Published in IET Control Theory and Applications
Received on 20th December 2009
Revised on 24th April 2010
doi: 10.1049/iet-cta.2009.0650

ISSN 1751-8644

Model predictive control of non-linear discrete time systems: a linear matrix inequality approach

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Abstract: Using a non-linear model in model predictive control (MPC) changes the control problem from a convex quadratic programme to a non-convex non-linear problem, which is much more challenging to solve. In this study, we introduce an MPC algorithm for non-linear discrete-time systems. The systems are composed of a linear constant part perturbed by an additive state-dependent non-linear term. The control objective is to design a state-feedback control law that minimises an infinite horizon cost function within the framework of linear matrix inequalities. In particular, it is shown that the solution of the optimisation problem can stabilise the non-linear plants. Three extensions, namely, application to systems with input delay, non-linear output tracking and using output-feedback, are followed naturally from the proposed formulation. The performance and effectiveness of the proposed controller is illustrated with numerical examples.

1 Introduction

Receding horizon control (RHC), also known as model predictive control (MPC) [1], is a well-established control strategy for different industrial plants. In this strategy, at each instant of time, the first element of an input trajectory is chosen to optimise a performance index. Since there is no restriction on the type of model used in the prediction, many formulations have been developed for linear or non-linear systems (e.g. [2–5]), and found wide applications in different industries in recent years [6].

Although most industrial processes are inherently non-linear, the MPC applications are widely based on linear dynamic models. By using a linear model and a quadratic objective, the nominal MPC algorithm takes the form of a structured convex quadratic programme (QP), for which reliable solution algorithms can easily be found. This is important because the solution algorithms converge properly to the optimum. Nevertheless, there are cases where non-linear effects are significant enough to justify the use of non-linear MPC.

On-line computational complexity is a major concern in MPC of non-linear systems. For fast sampling applications, high-dimensional systems and control problems that demand the use of large prediction horizons, this concern is more stringent. Since, the performance costs and constraints are generally non-convex functions of the predicted inputs, the numerical techniques used to solve the optimisation problem may exceed the available time for an on-line computation. It is therefore essential to look for suboptimal solutions. A convex problem that is efficiently solvable via semi-definite programming can be extended to MPC for non-linear systems, either through linear dynamic approximation together with bounds on the error of approximation [7, 8], or by using a linear differential inclusion [9, 10] in place of the original non-linear system. In this case, an invariant ellipsoid for an uncertain linear time-varying system is determined (e.g. [10, 11]). In fact, for on-line implementation of MPC synthesis, there is a need for computationally effective techniques that allow incorporation of a broad class of non-linear models.

More recently, min–max formulations with quadratic criteria are addressed in the powerful framework of LMI optimisation (e.g. [11–14]). The LMI methods are flexible in permitting the inclusion of a wide variety of additional design requirements, such as the size and the structure of matrices, degree of exponential stability, time delay and recasting much of the existing robust control theory. Our special interest in this paper is to exploit the ability of the LMI approach to accommodate an MPC-based technique for the control of plants with non-linearity, through linear dynamic approximation, together with Lipschitz bounds on the error approximation. In the proposed algorithm at each time step, a state-feedback that minimise a 'worst-case' infinite horizon performance objective is obtained, and the problem of minimising an upper bound on the 'worst-case' performance objective function is reduced to a convex optimisation involving linear matrix inequalities (LMIs). By this means a computationally cost effective algorithm is proposed, which is implementable for a wide range of non-linear plants.

The paper is structured as follows. Section 2 presents some preliminaries, while Section 3 describes the mathematical formulation of the proposed MPC problem with state-feedback as an LMI problem. Then, an extended formulation is given to incorporate input constraint. In Section 4, the proposed formulation is extended to the systems with non-linear output and input delay. Also, an extension to the output-feedback case is presented in the framework of state-feedback. In Section 5, several numerical examples are presented to illustrate the design procedure, and the effectiveness of the method. In the last two examples, both the proposed approach and the extended dynamic matrix control (EDMC) are applied to benchmark processes and the performances of the two controllers are compared. Finally, the benefits of the proposed controller are concluded in Section 6.

2 Preliminaries and problem statement

Consider the non-linear discrete-time dynamic

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

where k is the discrete time index, $x(k) \in R^n$ the state, $u(k) \in R^m$ the input, $f(\cdot, \cdot) \in C^2$, and $f(0, 0) = 0$. Let, $A = \partial f/\partial x\,(0, 0)$, $B = \partial f/\partial x\,(0, 0)$ then the dynamic system (1) can be reformulated as

$$x(k+1) = Ax(k) + Bu(k) + \tilde{f}(x(k), u(k)) \tag{2}$$

where

$$\tilde{f}(x(k), u(k)) = f(x(k), u(k)) - (Ax(k) + Bu(k))$$

and $\tilde{f}(\cdot, \cdot)$ is a Lipschitz non-linearity. The state and control variables are required to satisfy the following constraints

$$\begin{aligned} x(k+i|k) &\in \bar{\mathbb{X}} \\ u(k+i|k) &\in \bar{\mathbb{U}}, \quad i \geq 0 \end{aligned} \tag{3}$$

where $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are compact subsets of R^n and R^m, respectively, both containing the origin as an interior point. In order to design a state-feedback control law $u(k+i|k) = L(k)x(k+i|k)$ $(i \geq 0)$ for (2), one may consider the minimisation problem with respect to $u(\cdot)$ of the infinite horizon cost function.

$$J(k) = \sum_{i=0}^{\infty} x(k+i|k)^{\mathrm{T}} \mathcal{Q} x(k+i|k) + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) \tag{4}$$

subject to (2) and (3), in which, $\mathcal{Q}$ and $\mathcal{R}$ are positive definite weighting matrices. Let us introduce a quadratic function $V(x) = x^{\mathrm{T}}Px$, $P > 0$ of the state $x(k|k)$ of the system (2), with $V(0) = 0$. At sampling time k, suppose the following inequality is satisfied

$$\begin{aligned} V(k+i+1|k) - V(k+i|k) \geq -(x(k+i|k)^{\mathrm{T}} \mathcal{Q} x(k+i|k) \\ + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k)) \end{aligned} \tag{5}$$

Summing (5) from $i = 0$ to $i = \infty$, we have

$$x(\infty|k)^{\mathrm{T}} Px(\infty|k) - x(k|k)^{\mathrm{T}} Px(k|k) \geq -J$$

If the resulting closed-loop system for (2) is stable, $x(\infty|k)$ must be zero and result in

$$J \leq x(k|k)^{\mathrm{T}} Px(k|k) \leq -\gamma \tag{6}$$

where γ is a positive scalar and is regarded as an upper bound of the objective in (4)

$$\sum_{i=0}^{\infty} x(k+i|k)^{\mathrm{T}} \mathcal{Q} x(k+i|k) + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) \leq \gamma \tag{7}$$

Let us present the following technical lemmas for later use.

Lemma 1 (Schur complements): The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^{\mathrm{T}} & R(x) \end{bmatrix} > 0 \tag{8}$$

in which, $Q(x) = Q(x)^{\mathrm{T}}$, $R(x) = R(x)^{\mathrm{T}}$ and $S(x)$ are affine functions of x, and is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^{\mathrm{T}} > 0$$

or, equivalently

$$Q(x) > 0, \quad R(x) - S(x)^{\mathrm{T}} Q(x)^{-1} S(x) > 0$$

Proof: see [10]. □

Lemma 2: Let M, N be real constant matrices and P be a positive matrix of compatible dimensions.

Then

$$M^{\mathrm{T}} P N + N^{\mathrm{T}} P M \leq \varepsilon M^{\mathrm{T}} P M + \varepsilon^{-1} N^{\mathrm{T}} P N \tag{9}$$

holds for any $\varepsilon > 0$.

Proof: The proof follows from the condition

$$\left(\sqrt{\varepsilon} M^{\mathrm{T}} - \frac{1}{\sqrt{\varepsilon}} N^{\mathrm{T}}\right) P \left(\sqrt{\varepsilon} M - \frac{1}{\sqrt{\varepsilon}} N\right) \geq 0$$

□

3 Model predictive control via LMI

In this section, we discuss the MPC problem formulation for non-linear systems and then, we incorporate an input constraint.

3.1 State-feedback MPC

In the previous section, a convenient form of representing the processes to be controlled by MPC is presented. Now, we propose a convex optimisation method to solve the MPC problem. The idea behind this method is to solve the minimisation problem in order to determine the update of iterative input. In this method, instead of minimising J in (4) an upper bound of J is minimised. We minimise this upper bound with a state-feedback control law $u(k+i|k) = L(k)x(k+i|k)$ $(i \geq 0)$ for non-linear discrete-time system (2), and then give a representation of MPC law in terms of feasible solutions to LMIs. The following theorem is devoted to constructing the state-feedback matrix L.

Theorem 1: Consider the discrete-time system (2) at each time k and let $x(k|k)$ be the measured state $x(k)$. Then, the state-feedback matrix L in the control law that minimise the upper bound $V(x(k|k))$ of objective function at instant k is given by $L = YX^{-1}$, where $X > 0$ and Y are obtained from the solution of the following optimisation problem with variables γ, ξ, X, Y and $Z = [X; Y]$

$$\min_{\gamma,\xi,X,Y} \gamma$$

subject to

$$\begin{bmatrix} -I & * \\ x(k) & -X \end{bmatrix} \tag{10}$$

and

$$\begin{bmatrix} -X & * & * & * & * \\ \sqrt{(1+\varepsilon)}(AX+BY) & -X & * & * & * \\ \sqrt{\left(1+\dfrac{1}{\varepsilon}\right)WZ} & 0 & -\xi I & * & * \\ Q^{1/2}X & 0 & 0 & -\gamma I & * \\ \mathcal{R}^{1/2}Y & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0 \tag{11}$$

Proof: To obtain (11), the modified quadratic function V is required to satisfy

$$V(k+i+1|k) - V(k+i|k) \leq -(x(k+i|k)^{\mathrm{T}} \mathcal{Q} x(k+i|k) + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k)) \tag{12}$$

Substituting the state space (2), in inequality (12) results in

$$\begin{aligned} & u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) + x(k+i|k)^{\mathrm{T}} \mathcal{Q} x(k+i|k) \\ & - x(k+i|k)^{\mathrm{T}} P x(k+i|k) + \{Ax(k+i|k) \\ & + Bu(k+i|k) + \tilde{f}(x(k+i|k), u(k+i|k))\}^{\mathrm{T}} \\ & \times P\{Ax(k+i|k) + Bu(k+i|k) \\ & + \tilde{f}(x(k+i|k), u(k+i|k))\} < 0 \end{aligned} \tag{13}$$

Defining the function $g(x, u)$ as

$$\begin{aligned} g(x, u) = {} & \{Ax(k+i|k) + Bu(k+i|k) \\ & + \tilde{f}(x(k+i|k), u(k+i|k))\}^{\mathrm{T}} P \\ & \times \{Ax(k+i|k) + Bu(k+i|k) \\ & + \tilde{f}(x(k+i|k), u(k+i|k))\} \\ = {} & \{Ax(k+i|k) + Bu(k+i|k)\}^{\mathrm{T}} P \{Ax(k+i|k) \\ & + Bu(k+i|k)\} + \{Ax(k+i|k) + Bu(k+i|k)\}^{\mathrm{T}} \\ & \times P\{\tilde{f}(x(k+i|k), u(k+i|k))\} \\ & + \{\tilde{f}(x(k+i|k), u(k+i|k))\}^{\mathrm{T}} P\{Ax(k+i|k) \\ & + Bu(k+i|k)\} + \{\tilde{f}(x(k+i|k), u(k+i|k))\}^{\mathrm{T}} \\ & \times P\{\tilde{f}(x(k+i|k), u(k+i|k))\} \end{aligned} \tag{14}$$

and applying Lemma 2, the upper bound of $g(x, u)$ becomes

$$\begin{aligned} g(x, u) \leq {} & (1+\varepsilon)\{Ax(k+i|k) + Bu(k+i|k)\}^{\mathrm{T}} P \\ & \times \{Ax(k+i|k) + Bu(k+i|k)\} \\ & + (1+\varepsilon^{-1})\{\tilde{f}(x(k+i|k), u(k+i|k))\}^{\mathrm{T}} P \\ & \times \{\tilde{f}(x(k+i|k), u(k+i|k))\} \end{aligned} \tag{15}$$

doi: 10.1049/iet-cta.2009.0650

Consider

$$P \leq \lambda_{max} I \leq \mu I$$

where λ_{max} is the maximum eigenvalue of P and μI is the corresponding upper bound, then

$$\begin{aligned} g(x, u) \leq & (1+\varepsilon)\{Ax(k+i|k) + Bu(k+i|k)\}^{T} P \\ & \times \{Ax(k+i|k) + Bu(k+i|k)\} \\ & + (1+\varepsilon^{-1})\mu\{\tilde{f}x(k+i|k), u(k+i|k))\}^{T} \\ & \times \{\tilde{f}(x(k+i|k), u(k+i|k))\} \end{aligned}$$

The term involving $\tilde{f}(\cdot, \cdot)$ in the above equation is bounded as

$$\begin{aligned} & \tilde{f}(x(k+i|k), u(k+i|k))^{T} \tilde{f}(x(k+i|k), u(k+i|k)) \\ & \quad \leq [x(k+i|k)^{T} \; u(k+i|k)^{T}] W^{T} W [x(k+i|k); \; u(k+i|k)] \end{aligned} \tag{16}$$

Then

$$\begin{aligned} g(x, u) \leq & (1+\varepsilon)\{Ax(k+i|k) + Bu(k+i|k)\}^{T} P \\ & \times \{Ax(k+i|k) + Bu(k+i|k)\} \\ & + (1+\varepsilon^{-1})\mu[x(k+i|k)^{T} \; u(k+i|k)^{T}] W^{T} W \\ & \times [x(k+i|k); \; u(k+i|k)] \end{aligned}$$

In order to satisfy (12) for all $i \geq 0$, we should guarantee that the following equation is negative

$$\begin{aligned} & u(k+i|k)^{T} \mathcal{R} u(k+i|k) + x(k+i|k)^{T} \mathcal{Q} x(k+i|k) \\ & \quad - x(k+i|k)^{T} P x(k+i|k) \\ & \quad + (1+\varepsilon)\{Ax(k+i|k) + Bu(k+i|k)\}^{T} P \\ & \quad \times \{Ax(k+i|k) + Bu(k+i|k)\} \\ & \quad + (1+\varepsilon^{-1})\mu[x(k+i|k)^{T} \; u(k+i|k)^{T}] W^{T} W \\ & \quad \times [x(k+i|k); \; u(k+i|k)] < 0 \end{aligned} \tag{17}$$

Replacing $u(k+i|k)$ by $Lx(k+i|k)$, (17) is rewritten as

$$\begin{aligned} & (1+\varepsilon)x(k+i|k)^{T}(A+BL)^{T} P (A+BL) x(k+i|k) \\ & \quad - x(k+i|k)^{T} P x(k+i|k) \\ & \quad + x(k+i|k)^{T} \mathcal{Q} x(k+i|k) + x(k+i|k)^{T} L^{T} \mathcal{R} L x(k+i|k) \\ & \quad + (1+\varepsilon^{-1})\mu x(k+i|k)^{T} [I \; L^{T}] W^{T} W [I; L] \\ & \quad \times x(k+i|k) < 0 \end{aligned} \tag{18}$$

That is satisfied for all $i \geq 0$ if

$$\begin{aligned} & (1+\varepsilon)(A+BL)^{T} P (A+BL) - P + \mathcal{Q} + L^{T} \mathcal{R} L \\ & \quad + (1+\varepsilon^{-1})\mu[I \; L^{T}] W^{T} W [I; \; L] < 0 \end{aligned} \tag{19}$$

Substituting $X = \gamma P^{-1}$, $X > 0$, $Y = LX$ and $\xi = \gamma\mu^{-1}$. Pre- and post-multiplying (19) by X, and then applying Schur complements, (19) becomes

$$\begin{bmatrix} -X & * & * & * & * \\ \sqrt{(1+\varepsilon)}(AX+BY) & -X & * & * & * \\ \sqrt{\left(1+\dfrac{1}{\varepsilon}\right)} WZ & 0 & -\xi I & * & * \\ \mathcal{Q}^{1/2} X & 0 & 0 & -\gamma I & * \\ \mathcal{R}^{1/2} Y & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0$$

$$-X + \xi I \leq 0$$

where the symbol '$*$' stands for symmetric terms in the matrix. Applying Schur complements to (6), we derive

$$\begin{bmatrix} -I & * \\ x(k) & -X \end{bmatrix} \leq 0$$

□

By solving the inequalities of (10) and (11) the solution of the convex programming problem (4) provides a feedback gain L. The control law achieved by this means guarantees the closed-loop stability for a non-linear system described by (1). In this algorithm, the stability domain that is defined by an ellipsoidal invariant set $\mathcal{S} = \{x | x^{T} X^{-1} x \leq 1\}$ is re-evaluated at a new iteration until it becomes constant. Thus, the algorithm converges to a local minimum for each sampling time.

Remark 1: Theorem 1 can cover different kinds of unstructured uncertainty that can be replaced by a function of states and inputs.

Remark 2: Although the LMI conditions (10) and (11) are formulated for a non-linear system, the results can be readily extended to a linear system.

Remark 3: During the derivation of (15), the coefficient ε is introduced so as to convert the original non-convex problem into a convex problem. For finding the optimal value of ε, let us consider the following inequality

$$\begin{aligned} g(x, u) \leq & [x(k+i|k)^{T} \; u(k+i|k)^{T}] \\ & \times \{(1+\varepsilon)(A+BL)^{T} P (A+BL) \\ & + (1+\varepsilon^{-1})\mu W^{T} W\}[x(k+i|k); \; u(k+i|k)] \end{aligned}$$

Since matrix $(A+BL)^{T} P (A+BL)$ is symmetric and positive semi-definite, therefore, based on [15], the following condition can be derived

$$
\begin{aligned}
g(x, u) \leq\ & [x(k+i|k)^{\mathrm{T}}\ u(k+i|k)^{\mathrm{T}}]\mu\{(1+\varepsilon) \\
& \times \lambda_{\max}([A\ B]^{\mathrm{T}}[A\ B]) + (1+\varepsilon^{-1})\lambda_{\max}(W^{\mathrm{T}}W)\} \\
& \times [x(k+i|k);\ \ u(k+i|k)] \\
=\ & [x(k+i|k)^{\mathrm{T}}\ u(k+i|k)^{\mathrm{T}}]\mu\{(1+\varepsilon)\sigma_{\max}^2([A\ B]) \\
& + (1+\varepsilon^{-1})\sigma_{\max}^2(W)\}[x(k+i|k);\ \ u(k+i|k)] \\
=\ & s(\varepsilon)\mu\big\|[x(k+i|k);\ \ u(k+i|k)]\big\|_2^2
\end{aligned}
$$

In which, the scalar function $s(\varepsilon)$ is given by

$$
s(\varepsilon) + (1+\varepsilon)\sigma_{\max}^2([A\ B]) + (1+\varepsilon^{-1})\sigma_{\max}^2(W)
$$

and this function possesses its minimum at

$$
\varepsilon_{\max} = \frac{\sigma_{\max}(W)}{\sigma_{\max}([A\ B])} \tag{20}
$$

this value can be used as the optimal value of ε in the process of controller design.

3.2 Input constraint

Inherent physical limitations in the process impose hard constraints on the manipulated variables. Imposing two-norm hard constraint in the problem discussed in Theorem 1, Kothare *et al.* have developed a routine to incorporate constrained inputs in the optimal infinite horizon MPC [11]. This routine can be used in our proposed controller design scheme to incorporate input hard constrains in the optimisation solution. Consider an input two-norm constraint in the form of

$$
\|u(k+i|k)\|_2 \leq u_{\max,2}, \quad i \geq 0 \tag{21}
$$

From (6), we know that the states $x(k+i)$, $i \geq 0$ determine an ellipsoidal invariant set

$$
\mathcal{S} = \{x | x^{\mathrm{T}} X^{-1} x \leq 1\} \tag{22}
$$

Therefore

$$
\begin{aligned}
\big\|u(k+ik|k)\big\|_2^2 &= \big\|Lx(k+ik|k)\big\|_2^2 \\
&= \big\|YX^{-1/2}(X^{-1/2}x(k+ik|k))\big\|_2^2 \\
&\leq \big\|YX^{-1/2}\big\|_2^2
\end{aligned} \tag{23}
$$

from (21) and (23), the input two-norm constraint in (21) is rewritten as

$$
Y^{\mathrm{T}}X^{-1}Y - u_{\max}^2 I \leq 0 \tag{24}
$$

Applying Schur complements, (24) is equivalent to

$$
\begin{bmatrix} -u_{\max,2}^2 I & * \\ Y & -X \end{bmatrix} \leq 0 \tag{25}
$$

This is an LMI and can easily be combined with problem (4).

4 Formulation extensions

In the previous section, an LMI formulation of the predictive controller that uses a state space representation of the system was derived. In this section, we will extend the preceding development to several problems.

4.1 System with non-linear output

In many industrial applications penalising the output tracking error in the cost function is of utmost importance. As a general formulation consider that the outputs are measured as a non-linear function of the states. In order to incorporate this case into the original problem, the output prediction is transformed into a new state prediction. Assume that the plant can be described by the following discrete-time, non-linear, state-space model

$$
\begin{aligned}
x(k+1) &= f(x(k), u(k)) \\
y(k) &= h(x(k))
\end{aligned} \tag{26}
$$

where $y(k) \in R^q$ is a vector of process outputs, which is measured as a non-linear function of states. The MPC control algorithms described in this section solve a non-linear programme of the form

$$
\min_{u(k)} \sum_{i=0}^{\infty} y(k+i|k)^{\mathrm{T}} \mathcal{Q} y(k+i|k) + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) \tag{27}
$$

subject to model trajectories. Define an augmented state $\tilde{x}(k) = [x^{\mathrm{T}}(k)\, y^{\mathrm{T}}(k)]^{\mathrm{T}}$. As mentioned above, in terms of the new variables of the system, the objective function is given by

$$
\begin{aligned}
J_1(k) &= \sum_{i=0}^{\infty} \tilde{x}(k+i|k)^{\mathrm{T}} \mathcal{Q}_1 \tilde{x}(k+i|k) \\
&\quad + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) \\
u(k+i|k) &= \tilde{L}(k)\tilde{x}(k+i|k)\ (i \geq 0)
\end{aligned} \tag{28}
$$

where

$$
\tilde{L}(k) = \begin{bmatrix} L_1 & \dots & L_n & 0 & \dots & 0 \end{bmatrix}
$$

$$
\mathcal{Q}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{Q} \end{bmatrix}
$$

As in Section 3, the problem of minimising the upper bound

doi: 10.1049/iet-cta.2009.0650

of function (27) can be reduced to an objective minimisation as in Theorem 1.

Note, for dynamic equation (26), quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P_1 x + y^{\mathrm{T}} P_2 y,\ P_1, P_2 > 0$, is defined to establish closed-loop stability.

Remark 4: The choice of Q_1 can be generalised to include pure state tracking error into the cost function.

4.2 Output-feedback controller

Most of the existing techniques for MPC assume a measurable state, and apply state-feedback that is selected off-line or optimised online. However, stabilising non-linear MPC algorithms dealing with the output-feedback problem are still well established. Our goal in this section is to extend the controller developed in the previous section to a static output-feedback controller that makes the closed-loop dynamics of system (26) regular and stable. In this section, by using a suitable state-space representation of the process the output-feedback controller converted to a state-feedback controller.

As discussed for the objective function (27), the state vector $\tilde{x}(k) = [x^{\mathrm{T}}(k)\ y^{\mathrm{T}}(k)]^{\mathrm{T}}$ makes problem (27) into the standard form (4). For the system (26), our output-feedback controller is of the following form

$$u(k+i-d|k) = Ly(k+i|k)\ (i \geq 0)$$

Note that with the state vector $\tilde{x}(k)$, output-feedback controller can be written as

$$u(k+i|k) = \bar{L}(k)\tilde{x}(k+i|k)\ (i \geq 0),$$
$$\bar{L} = \begin{bmatrix} 0 & \dots & 0 & L_{n+1} & \dots & L_{n+q} \end{bmatrix}$$

where $\bar{L}$ is the gain matrix with appropriate dimension. So, with a new structure of gain matrix the straightforward result can be obtained from Theorem 1. This reformulation of the output-feedback control law to a state-feedback is without any loss of performance, and, as it turns out, it simplifies the numerical solution of the problem.

4.3 System with input delay

Time delays are very common phenomena in many real industrial applications. Time delay can be named as a great source of causing instability and poor performance. During the past few decades, the control of systems with time delay has received considerable attention. However, only few reported results for non-linear systems have considered the time delay in inputs. In this section, a new MPC controller of non-linear systems with delayed input is studied. Consider the following discrete-time non-linear system with delay elements, described by the equation

$$x(k+1) = f(x(k), u(k-d)) \tag{29}$$

at sampling time $k \geq d$, we would like to design a state-feedback control law $u(k+i-d|k) = Lx(k+i-d|k)\ (i \geq 0)$ to minimise the following performance function

$$J_2(k) = \sum_{i=0}^{\infty} x(k+i-d|k)^{\mathrm{T}} Q x(k+i-d|k) + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) \tag{30}$$

Let us associate with dynamic (29) the following Lyapunov–Krasovskii

$$V(k) = x(k)^{\mathrm{T}} P_0 x(k) + \sum_{i=1}^{d} x(k-i)^{\mathrm{T}} P_i x(k-i) = \hat{x}(k)^{\mathrm{T}} P \hat{x}(k)$$

where $P = \mathrm{diag}\{P_0, P_1, \dots, P_d\}$, in which $P_i, i = 0, 1, \dots, d$ are appropriately positive definite matrices in terms of an augmented state which is defined by

$$\hat{x}(k) = [x(k)^{\mathrm{T}}\ \ x(k-1)^{\mathrm{T}}\ \ \cdots\ \ x(k-d)^{\mathrm{T}}]^{\mathrm{T}}$$

Therefore, the cost function (30) may be represented by

$$J_2(k) = \sum_{i=0}^{\infty} \hat{x}(k+i|k)^{\mathrm{T}} Q_2 \hat{x}(k+i|k) + u(k+i|k)^{\mathrm{T}} \mathcal{R} u(k+i|k) \tag{31}$$

$$u(k+i|k) = \hat{L}(k)\hat{x}(k+i|k)\ (i \geq 0)$$

where

$$\hat{L}(k) = \begin{bmatrix} 0 & \dots & 0 & L_{k-d} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}$$

As previously discussed, we can reduce the problem to the standard form proposed in Section 3.

5 Numerical examples

In this section, six numerical examples are presented to elaborate digital implementation of the algorithms. The first example is chosen from a model of a laboratory tank in order to illustrate the basic implementation of the proposed algorithm. In Examples 2 to 4, the control strategy proposed in Section 4 was implemented on a three-stage column. For the sake of comparison, we also show simulations with EDMC controller and the proposed MPC controller in the last two examples. For all examples, the LMI control toolbox [16] in the MATLAB environment was used to compute the solution of the objective minimisation problem, and we use Euler's first-order approximation for all derivatives.

5.1 Example 1

Consider a laboratory tank with the following dynamic behaviour [17]

$$\dot{x}_1(t) = -0.625x_2(t) + 0.625u(t)$$
$$\dot{x} = \frac{1}{A}\{F_{\text{in}} - F_{\text{out}}\}, \quad A = 36\pi \qquad (32)$$

where A is the tank area, F_{in} and F_{out} are the inlet and outlet flow rates. The flow equations are given by

$$F_{\text{in}} = (ae^{bx_1(t)} + c)\sqrt{2gh_{\text{pump}}}$$

$$F_{\text{out}} = k_v\sqrt{2g(x_2(t) + h_0)}$$

$$a = 0.0522, \quad b = 0.0325, \quad c = -0.0638$$

$$g = 981, \quad h_{\text{pump}} = 1100, \quad h_0 = 38.62$$

where $k_v = 0.5299$, g and h_0 are constant values representing acceleration of gravity and liquid initial height, respectively. a, b and c are parameters estimated by identification methods.

In these equations, states x_1 and x_2 denote the inlet valve position and tank height, respectively. The second state is considered as output, while the input is pressure signal. A discrete time model has been obtained from (32) using 1 (sec) sampling time. If the controller input is set on 44, the tank overflows and spills, and if it is 36, the tank is drained. In this example, the desired input is considered as 40 and then the set point for the tank height is 25.3 (Fig. 1).

As discussed in [18], we can define an incremental state $\tilde{x}(k) = x(k) - x_{ss}$ and incremental input $\tilde{u}(k) = u(k) - u_{ss}$, to reduce the problem into a standard regulation problem stated in Section 3.

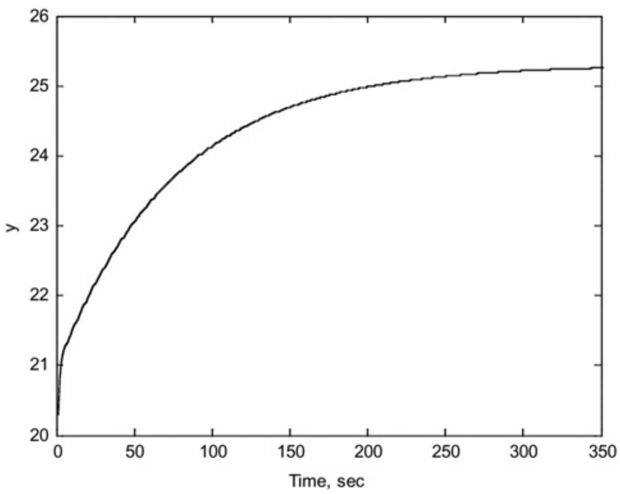

Figure 1 *Open-loop response for liquid level (u = 40)*

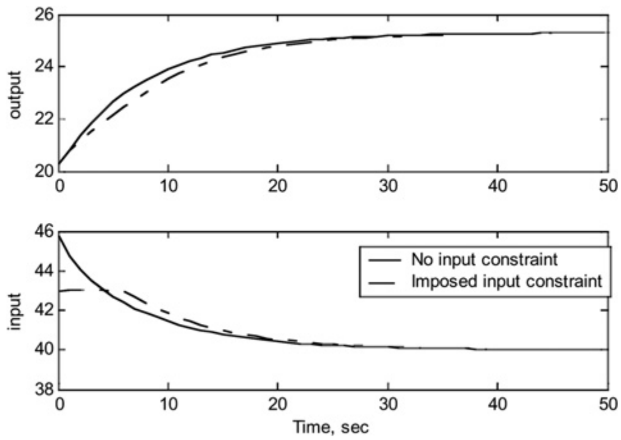

Figure 2 *Closed-loop response and control law for Example 1: no constraint (solid) and imposed constraint (dash-dotted)*

Fig. 2 shows the closed-loop response of the system corresponding to weighting matrices $\mathcal{Q} = I$, $\mathcal{R} = 0.5$. $\mathcal{Q}$, $\mathcal{R}$ are the tuning parameters used to scale the controlled and manipulated variables weights. They can be specified by the user on the basis of control objective priorities. By tuning these parameters we can obtain a modest manipulated variable move size. The other adjustable parameters in the proposed MPC include Lipschits weighting matrix W and constant ε.

Matrix W is the Lipschitz weighting matrix and is selected on the basis of process dynamics. This matrix should bind the non-linear part of system. In this example, we have selected it as $W = \text{diag}([0 \quad 0.1 \quad 0])$. Zero elements in this matrix illustrate less effective process variables in the Lipschitz condition and the non-zero elements will determine other process variables relative contribution in the non-linear dynamic behaviour of the system. Therefore, these weights illustrate the magnitude of each process variable and its importance in the Lipschitz condition. We know the changes of these weights can be effective on the response of the corresponding process variables. Therefore, by tuning these parameters using a trial and error tuning procedure, we can achieve a set point tracking with minimal overshoot. Note by increasing matrix W the feasible regions of inequalities (10) and (11) will decrease.

Constant ε is the only remaining adjustable parameter that represents the relative compromise between linear and non-linear terms whose minimum value is given in (20). In this example, this parameter is selected as $\varepsilon = 0.1$.

It is clearly seen from Fig. 2 that the response is stable and the performance is very well behaved, while the control effort is within the practical limits. The closed-loop response obtained from the proposed controller is more than ten times faster than the open-loop one. Fig. 2 also demonstrates the influence of the imposed input constraint ($\|u(k+i|k)\|_2 \leq 3, \; i \geq 0$) on the system performance. It can be seen that the control signal stays close to the

doi: 10.1049/iet-cta.2009.0650

constraint boundary, while a slower response is achieved compared to that of unconstrained MPC.

5.2 Example 2

For the second example, consider a simple column with only three stages proposed by Skogestad [19]. The continuous time of the model is given as

$$\dot{x}_1(t) = \mathcal{L}_2 x_2(t) - V_1 y_1(t) - B x_1(t)$$
$$\dot{x}_2(t) = F z_f + V_1 y_1(t) + \mathcal{L}_3 x_3(t) - V_2 y_2(t) - \mathcal{L}_2 x_2(t)$$
$$\dot{x}_3(t) = V_2 y_2(t) - \mathcal{L}_3 x_3(t) - D x_3(t)$$
$$y_i(t) = \frac{\alpha}{(1+(\alpha-1)x_i(x_i(t))^2}, \quad i = 1, 2 \tag{33}$$
$$\mathcal{L}_2 = \mathcal{L} + F, \quad \mathcal{L}_3 = \mathcal{L}, \quad V_1 = V$$
$$D = V - \mathcal{L}, \quad B = \mathcal{L} + F - V$$

The column separates a binary mixture with a relative volatility $\alpha = 10$, and has two theoretical stages ($N = 2$) plus a total condenser, namely, the liquid feed enters in stage 2 and the reboiler in stage 1. In these equations, index i is used to denote the stage number. Index B denotes the bottom product and index D denotes the distillate product. Feed rate F and feed composition z_f are considered 1 and 0.5, respectively.

Note that $\mathcal{L}_i$ and V_i denote liquid and vapour flow from stage i, x_i are the system states and y_i are the system outputs that are of the liquid and vapour composition in stage i. With these assumptions, the following discrete-time equations can be obtained from their continuous-time counterparts by discretisation, using a sampling time of 0.01 min.

$$x_1(k+1) = x_1(k) + 0.01(\mathcal{L}_2 x_2(k) - V_1 y_1(k) - B x_1(k))$$
$$x_2(k+1) = x_2(k) + 0.01(F z_f + V_1 y_1(k) + \mathcal{L}_3 x_3(k) - V_2 y_2(k) - \mathcal{L}_2 x_2(k))$$
$$x_3(k+1) = x_3(k) + 0.01(V_2 y_2(k) - \mathcal{L}_3 x_3(k) - D x_3(k))$$
$$y_i(k) = \frac{\alpha}{(1+(\alpha-1)x_i(k))^2} \quad i = 1, 2$$

where $\mathcal{L}$ and V are defined as inputs. The goal is to minimise the objective function given in (27), in which the weighting functions are assumed as $\mathcal{Q} = I$, $\mathcal{R} = I$, and the steady-state column data are summarised in Table 1. Fig. 3 illustrates four graphs, where the graphs stacked horizontally illustrate process output response and manipulated variable actions for initial state $x(0) = [0 \quad 0.17 \quad 0.7]^{\mathrm{T}}$, $\varepsilon = 0.001$ and $W = \mathrm{diag}([0.01 \quad 0 \quad 0 \quad 0.01 \quad 0])$. As shown in this figure through feasible control effort of the two manipulated variables very smooth set-point tracking is achieved for y_1 and y_2. This figure shows that the proposed controller forces the compositions to increase by decreasing liquid and increasing vapour flow in 2 min.

Table 1 Steady-state column data

Stage	i	$\mathcal{L}$	V	x_i	y_i
reboiler	1		3.55	0.1000	0.5263
feed stage	2	3.05	3.55	0.4737	0.9000
condenser	3	3.05		0.9000	

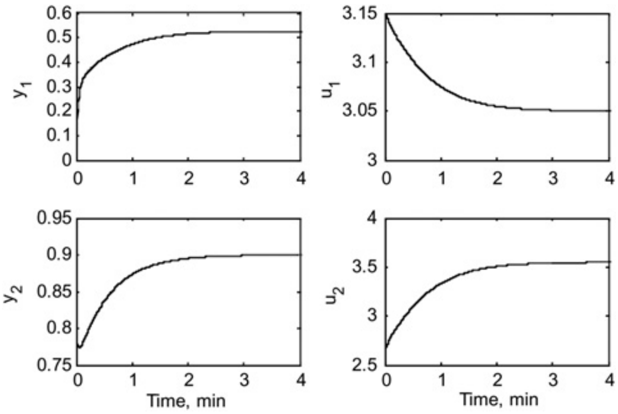

Figure 3 *Process output and manipulated variable for distillation column*

5.3 Example 3

To illustrate the application of proposed approach on output feedback, consider the three-stage distillation column described by (33) and by initial condition as Example 2 where $\varepsilon = 0.0001$ and $W = \mathrm{diag}([0 \quad 0.001 \quad 0 \quad 0 \quad 0.001])$. Fig. 4 shows time profiles for the closed-loop system. It can be seen that the MPC with output-feedback controller achieves the required tracking performance. When we use output-feedback controller, the input flows change sharply at the beginning of control time. Note that

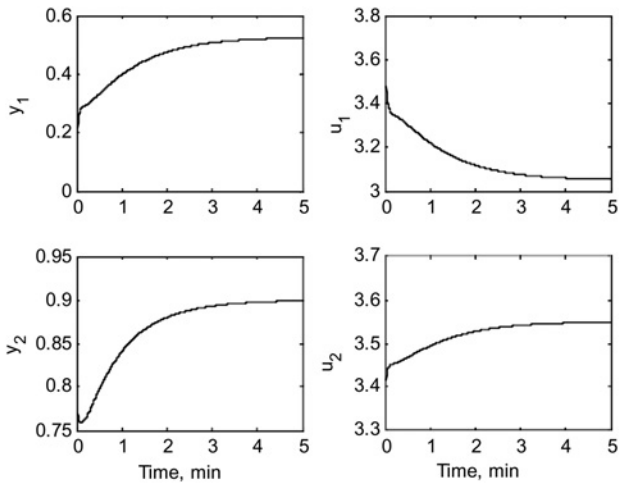

Figure 4 *Time profiles for the closed-loop system in Example 3*

doi: 10.1049/iet-cta.2009.0650

these changes immediately result in corresponding changes in outputs and we can see immediate response for distillate composition y_1. However, the effect of changes in bottom composition y_2 is much smaller.

5.4 Example 4

Let us consider the system (33) with delay in the input ($u(t-d)$). Set the delay time as $d = 0.5$. The performance objective function is given in (30), and $\mathcal{Q} = I$, $\mathcal{R} = 10I$. For the following given initial state $x(0) = [0 \quad 0.17 \quad 0.7]^{\mathrm{T}}$. Fig. 5 shows the states of discrete-time system with delay and the corresponding control action.

Note that during the first 0.5 min in which the control effort is not applied to the output, the slow zero-input response is obtained. While after the 0.5 min timeout delay the control effort is significantly affecting the response, and a very well-behaved tracking performance is obtained, while the input limits are satisfied.

5.5 Example 5

As shown by Peterson *et al.* [20], overall system with EDMC works properly for processes with single sign and slowly varying steady-state gain. Otherwise the iterative method used in algorithm converges to an unacceptable result. In this example, we want to show that this limitation is removed by the proposed MPC. Consider a DC/AC converter plant model that is borrowed from [21]

$$\begin{aligned} \dot{x}_1(t) &= \frac{x_2^2(t)}{x_1(t)} - 5x_1(t) + 5u(t) \\ \dot{x}_2(t) &= \frac{x_2^3(t)}{x_1^2(t)} - 7x_2(t) + \left(5\frac{x_2(t)}{x_1(t)} + 2x_1(t)\right)u(t) \\ y(t) &= x_2(t) \end{aligned} \tag{34}$$

The discrete-time model can be obtained by considering the sampling time as 0.01 min. As discussed in [22], this process

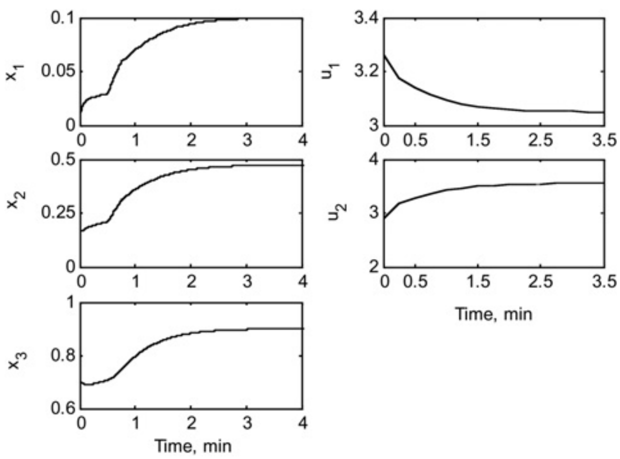

Figure 5 *States and control signal for three-stage column*

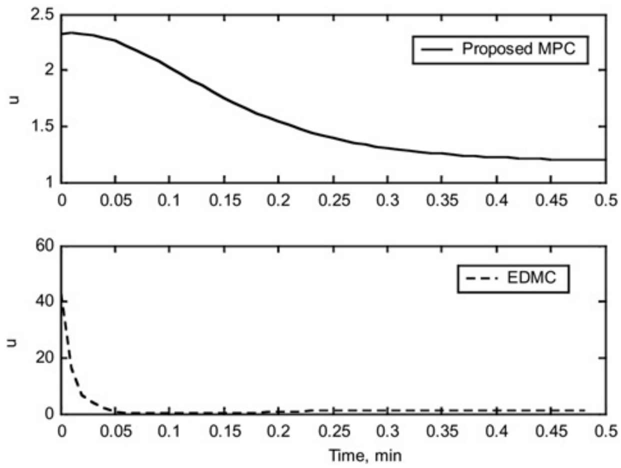

Figure 6 *Control signal of the controllers in Example 5: EDMC (dashed) and proposed MPC (solid)*

is stable in all its working condition. When the process input is positive, the process exhibits no sign changes in the output. However, its variation rate differs for different inputs. Figs. 6 and 7 show the results of both controllers for state initial condition $x(0) = [0.1 \quad 0]^{\mathrm{T}}$.

The parameters considered for EDMC is given by Shridhar and Cooper [23]

$$N = 2, \quad P = 20, \quad M = 3, \quad \lambda_S = 0.01, \quad \gamma = 1$$

and the following values is used for proposed controller

$$\mathcal{Q} = 1, \quad \mathcal{R} = 1, \quad \varepsilon = 0.001, \quad W = \mathrm{diag}([0.01 \quad 0.01 \quad 0])$$

Since steady-state gain is almost zero for small magnitude of the inputs, $u(k)$, EDMC produces large control signal (Fig. 6). As shown in Fig. 6, the iterative method used in EDMC causes the large initial move size for u which results unacceptably large overshoot in output. The

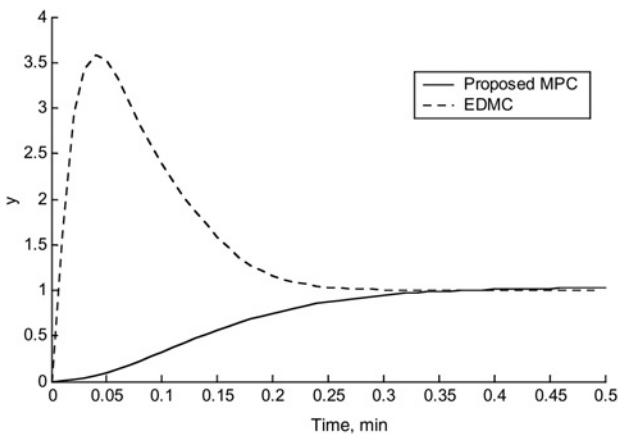

Figure 7 *Responses of the controllers in Example 5: EDMC (dashed) and proposed MPC (solid)*

IET Control Theory Appl., 2010, Vol. 4, Iss. 10, pp. 1922–1932
doi: 10.1049/iet-cta.2009.0650

maximum amplitude of output is 3.5 times more than steady state.

From Fig. 7, we can clearly verify the advantage of the method described in this paper. This figure shows that the proposed MPC achieves significant improvement in the closed-loop performance response despite less required actuator effort illustrated in Fig. 6. Generating a small control law with smooth variation is the best method for controlling this process, which is achieved by the proposed MPC.

5.6 Example 6

In this example, another comparison is drawn between EDMC and proposed MPC. A plant model is taken from [24], which is a model of an isothermal series/parallel Van de Vussue reaction in continuous stirred-tank reactor.

$$\begin{aligned}\dot{x}_1(t) &= -50x_1(t) - 10x_1^2(t) + (10 - x_1(t))u(t)\\ \dot{x}_2(t) &= 50x_1(t) - 100x_2(t) - x_2(t)u(t)\\ y(t) &= x_2(t)\end{aligned} \tag{35}$$

In this system, there is only one sign change in the steady-state gain on output 1.266. When set point is considered as 1.266, EDMC produces highly varying control signal because of almost zero steady-state gain. Control signal is shown in Fig. 8 using the following set of control parameters for EDMC

$$N = 2, \quad P = 21, \quad M = 3, \quad \lambda_S = 0.01, \quad \gamma = 1$$

Fig. 9 shows the proposed controller laws are generated for different choice of $\mathcal{R}$, while maintaining the other adjustable parameters constant at $Q = 1$, $\varepsilon = 0.1$ and $W = \mathrm{diag}([0.1 \ \ 0.15 \ \ 0])$.

In all cases, this figure shows that the proposed MPC achieves smooth control law behaviour irrespective of the choice of weighting matrix $\mathcal{R}$.

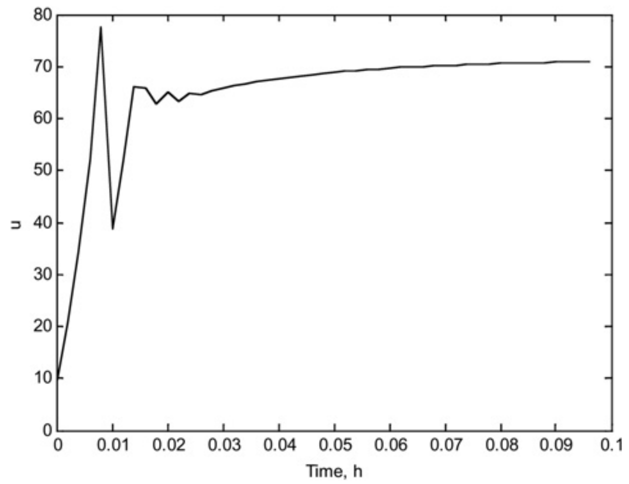

Figure 8 *Control signal of EDMC in Example 6*

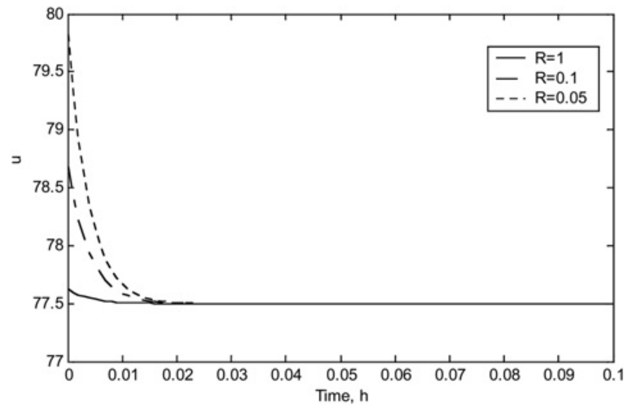

Figure 9 *Control signal of proposed MPC for $\mathcal{R} = 1$ (solid), $\mathcal{R} = 0.1$ (dashdot), and $\mathcal{R} = 0.05$ (dotted)*

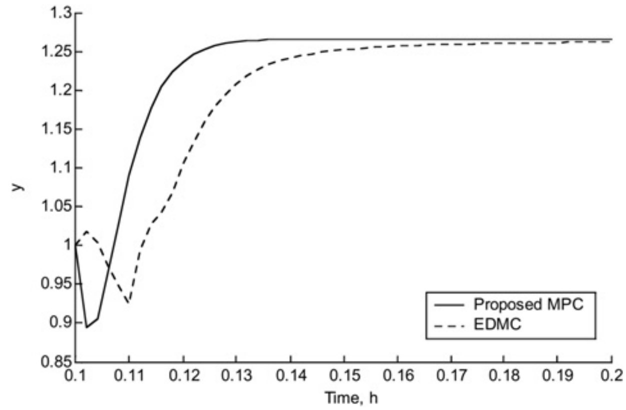

Figure 10 *Responses of the controllers in Example 6: EDMC (dashed) and proposed MPC (solid)*

By considering $\mathcal{R} = 1$ and sampling time 0.002 h, Fig. 10 shows the performance responses of both controllers corresponding to state initial condition of $x(0) = [2.5 \ \ 1]^{\mathrm{T}}$. It can be seen that the proposed MPC achieves better closed-loop performance and faster response compared to that of EDMC. The oscillation and long settling time seen in EDMC response are the result of more aggressive move in the manipulated variable.

6 Conclusions

In this paper, a sufficient synthesis condition is derived and formulated as an LMI optimisation, in order to generate an MPC effort for non-linear discrete-time systems. The stability of MPC is guaranteed as long as the optimisation problem is solvable at the initial step. As it is discussed, the proposed algorithm can be formulated into a QP form, resulting in a strictly convex non-linear programme. The only pay off is a moderate increment in the conservativeness of the obtained bounds. Moreover, the significant reduction of the computational burden opens new fields of applications to the MPC controllers. Most importantly, the proposed framework is suitable for systems with non-linear outputs and input delay. An additional result shows that stabilising output-feedback controller can be obtained by defining a state-feedback non-linear RHC law with a special structure. Several examples are given in this paper to illustrate different applications of the proposed control technique, and comparison to the EDMC shows superior performance, risk reduction of instability and less signal variation.

7 Acknowledgment

This research was supported by National Iranian Oil Products Distribution Company (NIOPDC).

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doi: 10.1049/iet-cta.2009.0650