambda$ for all $\lambda \in \Lambda_0$.*

$\lhd$ In fact, for all $\lambda \in (-1, 0] \cap \Lambda_0$ we have $\nu_{f_\lambda}(c(\lambda)) = \nu_{f_0}(c)$ and therefore

$$f_\lambda^n(c(\lambda)) = H_{\lambda,0}(f_0^n(c)).$$

Since both sides of the last equation are real-analytic in λ, it actually holds for all $\lambda \in \Lambda_0$, which implies the statement. $\rhd$

By the argument above, the complex box mapping G persists in a neighborhood $\Lambda_1 \subset \Lambda_0$ with a holomorphic motion $H_{\lambda,1}$ such that $H_{\lambda,1}|Q = H_{\lambda,0}$. Moreover, because (8) holds for all $\lambda \in (-1, 0)$, Theorem 7 implies that we may choose the holomorphic motion appropriately so that $H_{\lambda,1}$ conjugates G_0 to G_λ for all $\lambda \in \Lambda_1$. In particular, $\bigcup_{c \in \Omega_1} \{f_\lambda^n(c(\lambda))\}_{n=0}^\infty$ moves holomorphically with respect to λ.

Statement 2. *There exists a neighborhood $\Lambda_2 \subset \Lambda_1$ of 0 in $\mathbb{C}$ such that the complex box mapping $\hat{G}$ persists in Λ_2. Moreover, there exists a holomorphic motion $H_{\lambda,2} : \mathbb{C} \to \mathbb{C}$ based on Λ_2 such that $H_{\lambda,2}|Q = H_{\lambda,0}|Q$ and $H_{\lambda,2} \circ \hat{G} = \hat{G}_\lambda \circ H_{\lambda,2}$ hold on $\hat{A}$.*

$\lhd$ By Lemma 9, there exists a neighborhood Λ_2' of 0 in $\mathbb{C}$ such that for every component W of $\hat{A}$, there exists a holomorphic motion $H_{\lambda,W}$ based on Λ_2' such that $f_\lambda^{s(W)} \circ H_{\lambda,W} = H_{\lambda,0} \circ f_0^{s(W)}$ holds on W, where $s(W)$ is the positive integer such that $\hat{G}|W = f_0^{s(W)}$. Clearly, $H_{\lambda,W}|W \cap Q = H_{\lambda,0}|W \cap Q$.

Note that the endpoints of each component of $\hat{\mathbf{D}}$ belong to the set Q. So by Proposition 2, for each component W of $\hat{A}$ one has $W \subset D_{\alpha'}(W \cap \mathbb{R})$ for a constant α' close to α and therefore there exists $C > 0$ independent of W. We have $d_{\mathbb{C} \setminus Q}(\partial W, \partial \hat{B}) > C$ (where $d_{\mathbb{C} \setminus Q}$ denotes the hyperbolic metric on $\mathbb{C} \setminus Q$). By Lemma 10, there exists an open neighborhood $\Lambda_2 \subset \Lambda_2'$ of 0, such that for all $\lambda \in \Lambda_2$, $H_{\lambda,W}(W) \Subset H_{\lambda,0}(\hat{B})$.

Similarly, one proves that for any two distinct components W and W' of $\hat{A}$, we have $H_{\lambda,W}(W) \cap H_{\lambda,W'}(W') = \emptyset$ for all $\lambda \in \Lambda_2$.

Thus, due to the λ-lemma, we can define a holomorphic motion $\hat{H}_\lambda : \mathbb{C} \to \mathbb{C}$ such that $\hat{H}_\lambda|W = H_{\lambda,W}|W$ for each component W of $\hat{A}$, such that $\hat{H}_\lambda = H_{\lambda,0}$ holds on $Q \cup \partial\hat{B}$.

Let $\hat{A}_\lambda := \hat{H}_\lambda(\hat{A})$ and $\hat{B}_\lambda := \hat{H}_\lambda(\hat{B})$. Define $G_\lambda : \hat{A}_\lambda \to \hat{B}_\lambda$ to be such that for each component W of $\hat{A}$, we have $\hat{G}_\lambda|\hat{H}_\lambda(W) = f_\lambda^{s(W)}$. Then $\hat{G}_\lambda$ is a holomorphic family of complex box mappings without critical points. By Theorem 7, we can find a new holomorphic motion $H_{\lambda,2}$ conjugating $\hat{G}$ to $\hat{G}_\lambda$. $\triangleright$

Define

$$\hat{\mathbf{D}}' = \{x \in \mathbf{D}(Y) : \exists k \geq 1, R_Y(x), \ldots, R_Y^{k-1}(x) \in \mathbf{D}(Y), R_Y^k(x) \in \hat{\mathbf{D}}\} \cup \hat{\mathbf{D}}.$$

Given a component J of $\hat{\mathbf{D}}'$, let $k \geq 0$ be minimal such that $R_Y^k(J) \subset \hat{\mathbf{D}}$ and let K be the component of $\hat{\mathbf{D}}$ which contains $R_Y^k(J)$. Since $\hat{\mathbf{D}}$ is disjoint from the postcritical set of G, G^k maps a Jordan disk containing J conformally onto the component of $\hat{A}$ which contains K. Define $\hat{A}'$ to be the union of all Jordan disks obtained in this way.

STATEMENT 3. *There exists a neighborhood $\Lambda_3 \subset \Lambda_2$ of 0 in $\mathbb{C}$, such that*

$$H_{\lambda,i}(\hat{A}') \subset H_{\lambda,j}(\hat{B}) \tag{9}$$

holds for all $\lambda \in \Lambda_3$ and $i, j \in \{1, 2\}$.

$\triangleleft$ Let J be a component of $\hat{\mathbf{D}}'$ and W be the corresponding component of $\hat{A}'$. Consider two cases. First let J be a component of $\hat{\mathbf{D}}$ as well. We know that J cannot be a connected component of Y because then such a return domain would belong to $\mathbf{D}(Y)$. Therefore, $d_{\mathbb{C}\setminus Q}(\partial W, \partial\hat{B})$ is greater than some universal constant and the result of the lemma follows from Lemma 10.

Now let $J \subset \mathbf{D}(Y)$ and $\tilde{J}$ be the return domain of Y containing J. Then J is well-inside $\tilde{J}$. We know that $W \subset D_{\alpha'}(J)$ for a constant α' close to α, which implies that $d_{\mathbb{C}\setminus\partial\tilde{J}}(\partial W, \partial\hat{B})$ is bounded from below by a positive constant M, where $d_{\mathbb{C}\setminus\partial\tilde{J}}$ denotes the hyperbolic metric in $\mathbb{C} \setminus \partial\tilde{J}$. Since $\partial\tilde{J} \subset Q$, $d_{\mathbb{C}\setminus Q}(\partial W, \partial\hat{B}) > M$. By Lemma 10, the statement follows. $\triangleright$

Since there are finitely many critical points in Ω_1, by using continuity we can prove

STATEMENT 4. *There exists a neighborhood $\Lambda_4 \subset \Lambda_3$ of 0 in $\mathbb{C}$ such that for all $\lambda \in \Lambda_4$ and any $c \in \Omega_1$,*

$$H_{\lambda,1}(A_N(c)) \subset H_{\lambda,1}(\hat{B}) \cap H_{\lambda,2}(\hat{B}). \tag{10}$$

Let $\mathcal{J}$ denote the collection of all components of $\mathrm{Dom}'(I)$. For each $J \in \mathcal{J}$, let $s = s(J) \geq 0$ be the landing time of J into I, i.e., the minimal nonnegative integer such that $f_0^{s(J)}(J) \subset I$. Let $W = W(J)$ be the component of $f_0^{-s(J)}(V)$ which contains J. Note that $f_0^s|W$ is a conformal map onto a component of V and $W(J) \subset D_{\theta_1}(J)$.

Statement 5. *There exists a neighborhood $\Lambda_5 \subset \Lambda_4$ of 0 in $\mathbb{C}$ such that for each component J of $\mathrm{Dom}'(I)$, we can find a holomorphic motion $H_{\lambda,J} : \mathbb{C} \to \mathbb{C}$ such that for $\lambda \in \Lambda_5$,*

$$H_{\lambda,0} \circ f_0^{s(J)} = f_\lambda^{s(J)} \circ H_{\lambda,J} \text{ holds on } W$$

and such that $H_{\lambda,J}|Q = H_{\lambda,0}|Q$, where $s(J), W = W(J)$ are as above.

$\lhd$ Let $\mathcal{J}_1$ denote the subset of $\mathcal{J}$ consisting of all J's such that

$$f^j(J) \cap \tilde{Y} = \emptyset$$

for all $0 \leq j \leq s(J)$. By Lemma 9, we can find a desired holomorphic motion $H_{\lambda,J}$ for all $J \in \mathcal{J}_1$ which are based on a common neighborhood Λ_5' of 0.

Now assume that $J \notin \mathcal{J}_1$. Write $J_j = f^j(J)$, $W_j = f^j(W)$. Let $s < s(J)$ be maximal such that $J_s \cap \mathrm{Comp}_c(\tilde{Y}) \neq \emptyset$ for some $c \in \Omega_1$. From the definition of $\tilde{Y}$ we have $W_s \subset A_N(c)$. Since $J_{s+1} \in \mathcal{J}_1$, we have a desired holomorphic motion $H_{\lambda,W_{s+1}}$ defined on Λ_5'. Moreover, by shrinking Λ_5' we may assume that $W_{s+1,\lambda} := H_{\lambda,W_{s+1}}(W_{s+1})$ is disjoint from $\mathrm{orb}_{f_\lambda}(c'(\lambda))$ for any $c' \in \Omega_1$. Therefore, a desired holomorphic motion H_{λ,W_s} is defined on Λ_5'. Since $d_{\mathbb{C}\setminus Q}(\partial W_s, \partial A_N)$ is universally bounded away from zero, according to Lemma 10, by shrinking Λ_5' once more, we may assume that

$$W_{s,\lambda} := H_{\lambda,W_s}(W_s) \subset H_{\lambda,1}(\hat{B}) \cap H_{\lambda,2}(\hat{B}).$$

Let $s_1 < s_2 < \cdots < s_n = s$ be all the integers such that $J_{s_j} \subset Y$, and let $m \leq n$ be minimal such that $J_{s_j} \subset \mathbf{D}(Y)$ for all $m \leq j \leq n$. Let P be the component of $G^{-(n-m)}(A_N)$ which contains W_{s_m} and $P' = G^{n-m}(P)$. Then $G_\lambda^{n-m} : H_{\lambda,1}(P) \to H_{\lambda,1}(P')$ is a branched covering which is conjugate to G_0^{n-m} via $H_{\lambda,1}$. Note that $W_{s,\lambda} := H_{\lambda,W_s}(W_s)$ is disjoint from the postcritical set of G_λ. Now, we can define a desired holomorphic motion $H_{\lambda,W_{s_m}}$ based on Λ_5'. Moreover,

$$W_{s_m,\lambda} := H_{\lambda,W_{s_m}}(W_{s_m}) \subset H_{\lambda,1}(P) \subset H_{\lambda,1}(A_N) \subset H_{\lambda,1}(\hat{B}) \cap H_{\lambda,2}(\hat{B}).$$

Next define a holomorphic motion $H_{\lambda,W_{s_{m-1}}}$ based on Λ_5' using the family $\hat{G}_\lambda$ and the holomorphic motion $H_{\lambda,2}$. We have $W_{s_{m-1},\lambda} := H_{\lambda,W_{s_{m-1}}}(W_{s_{m-1}}) \subset H_{\lambda,2}(\hat{A}) \subset H_{\lambda,2}(\hat{A}') \subset H_{\lambda,1}(\hat{B}) \cap H_{\lambda,2}(\hat{B})$.

Let us show by induction that $W_{s_k} \subset \hat{A}'$ for $k = 1, \ldots, m-1$. We have already seen that $W_{s_{m-1}} \subset \hat{A} \subset \hat{A}'$. Assume that $W_{s_n} \subset \hat{A}'$ for $n = k, \ldots, m-1$.

If $W_{s_k} = \hat{G}(W_{s_{k-1}})$, then since $\hat{A}' \subset \hat{B}$ it follows that $W_{s_{k-1}} \subset \hat{A} \subset \hat{A}'$. The other case is $W_{s_k} = G(W_{s_{k-1}})$. Let $l \geq k$ be minimal such that $W_{s_l} \subset \hat{A}$. Such l exists because $W_{s_{m-1}} \subset \hat{A}$. Then $G^{l-k+1}(W_{s_{k-1}}) = W_{s_l}$ and the map $G^{l-k+1}|_{W_{s_{k-1}} \cap \mathbb{R}}$ is a restriction of a branch of the first entry map to $\hat{\mathbf{D}}$. This implies that $W_{s_{k-1}} \subset \hat{A}'$.

By further pull-back, using either G_λ or $\hat{G}_\lambda$ and using Statement (9), we define a desired holomorphic motion $H_{\lambda, W_{s_k}}$ based on Λ_5' for all $k = 1, 2, \cdots, m$.

Finally applying Lemma 9 once again (to landing domains of Y), we see that $H_{\lambda,J}$ can be defined in a neighborhood Λ_5 based on a possibly smaller neighborhood Λ_5 (independent of J). This completes the proof of this statement. $\triangleright$

Statement 6. *There exists a holomorphic motion $\hat{H}_{\lambda,0}$ based on a neighborhood Λ_6 of 0 in $\mathbb{C}$ such that $\hat{H}_{\lambda,0}|Q = H_{\lambda,0}|Q$ and such that the following holds: Let Q' be the union of the forward orbits of all endpoints of the components of $\mathbb{D}(I)$. Then for all $z \in Q' \setminus I$, $\hat{H}_{\lambda,0} \circ f_0(z) = f_\lambda \circ \hat{H}_{\lambda,0}(z)$.*

$\triangleleft$ First we observe that there exists $k_0 \in \mathbb{N}$ and $\delta > 0$ such that for each $x, y \in \partial I$, we have $d(f^k(x), y) > \delta$ for all $k \geq k_0$. Shrinking Λ_5 if necessary and assuming all $\lambda \in \Lambda_5$, we have $d(f_\lambda^k(H_{\lambda,0}(x)), H_{\lambda,0}(y)) \geq \delta/2$ for all $k \geq k_0$.

For each component J of $\mathrm{Dom}'(I)$, let $s(J)$ denote its entry time into I. By Statement 5, for each J, there exists a holomorphic motion $\hat{H}_{\lambda,J}$ based on Λ_5, such that $H_{\lambda,J} = H_{\lambda,0}$ on Q and such that $H_{\lambda,0} \circ f_0^{s(J)} = f_\lambda^{s(J)} \circ \hat{H}_{\lambda,J}$ holds on ∂J.

Let $\mathcal{J}$ be the collection of all components J of $\mathrm{Dom}'(I)$ which satisfies $s(J) \leq k_0$, and Q'' denote the forward orbits of the endpoints of components in $\mathcal{J}$. Note that $\overline{Q''}$ is a hyperbolic set. Thus there exists a holomorphic motion $H'_{\lambda,0}$ based on a neighborhood Λ_6 of 0, such that $H'_{\lambda,0}|Q = H_{\lambda,0}|Q$ and such that $H'_{\lambda,0} \circ f_0 = f_\lambda \circ H'_{\lambda,0}$ holds on Q''. In particular, for any distinct $z, z' \in Q''$, $H'_{\lambda,0}(z) \neq H'_{\lambda,0}(z')$ for all λ.

Now take $x, x' \in Q' \setminus I$ and suppose that they lie on the boundary of J and J' respectively. Let $x(\lambda) := H_{\lambda,J}(x)$ and $x'(\lambda) := H_{\lambda,J'}(x')$. If $x(\lambda) = x'(\lambda)$ for some $\lambda \in \Lambda_6$, Then $y(\lambda) := f_\lambda^{s(J)}(x(\lambda))$ and $y'(\lambda) = f_\lambda^{s(J')}(x'(\lambda))$ both belong to $H_{\lambda,0}(\partial I)$. Assume $s(J) \leq s(J')$. Then $f_\lambda^{s(J')-s(J)}(y(\lambda)) \in H_{\lambda,0}(\partial I)$, hence $s(J') - s(J) \leq k_0$. Let K (resp. K') denote the component of $\mathrm{Dom}'(I)$ which contains $f^{s(J)}(J)$ (resp. $f^{s(J)}(J')$), and let $z = f_0^{s(J)}(x) \in \partial K$, and $z' = f_0^{s(J)}(x')$. Then $z, z' \in Q''$, while $z(\lambda) = z'(\lambda)$, hence $z = z'$ which implies that $x = x'$. $\triangleright$

Statement 7. *There exists a neighborhood $\Lambda_7 \subset \Lambda_6$ of 0 in $\mathbb{C}$ such that for all $J \in \mathcal{J}$ and any $c \in \Omega_2 \cap U$,*

$$f_\lambda(c(\lambda) \notin H_{\lambda,J}(\partial W_J).$$

$\lhd$ Since $c \in U$ and $f_0(c)$ belongs to some W_J, this implies that the Euclidian distance from $f_0(c)$ to the boundary of $\cup_{J\in\mathcal{J}} W_J$ is positive. Hence the distance in the $\mathbb{C}\setminus Q$ from $f_0(c)$ to the boundary of $\cup_{J\in\mathcal{J}} W_J$ is also positive and we can use Lemma 10 and get the required property. Since we have finitely many points, the lemma follows. $\rhd$

We are now ready to finish the proof of Proposition 7. First let us assume that for any critical point c in Ω_2 we have $\Omega_2 \cap \omega(c) \neq \emptyset$ (which implies that we can use Statement 7). Let us prove that the complex box mapping $F : U \to V$ persists in a neighborhood of 0 in $\mathbb{C}$. In fact, replacing the holomorphic motion $H_{\lambda,0}$ with $\hat{H}_{\lambda,0}$ we may repeat the argument through Statements 1–6, and obtain for each component W of U, a holomorphic motion $H_{\lambda,W}$, based on a neighborhood Λ of 0 in $\mathbb{C}$ (independent of W), such that $\hat{H}_{\lambda,0} \circ f_0^{s(W)} = f_\lambda^{s(W)} \circ \hat{H}_{\lambda,W}$ holds on W and such that $H_{\lambda,W} = \hat{H}_{\lambda,0}$ on Q'. For any distinct components W_1 and W_2 of U, $d_{\mathbb{C}\setminus\overline{Q}'}(W_1, W_2)$ is bounded from below by a positive constant. Hence, according to Lemma 10 we can shrink Λ so that for all $\lambda \in \Lambda$, $H_{\lambda,W_1}(W_1) \cap H_{\lambda,W_2}(W_2) = \emptyset$. Similarly, by shrinking Λ once again, we may assume that $\hat{H}_{\lambda,0}(\partial V) \cap H_{\lambda,W}(\partial W) = \emptyset$ for each $W \Subset V$. Thus there exists a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ such that $H_\lambda = \hat{H}_{\lambda,0}$ outside V and such that $H_\lambda = H_{\lambda,W}$ on each component W of U. Defining $F_\lambda : U_\lambda := H_\lambda(U) \to V_\lambda := H_\lambda(V)$ as the appropriate iterates of f_λ, we obtain a holomorphic family which includes F.

Since $\omega_{f_0}(c) \cap \Omega_2 \neq \emptyset$ for all $c \in \Omega_2$ we have $\mathrm{Crit}(F) = \Omega_2$ is contained in the filled Julia set of F. By Theorem 7, it follows that the itinerary of each $c \in \Omega_2$ is constant when λ varies in a small neighborhood of 0.

Now assume that we are not in this case. Let $\Omega_{21} = \{c \in \Omega_2 : \omega_{f_0}(c) \cap \Omega_2 = \emptyset\}$, and let $\Omega_{22} = \Omega_2 \setminus \Omega_{21}$.

Let us first prove that $\nu_{f_\lambda}(c(\lambda))$ does not change in a neighborhood of 0 for $c \in \Omega_{21}$. Let $n \geq 0$ be maximal such that $c' := f_0^n(c)$ is again a critical point in Ω_2. It suffices to show that $\nu_{f_\lambda}(c'(\lambda))$ is constant in a neighborhood of 0. Note that c' is a nonrecurrent critical point. If $\mathrm{Forw}(c') = \{c'\}$, then $f(c')$ is contained in the hyperbolic set Q defined as above, thus by Statement 1, $\nu_{f_\lambda}(c'(\lambda))$, and hence $\nu_{f_\lambda}(c(\lambda))$, does not change in a small neighborhood of 0. If $\mathrm{Forw}(c') \neq \{c'\}$, then there exists $k \geq 1$ such that $R_Y^n(f_0^k(c')) \in \mathbf{D}(Y)$ for all $n \geq 0$, or the forward orbit of c' enters a component of $Y \setminus \mathbf{D}(Y)$ infinitely many times. In the former case, the statement follows from the argument at the beginning of this section. Assume that we are in the latter case. Let $K \subset Y \setminus \mathbf{D}(Y)$ be a nice interval which intersects the forward orbit of c' infinitely many times and let $J_1, J_2, \ldots$ be the return domains of K which intersect $\mathrm{orb}(c')$. Then $R_K : J_i \to K$ is a diffeomorphism for each i. Choosing K appropriately, one can prove, using a similar idea as above, that the first return map $R_K : \bigcup_i J_i \to K$ extends to a complex box mapping P which has no critical point and persists in a neighborhood $\Lambda \subset \mathbb{C}$ of 0. By Theorem 7, it follows that $\nu_{f_\lambda}(c(\lambda))$ is constant in a neighborhood of 0.

It follows that iterates of all points in Ω_{21} move holomorphically with respect to λ. As a result, Statement 7 can be extended:

STATEMENT 8 (7′). *There exists a neighborhood $\Lambda_8 \subset \Lambda_7$ of 0 in $\mathbb{C}$ such that for all $J \in \mathcal{J}$ and any $c \in \Omega_2$,*

$$f_\lambda(c(\lambda) \notin H_{\lambda,J}(\partial W_J).$$

$\lhd$ We have already proved this statement for points in $\Omega_2 \cap U$. Let $c \in \Omega_2 \setminus U$, in particular, $c \in \Omega_{21}$. As we just have seen, there exists a holomorphic motion $H_{\lambda,3} : \mathbb{C} \to \mathbb{C}$ defined on some $(\Lambda, 0)$ such that $f_\lambda^n \circ H_{\lambda,3}(c) = H_{\lambda,3} \circ f_0^n(c)$ for all $n \geq 0$. By shrinking I if necessary we can assume that $\overline{I} \cap \omega(c) = \emptyset$, and $f^n(c) \notin \overline{I}$ for all $n > 0$. Using Lemma 10 once again we can conclude that the sets ∂V_λ and $f_\lambda^n(c_\lambda)$ for $n > 0$ never intersect for λ in some $\Lambda' \subset \Lambda$. On the other hand, if $f_\lambda(c_\lambda) \in \partial W_{\lambda,J}$ for some $J \in \mathcal{J}$ and $\lambda \in \Lambda'$, then, $f_\lambda^{s+1}(c_\lambda) \in \partial V_\lambda$, where s is the entry time of J. This is a contradiction. $\rhd$

Having this generalized statement we can construct a holomorphic family $F_\lambda : U_\lambda \to V_\lambda$ for F_0 (no longer assuming that $\Omega_2 \cap \omega(c) \neq \emptyset$ for all $c \in \Omega_2$). The map F_0 can have critical points which escape the domain of the definition of F_0, so we cannot apply Theorem 7. To be able to apply this theorem we first construct from F_0 a new complex box mapping $\hat{F}_0 : \hat{U} \to V$ with $\mathrm{Crit}(\hat{F}) = \Omega_{22}$, as follows. For each component U' of U which does not intersect $\mathrm{Crit}(F) \cap \Omega_{21}$, U' is also a component of $\hat{U}$ and $\hat{F}|U' = F|U'$. For each $c \in \Omega_{21} \cap \mathrm{Crit}(F)$, let $n \geq 0$ be maximal such that $c' := F^n(c) \in \mathrm{Crit}(F)$, define $\hat{U} \cap U(c)$ to be the union of the components of $F^{-n}(V(c') \cap U)$, and for each component of $\hat{U} \cap U(c)$, define $\hat{F}|U' = F^{n+1}|U'$. Because of Statement 8 we can apply the same procedure for each F_λ, and obtain a holomorphic family of complex box mappings $\hat{F}_\lambda$ induced by f_λ exactly as in the case when $\Omega_2 \cap \omega(c) \neq \emptyset$ for each $c \in \Omega_2$. Again by Theorem 7, we obtain that $\nu_{f_\lambda}(c(\lambda))$ does not change in a neighborhood of 0. This completes the proof of Proposition 7.

5. Perturbations with more critical relations

Let f be a real polynomial. We want to find hyperbolic polynomials of the same degree arbitrarily close to f.

We may assume (see Lemma 11 below) that all critical points of f (including complex ones) are nondegenerate and that f has no neutral periodic points (again including complex). Such polynomials we will call *admissible.*

Now we will describe an inductive procedure which will allow us to obtain a hyperbolic polynomial from the given polynomial in finitely many steps. First we introduce a few definitions.

By a *critical relation* for f we mean a triple (n, c_i, c_j) such that c_i, c_j are critical points of f, $f^n(c_i) = c_j$ and $n > 0$. As before, if the iterates of a real critical point c of f converge to a hyperbolic attracting cycle or some iterate of c lands on a critical point of f, then we say that c is *controlled*.

We say that a real polynomial f *defines an interval map* if $f(X) \subset X$ and $f(\partial X) \subset \partial X$, where $X = [0, 1]$.

PROPOSITION 8. *Suppose f is a real polynomial with K controlled real critical points and suppose that K is less than the number of real critical points of f. Then, arbitrarily close to f in the space of real polynomials of the same degree, one can find an admissible real polynomial g of the same degree with $K+1$ controlled real critical points. Moreover, if f defines an interval map, then so does g.*

This proposition clearly implies the main theorem (density of hyperbolicity). Indeed, in a few steps we obtain an admissible polynomial with all real critical points controlled, which means it is Axiom A.

For each real polynomial f and real critical point c, let $n(c) \geq 0$ be maximal such that $f^{n(c)}(c)$ is again a critical point, and let

$$T(f) = \bigcup_{c \in \mathrm{Crit}(f)} \{c, f(c), \ldots, f^{n(c)}(c)\}. \tag{11}$$

5.1. *Destroy neutral cycles.*

LEMMA 11. *Any real polynomial g can be approximated by an admissible real polynomial $\hat{g}$ of the same degree in such a way that the number of controlled critical points of $\hat{g}$ is larger than or equal to the number of controlled critical points of g. Moreover, if g defines an interval map, then so does $\hat{g}$.*

To prove this lemma, we will need the following:

LEMMA 12. *For any real polynomial f and a neighborhood W of this polynomial (in the space of polynomials of the same degree) there exist $R > 0$ and $\delta > 0$ such that the following holds.*

Let $g : \mathbb{D}_R \to \mathbb{C}$ be a real-symmetric holomorphic map such that

$$\sup_{z \in \mathbb{D}_R} |g(z) - f(z)| < \delta.$$

Then there exists a real polynomial $\tilde{f} \in W$ conjugate to g in $\mathbb{D}_{R/2}$.

Proof. The proof of this lemma is the same as the proof of the Straightening Theorem [DH85]. One should notice that in the case of this lemma it is possible to construct a real-symmetric q.c. conjugating homeomorphism with an arbitrarily small dilation. $\square$

Proof of Lemma 11. It is clear that we can approximate g by a real polynomial $\tilde{g}$ of the same degree which have only nondegenerate critical points. So we may assume that g has only nondegenerate critical points.

Let W be a small neighborhood of g in the space of real polynomials of the same degree so that all maps in W have only nondegenerate critical points. Assume that g has a neutral cycle. We claim that one can find $g_1 \in W$ so that the number of controlled real critical points of g_1 is large or equal to that of g and the number of hyperbolic attracting cycles of g_1 is larger than that of g.

To this end, let R and δ be as in the previous lemma. Let $T = T(g)$ be as in (11) and let P be the set of all attracting or neutral periodic points of g. It suffices to prove that there exists a holomorphic map $\tilde{g} : \mathbb{D}_R \to \mathbb{C}$ such that $\sup_{z\in\mathbb{D}_R} |\tilde{g}(z) - g(z)| < \varepsilon$, $\tilde{g}|T = g|T$, $\tilde{g}'|T = g'|T$, and the number of hyperbolic attracting cycles of $\tilde{g}$ is larger than that of g. To this end, notice that T and P are both finite sets and they are symmetric with respect to complex conjugation (since g is real). So there exists a real polynomial (possibly with a large degree) $h(z)$ with the following properties:

- For all $z \in T$, $h(z) = 0$, $h'(z) = 0$;
- For all $z \in P$, $h(z) = 0$, $h'(z) = -P'(z)$.

Let $\tilde{g} = g(z) + \varepsilon h(z)$, where $\varepsilon > 0$ is a small constant such that $\sup_{z\in D} |g(z) - \tilde{g}(z)| < \delta$. Note that $\tilde{g}|(T \cup P) = g|(T \cup P)$ and $\tilde{g}'|T = g'|T$. Moreover, if $z \in P$ has period s, then

$$|(\tilde{g}^s)'(z)| = |(g^s)'(z)(1-\varepsilon)^s| < 1.$$

Now, z is a hyperbolic attracting periodic point of $\tilde{g}$. This completes the proof of the claim.

If g_1 has no neutral cycle, then the proof of the lemma is completed. Otherwise, repeating the same argument for g_1, we obtain a real polynomial $g_2 \in W$ with more hyperbolic attracting cycles and without decreasing the number of controlled critical points. Since the number of hyperbolic attracting cycles of a polynomial of degree d is bounded from above by $2d-2$, we find the desired approximation within $2d-2$ steps.

Now assume that g defines an interval map. In the above construction of $\tilde{g}$, let us also require that $\tilde{g}|\partial X = g|\partial X$. Then $\tilde{g}$ defines an interval map. Since g_1 is conjugate to $\tilde{g}$ via a q.c. map close to the identity, by an appropriate rescaling, we may assume that g_1 defines an interval map as well. This proves the last statement of the lemma. ☐

5.2. *Construction of special families.*

PROPOSITION 9. *Let $f : X \to X$ be a real analytic nondegenerate interval map without neutral cycle. Assume that no critical point of f is contained in* *the boundary of the basin of a hyperbolic attracting cycle and that f is not hyperbolic, and that f has a noncontrolled critical point. Then there exists a real polynomial h such that*

$$f_\lambda = f + \lambda h, \lambda \in (-1, 1),$$

is a special family of nondegenerate interval maps satisfying the assumption of Theorem 5.

Let us first deal with the case when f has a recurrent critical point which has a minimal ω-limit set. This case is easier since we do not need to care about the regularity of the maps f_λ.

Proof of Proposition 9 *in the minimal case.* Let c be a recurrent critical point of f such that $\omega(c)$ is minimal. Let us fix a small neighborhood U of c. Let $\hat{g} : X \to X$ be a C^∞ function such that

- $\hat{g} = f$ outside of U,
- $\hat{g}$ and f have the same critical points as f,
- $\hat{g}(c_0) \in \partial X$.

So the itineraries of c_0 maps f and $\hat{g}$ are different. Note that controlled critical points of f are also controlled critical points of g provided that U was chosen sufficiently small.

Now we approximate $\hat{g}$ on X by a real polynomial g_1 in the C^2 topology such that $g_1 = g$ and $g_1' = g'$ on $\partial X \cup T(g)$, so that all controlled critical points of $\hat{g}$ are controlled critical points of $g_1 : X \to X$.

There exists $\varepsilon > 0$ such that the function $(1 + \lambda)f - \lambda g_1$ for $\lambda \in [0, \varepsilon]$ has only nondegenerate critical points. The family $g_\lambda = (1 - \lambda\varepsilon)f + \lambda\varepsilon g_1$, $\lambda \in (-1, 1)$, is the required special family passing through f. $\square$

To deal with the remaining case, we need to guarantee all maps f_λ we shall construct are regular. For this purpose, we will use the following.

LEMMA 13. *Let $f : X \to X$ be a C^3 nondegenerate interval map without neutral cycle. There there exists a C^3 neighborhood W of f consisting of regular interval maps.*

Proof. The proof of this statement for multimodal maps is the same as in [Koz03, Lemma 4.6], where instead of the results for the negative Schwarzian condition of [Koz00], one uses its generalization [vSV04]).

Proof of Proposition 9 *in the remaining case.* First, we notice that it suffices to prove that f can be approximated in the C^3 topology by C^3 interval maps $g : X \to X$ such that $g = f$ and $g' = f'$ on $\partial X \cup T(f)$ and such that g has at least one more critical relation than f. In fact, once this has been done, we can actually choose the approximation maps g to be real analytic. Then $f_\lambda = (1 - \lambda/2)f + (\lambda/2)g$ defines a special family passing through f satisfying the assumption of Theorem 5.

In the case that f has a nonrecurrent noncontrolled critical point, it is well-known that the required C^3 approximation exists; see for example Lemmas 3.10 and 3.12 in [BM00]. So let us assume that f has a noncontrolled recurrent critical point c_0 (with a nonminimal ω-limit set).

Let us construct a C^3 perturbation of f (in the same way as in [Koz03]). Due to Theorem 3, there exists a box mappings $F : U \to V$ for the map f such that $c_0 \in U$, and there are universal constants $\theta_1 \in (0, \pi)$, $C_1 > 0$, such that for any connected component U' of U, we have that $f(U')$ is contained in $D_{\theta_1}(f(U') \cap \mathbb{R})$. Moreover, if $U' \subset \mathrm{Comp}_{c_0}(V)$ then $\mathrm{mod}(V \setminus U') > C_1$.

Let a be a real boundary point of the domain $\mathrm{Comp}_{c_0} V$. Consider the following perturbation of the map f:

$$f_\lambda(x) = \begin{cases} f(x) & , \quad x \notin \mathrm{Comp}_{c_0} V, \\ f(x) + \lambda \frac{(f(x) - f(a))^4}{(f(c_0) - f(a))^3} & , \quad x \in \mathrm{Comp}_{c_0} V. \end{cases}$$

Notice that for all λ the map f_λ is C^3. Note also that provided that V is small enough, all controlled critical points of f are still controlled for all maps f_λ.

For constants θ_1 and C_1 there exists $\lambda_1 > 0$ such that for any $\lambda \in \mathbb{D}_{\lambda_1}$, the map f_λ induces a complex box mapping F_λ with the same domain V as for the map f_0 and a deformed domain U^λ.

Let us prove that there exists an arbitrarily small $\lambda \in \mathbb{R}$ such that f_λ is not essentially combinatorially equivalent to f_0. Arguing by contradiction, assume that this is not true. Let $\Lambda = \{\lambda : f_\lambda(c_0) \in f_0(U(c_0))\}$, which is a topological disk. By choosing the complex box mapping F appropriately, we can assume that $|f(\mathrm{Comp}_{c_0} U) \cap \mathbb{R}| / |f(\mathrm{Comp}_{c_0} V) \cap \mathbb{R}|$ is very small, and so $\Lambda \Subset \mathbb{D}_{\lambda_1}$. For $\lambda \in \Lambda$, $U_\lambda \ni c_0$, so that $F_\lambda : U_\lambda \to V$, $\lambda \in \Lambda$ is a holomorphic family of complex box mappings. By Theorem 7, it follows that all the maps F_λ, $\lambda \in \Lambda$ are q.c. conjugate. In particular, f_λ and f_0 are essentially combinatorially equivalent for all $\lambda \in \Lambda \cap \mathbb{R}$. But by Remark 4.1, it follows that the same holds for $\lambda = \lambda_0 \in \partial\Lambda \cap \mathbb{R}$. However, c_0 is a nonrecurrent critical point of f_{λ_0}, a contradiction. This completes the proof. □

Proof of Proposition 8. First let us assume that f defines an interval map $f : X \to X$. By Lemma 11 we may assume that f is admissible. Let us now consider two cases.

Case 1. There exists $c \in \mathrm{Crit}(f)$ and $n \geq 1$ such that $p := f^n(c)$ lies on the boundary of the immediate basin B of a hyperbolic attracting cycle O. Let $q \in O$ be such that $[q, p) \subset B$. Without loss of generality assume that $q < p$. Let

h be a polynomial such that $h(z) = h'(z) = 0$ on $T(f) \cup \{c, f(c), \dots, f^{n-2}(c)\}$ and $h(f^{n-1}(c)) = 1$, where $T(f)$ is as in (11). Then for ε small enough, $f_\varepsilon = f - \varepsilon h$ defines an interval map such that all controlled critical points of f are controlled by f_ε. Note that c becomes a new controlled critical point. The conclusion of the proposition follows by Lemma 12.

Case 2. f has no critical point whose orbit hits the boundary of the immediate basin of a hyperbolic attracting cycle. Then by Proposition 9, there exists a special family $f_\lambda = f_0 + \lambda h$ with $f_0 = f$ such that for some λ_0, f_{λ_0} has one more controlled critical point than f_0. By Theorem 5, there exists $\lambda_n \to 0$ such that the number of controlled critical points of f_{λ_n} is more than that of f_0. Again by Lemma 12, the proposition follows.

If f does not preserve X, then instead of interval endomorphisms, we consider a wider class of interval maps, i.e. maps of the form $g : Y' \to Y$, where $Y' \subset Y$ are compact intervals. In this case, the whole argument we have used applies except that we need to add to the definition of a controlled critical point the case of an escaping critical point, i.e., a critical point c is also called controlled if $g^n(c) \in Y \setminus Y'$ for some $n \geq 1$. More precisely, let Y be a large compact interval containing so that $Y' := f^{-1}(Y) \cap \mathbb{R}$ is a compact interval compactly contained in Y. Arguing as before we obtain a sequence of real polynomials f_n such that f_n has at least one more controlled critical point than f and such that $f_n \to f$ uniformly on any compact set in $\mathbb{C}$. By Lemma 12, we obtain the proposition. □

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