Mean Square Exponential Synchronization for Impulsive Coupled Neural Networks with Time-Varying Delays and Stochastic Disturbances

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In this article, the mean square exponential synchronization of a class of impulsive coupled neural networks with time-varying delays and stochastic disturbances is investigated. The information transmission among the systems can be directed and lagged, that is, the coupling matrices are not needed to be symmetrical and there exist interconnection delays. The dynamical behaviors of the networks can be both continuous and discrete. Specially, the time-varying delays are taken into consideration to describe the impulsive effects of the system. The control objective is that the trajectories of the slave system by designing suitable control schemes track the trajectories of the master system with impulsive effects. Consequently, sufficient criteria for guaranteeing the mean square exponential convergence of the two systems are obtained in view of Lyapunov stability theory, comparison principle, and mathematical induction. Finally, a numerical simulation is presented to show the verification of the main results in this article. © 2015 Wiley Periodicals, Inc. Complexity 000: 00–00, 2015

Key Words: mean square exponential synchronization; neural network; stochastic disturbances; impulsive effects; time-varying delays

1. INTRODUCTION

During the last two decades, the topic of neural networks (NNs) is a hot issue which draws considerable attention from various fields of researchers from different subjects, such as mathematics, engineering, biology, physics, and even sociology science due to its extensive applications in different areas [1–7]. Up to now, there are many previous works which have devoted to stability analysis and performance behavior of NNs [8–12], and it has been shown that the NNs can perform some complicated dynamics, see [5] and the references therein.

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System scientists have extensively focused on the structure, dynamics, regulation, control, and systematism of dynamic systems for a long time. Among those properties of the systems, collective behaviors could not only reveal the relationship and attribute but also uncover the activity routines of the systems. In this regard, synchronization is undoubtedly one of the most significant collective behaviors of a dynamic system due to its potential applications in realistic world, as well as artificial virtual world [13–16]. Synchronization of systems signifies that some systems, which could be chaotic or periodic initially, share a common behavior eventually using the coupling strength among each system or some external forces, such as feedback effects, impulsive effects, intermittent effects, and so on. Its main purpose is to realize the coordination of events to force systems into unification. Until now, various kinds of synchronization phenomena have been discussed such as complete synchronization [17], cluster synchronization [18–20], lag synchronization [21,22], adaptive synchronization [23], impulsive synchronization, [24] and so on.

In the evolution of the networks, delays based on time are easy to occur due to the congestion of the communication channel and the speed of transmission information among the nodes in the complex dynamical networks, see [10,12,16,20,25–27] and references therein. The time delay included in the dynamics of each neuron describes the dynamical behaviors of the neuron model more precisely. For instance, the problem of designing decentralized impulsive controllers for synchronization of a class of complex dynamical networks with time delays about some prescribed goal function was studied in [10]. Also, the state estimation for discrete-time NNs with mixed time-varying delays was investigated in [12]. Similarly, the synchronization control problem was considered in [16] for the delayed hybrid-coupled heterogeneous networks, some synchronization criteria were developed for the complex network by applying the impulses to the open-loop network and using the improved Halanay inequality. In addition, the problem of passivity analysis for uncertain stochastic NNs with discrete interval and distributed time-varying delays was studied in [25]. Besides, the synchronization problem of uncertain complex networks with multiple coupled time-varying delays was studied in [26]. The synchronization criterion was deduced for complex dynamical networks based on Lyapunov stability theory and robust adaptive principle.

However, most natural networks such as the metabolic networks and NNs and artificial networks like software network and scientific citation network are generated randomly. It means that the networks are often subject to instantaneous disturbances and experience abrupt changes due to random uncertainties [28–30], which may be caused by switching phenomenon, frequency change, or other sudden noise. Therefore, one should analyze a more generalized model by considering the stochastic phenomena in case of studying the synchronization of networks. For example, authors in [28] presented an adaptive nonlinear controller to achieve stochastic synchronization of complex networks using the properties of Weiner process. As another example, Pototsky and Janson [30] investigated the synchronization issue of a large number of one-dimensional continuous stochastic elements with coupled stochastic units, and obtained approximate results by expanding the solution of the linearized Fokker–Planck equation into a series of eigenfunctions of the stationary Fokker–Planck operator.

It is known to all that impulse is one of the most important phenomena no matter in engineering field or manufacture region. In the impulsive communication framework, the dynamics of each node are only affected by its neighbors at the impulsive instants and there exist impulsive effects in the dynamical behavior of nodes. During the evolution of the system interchange information with each other, impulsive effect could play both a positive role and a negative role. It means that the impulse effects in the system could either accelerate the speed of synchronization or hamper and even destroy the whole system. Some prior works have been focused on the synchronization of complex works with impulses effects, see [31–33] and the references therein. In fact, it should be noted that the theoretical and technical analysis in studying the synchronization for impulsive coupled NNs with stochastic disturbances is difficult due to the definition of average impulsive interval and time-series of the impulsive sequence. To the best of our knowledge, there are seldom works to focus on the exponential synchronization of impulsive coupled NNs with stochastic disturbances, this motives us to study this work.

In this article, we consider a kind of impulsive coupled NNs with time-varying delays and stochastic disturbances. The problem of the mean square global exponential synchronization of the networks will be discussed. Importantly, the impulsive effects with time-varying delay will be taken into consideration when we model the network. An effective control scheme is proposed and imposed on the networks. Then, some criteria for the mean square exponential synchronization of two coupled NNs are obtained, where the Lyapunov stability theorem, the comparable principle and mathematical induction will be utilized conjointly.

The rest of the article is organized as follows. In section 2, we first present the impulsive coupled NN with time-varying delayed couplings and stochastic disturbances, and then we give some preparatory works, such as definitions and lemmas which will be used throughout the article. In section 3, we analyze the mean square global exponential synchronization of the NN by applying the
designed controllers. In section 4, a numerical simulation is given to verify the theoretical results of this article. Finally, we make the conclusion in section 5.

Notations: $L^T$ denotes the transpose of the matrix $L$, $L^T = \frac{1}{2}(L + L^T)$. $R^n$ denotes the $n$-dimensional Euclidean space. $R^{n \times n}$ is the set of $n \times n$ real matrices. diag{⋯} stands for a diagonal matrix. The symbol $|| \cdot ||$ stands for the Euclid norm of the matrix or the vector. A symmetric real matrix $L$ is positive definite (semidefinite) if $x^T L x > 0$ ($\geq 0$) for all non-zero $x$, then we denote this as $L > 0$ ($L \geq 0$). $L_N$ stands for the identity matrix with $N$ dimension. $\max(\cdot)$ is used to denote the maximum eigenvalue of a real symmetric matrix.

2. MODEL DESCRIPTION AND PRELIMINARIES

In this article, we consider the following coupled time-varying delayed NN model with stochastic disturbances and impulsive effects

$$
\frac{dx_i(t)}{dt} = [Ax_i(t) + B_1 \tilde{f}_1(x_i(t)) + C_1 \tilde{f}_2(x_i(t-\tau(t))) + I(t)] + c_i \sum_{j=1}^{N} g_{ji} \Gamma x_j(t-\tau_j(t))dt
$$

$$
+ \tilde{h}_i(x_i(t-\tau_i(t)), \ldots, x_N(t-\tau_N(t))), \ldots, x_N(t-\tau_N(t)), x_i(t-\tau(t)))]d\tau_i(t)
$$

$$
\Delta x_i(t_k) = \xi_i(t_k^-) - \xi_i(t_k^+),
$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in R^n$ for $i = 1, 2, \ldots, N$ stands for the state vector of the $i$th NN at time $t$; $A = \text{diag} \{a_1, a_2, \ldots, a_n\} \in R^{n \times n}$ denotes the rate with which the neural cell $j$ resets its potential to the resting state when isolated from other neural cells and inputs; $B = (b_{ij}) \in R^{n \times n}$ and $C = (c_{ij}) \in R^{n \times n}$ are constant matrices which denote connection weight matrix and the delayed connection weight matrix, respectively, where $b_{ij}$ and $c_{ij}$ stands for the strengths of connectivity between the $i$th neural cell and $j$th neural cell within the $i$th node at time $t$ and $t-\tau(t)$, respectively; $\tau_i(t_1), \tau_2(t), \tau_3(t)$ are the transmission time-varying delays satisfying $0 \leq \tau_i(t) \leq \tau_i, i = 1, 2, 3, \tau = \max(\tau_i, \tau_i, \tau_i)$; the positive constants $c_i$, $c_2$ denote the coupling strength of the NN and $\Gamma = \text{diag}\{\gamma_1, 1, \ldots, \gamma_n\} \in R^{n \times n}$ is the inner-linking matrix between two connected nodes, which is a diagonal matrix with $\gamma_i \geq 0$; functions $\tilde{f}_1(\cdot)$ and $\tilde{f}_2(\cdot)$ are memoryless nonlinear vector-valued function which present the neural cell activation functions and they are continuously differentiable on $R$ with initial value $\tilde{f}_1(t) = 0, \tilde{f}_2(t) = 0$, and $\tilde{f}_1(x_i(t)) = \tilde{f}_1'(x_i(t)), \tilde{f}_2(x_i(t)) = \tilde{f}_2'(x_i(t))$, $\tilde{f}_2(x_i(t))/\tilde{f}_2'(x_i(t))$, $I(t) = [I_1(t), I_2(t), \ldots, I_n(t)]^T \in R^n$ is an external input vector. $L = (l_{ij}), G = (g_{ij}) \in R^{n \times N}$ are the coupling matrices, which describe the connection topology and they are decided by the NN structure satisfying $l_{ij} > 0, g_{ij} > 0$ if there is a connection from node $i$ to node $j(i \neq j)$, otherwise $l_{ij} = 0, g_{ij} = 0$ and it is assumed that they are zero-sum-row matrices, that is, $l_{ij} = \sum_{j=1}^{N} l_{ij}(g_{ij} - \sum_{j=1}^{N} g_{ij})$; it is noted that here $l_{ij} \neq l_{ij}$ ($g_{ij} \neq g_{ij}$), which means the network is a directed NN and we assume it is irreducible; $\tilde{h}_i(x_i(t), \ldots, x_N(t), x_i(t-\tau_i(t)), \ldots, x_N(t-\tau_N(t)), x_N(t-\tau_N(t))) \in R^{n \times n}$ is the noise intensity matrix; $u_i(t) = [u_{i1}(t), u_{i2}(t), \ldots, u_{in}(t)]^T \in R^n$ is the $n$-dimensional bounded vector form Wiener process (Brownian motion) which is defined on probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$, and satisfying $E[u_i^2(t)] = E[u_i^2(t)]^3 = 1$, $E[u_i(t)u_j(t)] = 0$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$; the time series $\{z_t = (z_{1t}, z_{2t}, \ldots, z_{nt})\}$ is a sequence of strictly increasing impulsive moments; lastly, the constant $\mu_k$ and $\rho_k$ are the corresponding impulsive effects at time $t_k$ and $t_k - \tau_i(t_k)$, respectively, $k = 1, 2, \ldots$.

Throughout this article, we always assume that $x_i(t)$ is right-hand continuous at $t = t_k$, that is, $x_i(t_k^-) = x_i(t_k)$. Therefore, according to above assumption, we can get that the solutions of system (1) are piecewise right-hand continuous functions with discontinuities at $t = t_k, k = 1, 2, \ldots$

Suppose $C([-\tau, 0], R^n)$ be the Banach space of continuous vector-valued functions mapping the interval $[-\tau, 0]$ into $R^n$ with the norm $||\phi|| = \sup_{-\tau \leq s \leq 0} ||\phi(s)||$. For the functional differential equation (1), its initial conditions are given by

$$
x_i(t) = \phi_i(t), \quad -\tau \leq t \leq 0
$$

In this article, we consider the system (1) as the master system, and the slave system is given as
\[
\begin{align*}
\dot{y}_i(t) &= |Ay_i(t) + B\tilde{f}_i(y_i(t)) + C\tilde{F}(y_i(t - \tau_1(t))) + I(t) + c_i \sum_{j=1}^{N} y_j(t)| dt + \dot{h}_i(y_i(t)), \\
&= c_i \sum_{j=1}^{N} g_{ij} y_j(t - \tau_2(t)) + u_i(t) dt + \dot{\psi}_i(t), \\
\end{align*}
\]

where \( \dot{\psi}_i(t) \) is the initial value, and \( u_i(t) \) is the control scheme to be designed later for \( i = 1, 2, \ldots, N \).

Define the error vector as

\[
e_i(t) = x_i(t) - y_i(t), i = 1, 2, \ldots, N.
\]

To force error systems to be stable, we design the following control scheme

\[
u_i(t) = d_i \Gamma_0 e_i(t) + \sum_{k=1}^{N} [\mu_k y_k(t_k) + \rho_k y_k(t_k - \tau_3(t_k))] \cdot \delta(t - t_k)
\]

where \( \delta(t) \) is the Dirac function, and \( d_i \) are the control gains for \( i = 1, 2, \ldots, N \).

By the definition of error vector, the following error system is given:

\[
\begin{align*}
\dot{e}_i(t) &= |A e_i(t) + B f_i(e_i(t)) + C f_i(e_i(t - \tau_1(t)))| dt \\
&+ h_i(e_i(t), e(t - \tau_1(t))) dt \\
&+ \Delta e_i(t_k) = \mu_k e_i(t_k) + \rho_k e_i(t_k - \tau_3(t_k)), \\
&\Delta e_i(t_k) = C(\tau_2, 0, R^p), \\
\end{align*}
\]

where \( \phi_i(t) = \psi_i(t) - \psi_i(t), f_i(e_i(t)) = \dot{f}_i(x_i(t)), \dot{f}_i(x_i(t)) = \dot{x}_i(t), \) and \( h_i(e_i(t)) = \dot{h}_i(\chi(t)) \).

Here, some preliminaries such as definitions and lemmas are needed, which are used to get main results throughout the article.

Definition 1

For some designed control schemes, the stochastic master system (1) and the slave system (2) are said to be globally exponentially synchronized in mean square if there exist constants \( \lambda_0 > 0, M_0 > 0, T > 0 \) such that for any initial values \( \psi_i(0) (i = 1, 2, \ldots, N) \),

\[
\mathbb{E}[|x_i(t) - y_i(t)|^2] \leq M_0 \sup_{n \in [t - \tau, t]} \mathbb{E}[|\phi_i(s)|^2] e^{-\lambda_0 t}
\]

holds for all \( t > T > 0 \), where \( i = 1, 2, \ldots, N \), and the constant \( \lambda_0 \) is the convergence rate of the exponential synchronization.

Definition 2 [31]

If there exist a positive integer \( N_0 \) and a positive number \( N_\omega \) then the average impulsive interval of the impulsive sequence \( \zeta = \{t_1, t_2, \ldots\} \) is not larger than \( N_0 \) such that

\[
\frac{T - t}{N_\omega} - N_0 \leq N_i(T, t) \leq \frac{T - t}{N_\omega} + N_0, \quad \forall T \geq t \geq 0
\]

where \( N_i(T, t) \) denotes the number of impulsive times of the impulsive sequence \( \zeta \) in the time interval \( (t, T) \).

Lemma 1 [34]

Suppose \( \alpha > \beta \geq 0 \) and \( V(t) \) satisfies the scalar impulsive differential inequality

\[
\begin{align*}
D^+ V(t) &\leq -z V(t) + \beta \sup_{n \in [-t, t]} V(s), \\
V(t) &= \phi(t), \\
t \in [t_0, t_0 + T],
\end{align*}
\]

where initial condition \( \phi(t) \) is piecewise continuous, and denote \( \bar{V}(t) = \sup_{n \in [-t, t]} V(s) \). Then,

\[
V(t) \leq \bar{V}(t)e^{-\lambda(t - t_0)}, \quad t \geq t_0
\]

where \( \lambda > 0 \) is the unique solution of the equation, \( -z + \beta \cdot e^\lambda = 0 \).

To analyze the stochastic differential equation, we present the following It\'o operator.

Consider a stochastic differential equation with \( n \) dimension

\[
dx(t) = f(x(t), x(t - \tau)) dt + \theta(x(t), x(t - \tau)) dw(t).
\]

Define \( C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) \) as the family of all non-negative functions \( V(t, x) \) on \( \mathbb{R}_+ \times \mathbb{R}^n \) which are continually once differentiable in \( t \) and twice in \( x \). If \( V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) \), define an operator \( \mathbb{E} \) on function \( V \) from \( \mathbb{R}_+ \times \mathbb{R}^n \) to \( \mathbb{R} \) by

\[
\mathbb{E} V(t, x) = V_0(t, x) + V_2(t, x)f(t, x, x_x) + \frac{1}{2} \text{Trace} [\theta(x(t), x(t - \tau)) \theta(t, x(t), x(t - \tau)) dt dt]
\]

where \( x_x = x(t - \tau), V_0(t, x) = \frac{\partial V_0(t, x)}{\partial x}, V_2(t, x) = \frac{\partial V_2(t, x)}{\partial x}, \ldots, \frac{\partial^2 V_2(t, x)}{\partial x_x \partial x_x} \) and \( V_0(t, x) = \frac{\partial V_0(t, x)}{\partial x_x} V_2(t, x) \).

Lemma 2

For \( V(t, x) \) satisfying Lemma 1 with \( \alpha > \beta \geq 0 \) and any time \( t > t_0 \geq 0 \), the following inequality holds

\[
\mathbb{E} V(t, x(t)) \leq \mathbb{E} V(t_0) e^{-\lambda(t - t_0)},
\]

where \( \lambda > 0 \) is the unique solution of the equation, \( -z + \beta \cdot e^\lambda = 0 \).
where \( \dot{x} = -x + \beta e^{\frac{t}{2}} \), and 
\( \dot{V}(t) = \sup_{t_0 \leq s \leq t} \{ V(s, x(s)) \} \).

**Proof**

According to the Itô differential formula

\[
dV(t, x(t)) = \mathcal{L}V(t, x(t))dt + V'_x(t, x(t)) \sigma(t, x(t), x(t-\tau))d\omega(t),
\]

by integrating both sides of above equation from \( t_0 \) to \( t \), then one can find

\[
V(t, x(t)) = V(t_0, x(t_0)) + \int_{t_0}^{t} \mathcal{A}V(u, x(u))du \\
+ \int_{t_0}^{t} V'_x(u, x(u)) \sigma(u, x(u), x(x(t-\tau)))d\omega(u).
\]

Taking the mathematical expectation on the both sides of the above equation and considering \( \mathbb{E}\omega(t) = 0 \) gives

\[
\mathbb{E}V(t, x(t)) = \mathbb{E}V(t_0, x(t_0)) + \int_{t_0}^{t} \mathbb{E}\mathcal{A}V(u, x(u))du \\
\leq \mathbb{E}V(t_0) + \int_{t_0}^{t} \mathbb{E}\mathcal{A}V(u, x(u))du.
\]

Then, according to the definition of Dini derivative, we have

\[
\mathbb{E}V(t_0, x(t_0)) \leq \mathbb{E}V(t_0) e^{-\lambda(t-t_0)}
\]

where \( \mathbb{E}V(t) = \sup_{t_0 \leq s \leq t} \{ V(s, x(s)) \} \) and \( \mathbb{E}V(t_0) \neq \mathbb{E}V(t_0, x(t_0)) \).

To derive the main results of the impulsive NN, we make the following hypothesis.

**Hypothesis 1**

The nonlinear neural cell activation functions \( f_i(\cdot) \) and \( f_i(\cdot) \) are assumed to satisfy the Lipschitz condition, that is, there exist constants \( l_{ij}^1 > 0 \) and \( l_{ij}^2 > 0 \), such that

\[
|f_{ij}^1(u) - f_{ij}^1(v)| \leq l_{ij}^1 |u - v|, \\
|f_{ij}^2(u) - f_{ij}^2(v)| \leq l_{ij}^2 |u - v|
\]

hold for any \( u, v \in \mathbb{R}, j = 1, 2, \cdots, n \). Here, we denote \( L_1 = \text{diag}(l_1^1, l_2^1, \cdots, l_n^1) \) and \( L_2 = \text{diag}(l_1^2, l_2^2, \cdots, l_n^2) \).

**Hypothesis 2**

For the noise intensity matrix \( h_i(e(t), e(t-\tau_1(t)), e(t-\tau_2(t))) \), there are suitable dimensional positive definite constant matrices \( A_{1i}, A_{2i}, A_{3i}, i = 1, 2, \cdots, N \) such that the following inequality holds

\[
\text{Trace}[h_i(e(t), e(t-\tau_1(t)), e(t-\tau_2(t)))h_i(e(t), e(t-\tau_1(t)), e(t-\tau_2(t)))] \\
\leq \sum_{j=1}^{N} [e_i^T(t)A_{1j}e_j(t) + e_i^T(t-\tau_1(t))A_{2j}e_j(t-\tau_1(t)) + e_i^T(t-\tau_2(t))A_{3j}e_j(t-\tau_2(t))].
\]

**3. Exponential Synchronization Analysis for Impulsive Coupled Neural Networks**

In this section, we are about to present the main result of this article in the form of Theorem 1 in the following which guarantees the exponential synchronization in mean square between the master (1) and slave networks (2).

**Theorem 1**

Assume that Hypothesis 1 and 2 hold and the average impulsive interval of the impulsive sequence \( \zeta = \{\tau_1, \tau_2, \cdots\} \) is less than \( N_0 \). Then, the master system (1) and the slave system (2) with impulsive mixed time-varying delays and stochastic disturbances under the control scheme (3) are said to be globally and exponentially synchronization in mean square if there exist positive constants \( a, b, q \) and non-negative constants \( d_i, i = 1, 2, \cdots, N \), such that the following conditions hold (i) the matrix inequality

\[
X = \begin{pmatrix} c_1L^T + aI & -\frac{1}{2}c_2G \\ \frac{1}{2}c_2G^T & -bI \end{pmatrix} \leq 0,
\]

where \( D = \text{diag}(d_1, d_2, \cdots, d_N) \); (ii) for \( \tau = \max\{\tau_1, \tau_2, \tau_3\} \) and \( 0 \leq \tau_i(t) \leq \tau_j, j = 1, 2, 3, \) it satisfies

\[
\max\{e_i^T, m_k, n_k e^{\frac{t}{2}}\} \leq q.
\]

where \( m_k = (1 + \mu_i)(1 + \mu_k), n_k = p_k(1 + \mu_i + \mu_k), k = 1, 2, \cdots, \) and the constant \( \lambda > 0 \) is the unique solution of the equation

\[
\lambda - \alpha e^{\frac{t}{2}} = 0
\]

with \( \alpha > \beta > 0 \). 

Then, the solution of the master system (1) is globally exponential synchronized in mean square with the solution of the slave system (2) under the designed control strategy (3).
Proof

Construct the following Lyapunov function in the form of

\[ V(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) \phi_i(t). \]  

(15)

We consider the time interval \([t_{k-1}, t_k], k=1, 2, 3, \ldots\). First, at the impulsive instants \(t = t_k\), according to system (4) and the definition of error vector, we have

\[ e_i(t_k^+) = (1 + \mu_k) e_i(t_k^-) + \mu_k e_i(t_k^- - \tau_i(t_k^-)), \]

and it gives

\[
V(t_k^+)=\frac{1}{2} \sum_{i=1}^{N} e_i^T(t_k^-) \phi_i(t_k^-) \\
=\frac{1}{2} \sum_{i=1}^{N} [(1 + \mu_k) e_i(t_k^-) + \mu_k e_i(t_k^- - \tau_i(t_k^-))]^T [(1 + \mu_k) e_i(t_k^-) \\
+ \mu_k e_i(t_k^- - \tau_i(t_k^-)) + \gamma_k e_i(t_k^- - \tau_i(t_k^-))] \\
=\frac{1}{2} \sum_{i=1}^{N} [(1 + \mu_k)^2 e_i^T(t_k^-) \phi_i(t_k^-) + \mu_k (1 + \mu_k + \gamma_k) e_i^T(t_k^-) e_i(t_k^-) \\
+ \mu_k^2 e_i^T(t_k^- - \tau_i(t_k^-)) e_i(t_k^- - \tau_i(t_k^-)) + \gamma_k \mu_k e_i(t_k^- - \tau_i(t_k^-))] \\
\leq \frac{1}{2} \sum_{i=1}^{N} [(1 + \mu_k)^2 e_i^T(t_k^-) \phi_i(t_k^-) + \mu_k (1 + \mu_k + \gamma_k) e_i^T(t_k^-) e_i(t_k^-) \\
+ \mu_k^2 e_i^T(t_k^- - \tau_i(t_k^-)) e_i(t_k^- - \tau_i(t_k^-)) + \gamma_k \mu_k e_i(t_k^- - \tau_i(t_k^-))] \\
\leq \frac{1}{2} \sum_{i=1}^{N} (1 + \mu_k) (1 + \mu_k + \gamma_k) e_i^T(t_k^-) \phi_i(t_k^-) \\
+ \mu_k (1 + \mu_k + \gamma_k) e_i^T(t_k^- - \tau_i(t_k^-)) e_i(t_k^- - \tau_i(t_k^-)) \\
+ \mu_k^2 e_i^T(t_k^- - \tau_i(t_k^-)) e_i(t_k^- - \tau_i(t_k^-)) + \gamma_k \mu_k e_i(t_k^- - \tau_i(t_k^-)) \\
= \frac{1}{2} \sum_{i=1}^{N} (1 + \mu_k) (1 + \mu_k + \gamma_k) e_i^T(t_k^-) \phi_i(t_k^-) \\
+ \mu_k (1 + \mu_k + \gamma_k) e_i^T(t_k^- - \tau_i(t_k^-)) e_i(t_k^- - \tau_i(t_k^-)) \\
+ \mu_k^2 e_i^T(t_k^- - \tau_i(t_k^-)) e_i(t_k^- - \tau_i(t_k^-)) + \gamma_k \mu_k e_i(t_k^- - \tau_i(t_k^-)) \\
\leq \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) \phi_i(t) + \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) \phi_i(t_k^-) + \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) \phi_i(t_k^- - \tau_i(t_k^-)),
\]

(16)

where \(m_k = (1 + \mu_k) (1 + \mu_k + \gamma_k), n_k = \mu_k (1 + \mu_k + \gamma_k), \) for \(k = 1, 2, 3, \ldots\).

Taking the mathematical expectation on both sides of above inequality,

\[ \mathbb{E}V(t_k^-) \leq m_k \mathbb{E}V(t_k^-) + n_k \mathbb{E}V(t_k^- - \tau_i(t_k^-)), \quad k=1, 2, \ldots. \]

(17)

Second, for the time interval \(t \in [t_{k-1}, t_k], k=1, 2, \ldots, \) according to the Itô differential formula in Lemma 2, we calculate the derivative of \(V(t, e(t))\) in the form of

\[ dV(t, e(t)) = \frac{\partial V(t, e(t))}{\partial t} dt + \sum_{i=1}^{N} \frac{\partial V(t, e(t))}{\partial e_i} \dot{e}_i(t) dt. \]

(18)

Make the mathematical expectation on the both sides of Eq. (18), and by taking the definition of Dini derivative for \(t \in [t_{k-1}, t_k], k=1, 2, \ldots, \) one can obtain

\[ D^+ \mathbb{E}V(t, e(t)) = \mathbb{E}\{dV(t, e(t))\}, \]

(19)

where \(dV(t, e(t))\) is the operator defined previously, and

\[ \mathbb{E}V(t, e(t)) = \sum_{i=1}^{N} e_i^T(t) [A e_i(t) + B f_i(e_i(t)) + C f_2(e(t - \tau_i(t)))]. \]

(20)

by Hypothesis 2.

By considering Hypothesis 1, it yields that

\[ \sum_{i=1}^{N} e_i^T(t) B f_i(e_i(t)) \leq \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) B B^T e_i(t) + \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) f_i(e_i(t)) f_i(e_i(t)), \]

\[ \sum_{i=1}^{N} e_i^T(t) C f_2(e_i(t - \tau_i(t))) \leq \frac{N}{2} \sum_{i=1}^{N} e_i^T(t) C C^T e_i(t), \]

\[ \sum_{i=1}^{N} e_i^T(t) \sum_{j=1}^{N} g_{ij}^T e_j(t - \tau_j(t)) e_i(t - \tau_i(t)) \leq \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) C C^T e_i(t) + \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) \sum_{j=1}^{N} g_{ij}^T e_j(t) e_i(t), \]

(21)

For convenience written later, let \( \tilde{e}_1(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T, \)

\[ \tilde{e}_2(t) = [e_1^T(t - \tau_1(t)), e_2^T(t - \tau_2(t)), \ldots, e_N^T(t - \tau_N(t))]^T, j = 1, 2, \ldots, n, \text{ and then we have} \]

\[ \epsilon C_1 \sum_{j=1}^{N} \sum_{i=1}^{N} \gamma_{ij} e_i^T(t) e_j(t) - \sum_{j=1}^{N} \sum_{i=1}^{N} \gamma_{ij} \tilde{e}_1^T(t) D \tilde{e}_1(t), \]

\[ \epsilon C_2 \sum_{j=1}^{N} \sum_{i=1}^{N} \gamma_{ij} e_i^T(t) e_j(t) - \sum_{j=1}^{N} \sum_{i=1}^{N} \gamma_{ij} \tilde{e}_2^T(t) G \tilde{e}_2(t), \]

\[ - \sum_{j=1}^{N} d e_i^T(t) e_i(t) = - \sum_{j=1}^{N} \gamma_{ij} \tilde{e}_1^T(t) D \tilde{e}_1(t). \]

(22)
Substitute Eqs. (21) and (22) into (20), for some constants \( a \) and \( b \), it yields that
\[
\mathcal{V}(t, e(t)) \leq \frac{1}{2} \sum_{i=1}^{N} e_i^2(t) + 2A + BB^T + L_1 L_1^T + \frac{1}{2} CC^T
\]
\[
+ \sum_{i=1}^{N} \lambda_i - 2a \Gamma |e_i(t) |
\]
\[
+ \frac{1}{2} \sum_{j=1}^{N} e_j^2(t - \tau_1(t)) \left[ L_1 L_2^T + \sum_{i=1}^{N} \lambda_{i2} \right] |e_j(t - \tau_2(t)) |
\]
\[
+ \frac{1}{2} \sum_{j=1}^{N} e_j^2(t - \tau_2(t)) \left[ 2b \Gamma + \sum_{i=1}^{N} \lambda_{i1} \right] |e_j(t - \tau_2(t)) |
\]
\[
+ \frac{n}{2} \sum_{j=1}^{N} \gamma_j \tilde{e}_j^2(t - \tau_1(t))|e_j(t - \tau_2(t)) |
\]
\[
- \frac{b}{2} \sum_{j=1}^{N} \gamma_j \tilde{e}_j^2(t - \tau_2(t)) |e_j(t - \tau_2(t)) |
\]
\[
= \frac{1}{2} \sum_{j=1}^{N} e_j^2(t) + 2A + BB^T + L_1 L_1^T + \frac{1}{2} CC^T + \sum_{i=1}^{N} \lambda_i - 2a \Gamma |e_i(t) |
\]
\[
+ \frac{1}{2} \sum_{j=1}^{N} e_j^2(t - \tau_1(t)) \left[ L_1 L_2^T + \sum_{i=1}^{N} \lambda_{i2} \right] |e_j(t - \tau_2(t)) |
\]
\[
+ \frac{1}{2} \sum_{j=1}^{N} e_j^2(t - \tau_2(t)) \left[ 2b \Gamma + \sum_{i=1}^{N} \lambda_{i1} \right] |e_j(t - \tau_2(t)) |
\]
\[
+ \frac{n}{2} \sum_{j=1}^{N} \gamma_j \tilde{e}_j^2(t - \tau_1(t))|e_j(t - \tau_2(t)) |
\]
\[
- \frac{b}{2} \sum_{j=1}^{N} \gamma_j \tilde{e}_j^2(t - \tau_2(t)) |e_j(t - \tau_2(t)) |
\]
\[
(23)
\]
As the first condition (12) in Theorem 1, the inequality (22) can be transformed into
\[
\mathcal{V}(t, e(t)) \leq \lambda_{\text{max}} \left[ 2A + BB^T + L_1^T L_1 + \frac{1}{2} CC^T + \sum_{i=1}^{N} \lambda_i - 2a \Gamma |V(t) |ight]
\]
\[
+ \lambda_{\text{max}} \left[ \frac{1}{2} L_1^T L_2 + \sum_{i=1}^{N} \lambda_{i2} |V(t - \tau_1(t)) |ight]
\]
\[
+ \lambda_{\text{max}} \left[ 2b \Gamma + \sum_{i=1}^{N} \lambda_{i1} |V(t - \tau_2(t)) |ight]
\]
\[
- \gamma \mathcal{V}(t) + \beta \mathcal{V}(t - \tau_1(t)) + \beta \mathcal{V}(t - \tau_2(t))
\]
\[
\leq -\gamma \mathcal{V}(t) + \beta \mathcal{V}(t - \tau_1(t)) + \beta \mathcal{V}(t - \tau_2(t))
\]
\[
(24)
\]
where \( \gamma = -\lambda_{\text{max}} \left[ 2A + BB^T + L_1^T L_1 + \frac{1}{2} CC^T + \sum_{i=1}^{N} \lambda_i - 2a \Gamma |V(t) |ight]
\]
\( \beta = \lambda_{\text{max}} \left[ \frac{1}{2} L_1^T L_2 + \sum_{i=1}^{N} \lambda_{i2} |V(t - \tau_1(t)) |ight]
\]
\( \beta = \lambda_{\text{max}} \left[ 2b \Gamma + \sum_{i=1}^{N} \lambda_{i1} |V(t - \tau_2(t)) |ight]
\]
and \( 0 < \beta < \gamma \) holds.

Therefore, considering Lemma 1 and Lemma 2, for \( t \in [t_k - 1, t_k) \), we obtain
\[
\mathcal{V}(t) \leq \mathcal{V}(t_{k-1}) e^{-\theta(t - t_{k-1})}.
\]
\[
(25)
\]
Next, for a positive constant \( q \), we will prove that the following inequality holds for all \( t > t_0 > 0 \) by mathematical induction method
\[
\mathbb{E}V(t) \leq q^{k-1} \mathbb{E}V(t_0) e^{-\theta(t - t_0)}.
\]
\[
(26)
\]
As the first step, when \( k = 1 \), that is, \( t \in [t_0, t_1) \), for a positive constant \( q \) with \( q^0 = 1 \), then it follows from inequality (24) that
\[
\mathbb{E}V(t) \leq \mathbb{E}V(t_0) e^{-\theta(t - t_0)}.
\]
For the second step, we assume that the inequality (25) is satisfied for \( k = p \), the following purpose is to prove the inequality (25) also hold using the results derived in case of \( k = p \).

When \( k = p + 1 \), it follows from (16), (25) and the second condition (ii) in Theorem 1 that
\[
\mathbb{E}V(t) \leq \mathbb{E}V(t_0) e^{-\theta(t - t_0)}
\]
\[
= \max \left\{ \sup_{t \in [t_p - t_{p+1}]} \mathbb{E}V(s) e^{-\theta(t - t_0)} \right\}
\]
\[
\leq \max \{ q^{p+1} \mathbb{E}V(t_0) e^{-\theta(t - t_0)} \}
\]
\[
= \max \{ q^{p+1} \mathbb{E}V(t_0) e^{-\theta(t - t_0)} \}
\]
\[
\leq \max \{ q^{p+1} \mathbb{E}V(t_0) e^{-\theta(t - t_0)} \}
\]
\[
\leq \mathbb{E}V(t_0) e^{-\theta(t - t_0)}.
\]

Above all, we have verified that for \( t \in [t_k - 1, t_k) \),
\[
\mathbb{E}V(t) \leq q^{k-1} \mathbb{E}V(t_0) e^{-\theta(t - t_0)}.
\]
Recalling the definition of average impulsive interval for \( t > t_0 \), it gives
\[
\frac{t - t_0}{N_a} - N_0 \leq N_i(t, t_0) \leq \frac{t - t_0}{N_a} + N_0, \quad \forall t \geq t_0 \geq 0
\]
where \( N_i(t, t_0) \) denotes the number of impulsive times of the impulsive sequence \( \zeta = \{t_0, t_1, t_2, \cdots \} \) on the interval \( (t_0, t) \).

We will discuss the exponential convergence of the system by classifying \( \rho \).
1. When the positive constant \( q \geq 1 \), that is \( q^k \to +\infty \), \( k \to +\infty \). Thus, for \( t \in [t_{k-1}, t_k) \), \( k=1, 2, \cdots \)

\[
\mathbb{E} V(t) \leq q^{-1} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

\[
\leq q^{N_0} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

\[
\leq q^{N_0} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

\[
= q^{N_0} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

(27)

2. On the other side, if \( 0 < q < 1 \), that is, \( q^k \to 0 \), \( k \to +\infty \), for \( t \in [t_{k-1}, t_k) \), \( k=1, 2, \cdots \)

\[
\mathbb{E} V(t) \leq q^{-1} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

\[
\leq q^{-N_0} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

(28)

\[
= q^{-N_0} \mathbb{E} V(t_0) e^{-\lambda (t-t_0)}
\]

Remark 1

In this article, we have analyzed the outer exponential synchronization between two different systems, that is, the master system and the slave system. By imposing some suitable control strategy on the slave system, we have forced the two systems to act the same as time goes to infinity. However, in some previous works such as [20,23,29], the synchronization state is \( s(t) \) which is a solution of an isolated system. This means that the ultimate state of all the nodes of the network are forced to the fixed state \( s(t) \).

Remark 2

As most natural networks such as the metabolic networks, NNs, and artificial networks like software network and scientific citation network are generated randomly, the networks may be easy subject to instantaneous disturbances and experience abrupt changes [29,33], for instance, the white noise. Considering this situation, we have analyzed a more generalized model by modeling the stochastic phenomena, the Brownian motion (Wiener process) in this article when investigating the exponential synchronization between the master system and the slave system. Furthermore, time-varying delays [35–37] in the stochastic disturbances are considered. Hence, our study is a more general one compared with previous works.

Remark 3

Every system has both continuous-time dynamics and discrete-time dynamics which are described by coupled differential equations and impulsive quantitative relation equations, respectively. Thereinto, the "impulsive" means that the information interchanging between nodes occurs impulsively at discrete time instants. More importantly, the time-varying delay \( \tau (t_k) \) exists in the impulsive transmission, which means the differences between before impulsive happens and after impulsive occurs.

Now, we would like to give some corollaries based on Theorem 1. As a special situation, we consider the master system and the slave system without stochastic disturbances. Then, we can obtain the following error system by simply computing

\[
\begin{align*}
\dot{e}_i(t) &= Ae_i(t) + Bf_i(e_i(t)) + C(y_i(t) - y_i(t_0)) + e_1 \sum_{j=1}^{N} \tilde{y}_j(t) + e_2 \sum_{j=1}^{N} \tilde{y}_j(t) + e_3 \sum_{j=1}^{N} \tilde{y}_j(t) + \cdots \cdots \\
&+ e_1 \sum_{j=1}^{N} \tilde{y}_j(t) + e_2 \sum_{j=1}^{N} \tilde{y}_j(t) + e_3 \sum_{j=1}^{N} \tilde{y}_j(t) + \cdots \cdots \\
&+ \Delta e_i(t) = \mu_i e_i(t) + \rho_i e_i(t) - \tau_i(t) + e_i(t) \tau_i(t), \quad t \neq t_k, k=1, 2, \cdots \\
&\dot{e}_i(t) = \phi(t) e_i(t) \in C([-\tau, 0], \mathbb{R}^n), \quad -\tau \leq t \leq 0, i=1, 2, \cdots, N.
\end{align*}
\]

(30)

And, we give the following corollary with respect to Theorem 1. The proof is just a simplified process of the proof of Theorem 1, here we omit it due to the space limitation.

Corollary 1

Assume that Hypothesis 1 holds and the average impulsive interval of the impulsive sequence \( \zeta = \{1, 2, \cdots \} \) is less
than $N_p$. Then, the error system (29) with impulsive mixed time-varying delays under the control scheme (3) is said to be globally and exponentially synchronized in mean square if there exist positive constants $\alpha' > 0$, $b' > 0$, and $q'$ > 0 such that the following conditions hold

i. Condition (i) in Theorem 1;

ii. For the same $\tau_i, m_k, n_k, k=1, 2, \ldots$, the following satisfies

$$\max\{e^{\lambda t}, m_k + n_k e^{\lambda t}\} \leq q',$$  

(31)

where the constant $\lambda' > 0$ is the unique solution of the equation

$$\dot{x}' - \lambda' + b' e^{\lambda t} = 0$$

with $\alpha' > b' > 0$, $x' = -\lambda_{\text{max}}[A + BB^T + 2L_1^T L_1 + \frac{1}{2}CC^T - 2\alpha \Gamma]$, $\tilde{\beta} = \tilde{\beta}' + P \tilde{\beta}$, $\bar{\beta}' = \bar{\beta} + P \bar{\beta}$, and $\overline{P} = 2B' \max_{1 \leq s \leq n}(\gamma_s)$;

iii. The exponential convergence rate satisfies

$$\frac{\ln q'}{N_0} - \lambda' < 0.$$  

(32)

That is, the master system without stochastic disturbances is globally exponential synchronized in mean square to the slave system under the designed control strategy (3).

As another special case, let us consider the following impulsive coupled error system

$$\begin{cases}
\dot{e}_i(t) = [Ae_i(t) + Bf_i(e_i(t)) + CF_i(e_i(t - \tau(t)))] + c_1 \sum_{j=1}^{N} I_{ij} \Gamma e_j(t) + c_2 \sum_{j=1}^{N} g_{ij} \Gamma e_j(t - \tau_j(t))] dt \\
+ h_i(e_i(t), e(t - \tau(t)), e(t - \tau_j(t))) d\Omega_i(t) - d_i \Gamma e_i(t) dt, \ t \neq t_k, \\
\Delta e_i(t_k) = \mu_k e_i(t_k^+), \ t = t_k, \\
e_i(t) = e_i(t) \in C([-\tau, 0], \mathbb{R}^n), \ -\tau \leq t \leq 0.
\end{cases}$$

(33)

Similar to the above analysis, we give another corollary.

**Corollary 2**

Assume that Hypothesis 1 holds and the average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is less than $N_p$. Then, the error system (32) with impulsive mixed time-varying delays under the control scheme (3) is said to be globally and exponentially synchronized in mean square if

i. Condition (i) in Theorem 1;

ii. For the upper bound of the time delay $\tau = \max\{\tau_1, \tau_2\}, 0 \leq \tau_j(t) \leq \tau_j, j=1, 2, \ldots$, and the impulsive effect $\mu_k, k=1, 2, \ldots$, the following satisfies

$$\max\{e^{\lambda t}, m_k + n_k e^{\lambda t}\} \leq q',$$  

(31)

where the constant $\lambda' > 0$ is the unique solution of the equation

$$\dot{x}' - \lambda' + b' e^{\lambda t} = 0$$

with $\alpha' > b' > 0$, $x' = -\lambda_{\text{max}}[A + BB^T + 2L_1^T L_1 + \frac{1}{2}CC^T - 2\alpha \Gamma]$, $\tilde{\beta} = \tilde{\beta}' + P \tilde{\beta}$, $\bar{\beta}' = \bar{\beta} + P \bar{\beta}$, and $\overline{P} = 2B' \max_{1 \leq s \leq n}(\gamma_s)$;

iii. The exponential convergence rate satisfies

$$\frac{\ln q'}{N_0} - \lambda' < 0.$$  

(32)

where the constant $\lambda' > 0$ is the unique solution of the equation

$$\dot{x}' - \lambda' + \beta e^{\lambda t} = 0$$

with $\alpha' > \beta > 0$, $a' = \lambda_{\text{max}}[A + BB^T + 2L_1^T L_1 + \frac{1}{2}CC^T - 2\alpha \Gamma]$, $\tilde{\beta} = \tilde{\beta}' + P \tilde{\beta}$, $\bar{\beta}' = \bar{\beta} + P \bar{\beta}$, and $\overline{P} = 2B' \max_{1 \leq s \leq n}(\gamma_s)$;

iii. The exponential convergence rate satisfies

$$\frac{\ln q'}{N_0} - \lambda' < 0.$$  

(32)

**Remark 4**

In [38], authors investigated a kind of stochastic dynamical networks using impulsive strategy, and they derived sufficient conditions for synchronization of the networks so that the whole state-coupled dynamical network can be forced to some desired trajectory by placing impulsive controllers on the nodes in the networks. Even though, the results in [38] are effective, they did not consider the situation that there exist time delays due to the speed of the interconnection and the congestion of the information channels when the nodes transmit the information with each other. One can find that the model discussed in Corollary 2 in this article, to some extent, has expanded the model in [38]. Therefore, we have studied more realistic problems in this article.

**4. Numerical Simulation**

We present an example for illustrating the validity of the theory in the previous sections.

Consider the following chaotic NN with time delay [29]

$$dx(t) = [Ax(t) + Bf_1(x(t)) + CF_2(x(t - \tau_1(t))) + I(t)] dt$$

(36)

with parameters $f_1(x) = f_2(x) = \tan h(x), \tau_1(t) = 1$,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 4.5 \end{bmatrix}, \quad C = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix},$$

$$I(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

With simple calculation, it is clear that $L_1 = L_2 = I_2$ for the Lipschitz constant matrices in Hypothesis 1. Select the initial values as $x(t) = 0.2, x(t) = 0.3$, for $t \in [-1, 0]$, then we plot the phase trajectory of the isolated NN in Figure 1(a), which shows a chaotic attractor of system (35). And, Figure 1(b) is the state-variable evolution curves of each state in this system.
In the following, we choose a coupled NN consists with four nodes to simulate the main result in this article. Take \( \Gamma=I_2 \), \( \tau_2(t)\neq 0.1 \cdot e^{t/\tau} \), and select the coupling strength \( c_1=5, c_2=0.1 \), respectively. Also, the matrices for nontime delay and time-varying delay couplings are given as follows

\[
L=\begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -3 & 2 & 0 \\
0 & 3 & -4 & 1 \\
2 & 0 & 0 & -2
\end{pmatrix}, \quad G=\begin{pmatrix}
-2 & 1 & 1 & 1 \\
0 & -2 & 1 & 1 \\
1 & 0 & -2 & 1 \\
1 & 1 & 0 & -2
\end{pmatrix},
\]

and choose \( a=50, b=2 \). The stochastic matrix

\[
\hat{h}_i(x(t),x(t-\tau_1(t)),x(t-\tau_2(t)))=0.1 \cdot \text{diag}\{x_1^t-x_1^{t-1}, x_2^t-x_2^{t-1}\}.
\]

By simple calculation, we obtain constant matrices \( A_{ij} = 0.01I_2 \) for the Hypothesis 2, \( i=1, \ldots, 4, j=1, 2 \). In the simulation, we select the impulsive distance \( t_k-t_{k-1}=0.1 \), that is, the average impulsive interval is not larger than a constant \( N_0=0.1 \). Using LMI toolbox in MATLAB, we can calculate the parameters which satisfy the requirements in Theorem 1 with \( D=30I_4 \) for the feedback control gain matrix. Then, by solving inequality (13) for the second requirement in Theorem 1, we obtain \( z=58.7001, \beta=\hat{\beta}+\beta=4.5800 \). Further, we calculate \( \lambda=11.6478 \) which is the unique solution of equation \( \lambda-\beta+\beta e^{\lambda}/\lambda=0 \).

From (13), \( \mu_k=0.05 \) and \( \rho_k=0.01 \) are selected for all time instants, \( k=1,2, \ldots \). It follows that \( m_k=1.1130, n_k=0.0106 \). And choose the impulsive time delay \( \tau_3(t)=0.2 \), thus, \( \tau=\max\{\tau_1, \tau_2, \tau_3\}=1 \), \( e^{\tau}=1.2623 \), and \( m_k+n_k e^{\tau}=1.2218 \). According to the above analysis, we choose \( q>\max\{e^{\tau}, m_k+n_k e^{\tau}\}=1.3 \), and then by simply computing, it comes the result of the convergence rate that \( \lim \frac{\ln x}{N}=-\lambda = -9.0242 < 0 \), which implies that requirement (14) in Theorem 1 is satisfied. Therefore, in this situation, the master NN is forced to the slave system, that is, the two systems are achieved the exponential synchronization in mean square by applying the designed controllers.

We plot two state evolution curves for the two networks in Figure 2, it can be concluded that the master system and the slave system achieve the synchronization as all of the first state of every node act the same, and so does the second state. Here, it is noted that we give the Winner process in Figure 3(a). According to the definition of the error \( e_i(t)(i=1,2,3,4, j=1,2) \) for this impulsive controlled NN, some graphs are plotted in Figure 3(b). From the error evolution graph (b) in Figure 3, the two NNs consisting of eight nodes are achieved the synchronization with the convergence rate (14) presented in the main theorem.

**Remark 5**

Authors in [31,33,38] investigated the synchronization problem of complex dynamical networks or NNs with impulsive disturbances, but they did not consider the situation that the impulsive effects could either play a positive role or a negative role, and also, there were no time-varying delays within the impulsive process. As you can see in the simulation part, comparing with previous works, we set the impulsive delay as 0.2, that means the whole NN could be affected by impulsive effects due to the existing of delays. And finally, the NNs realize exponential synchronization in mean square within 13 s.
5. CONCLUSION

In this article, we have discussed the mean square global exponential synchronization of impulsive coupled NNs with mixed time-varying delays. The stochastic disturbances with time-varying delays have been taken into consideration when we model the network to describe more practical applications in real world. Considering the fact that the impulsive effect between every nodes in the network could play either a positive role or negative role, we have utilized an effective control strategy which contains both feedback effects and impulsive effects, and then sufficient criteria for the mean square exponential synchronization of two coupled NNs have been derived by applying the Lyapunov stability theorem, the comparable principle, and mathematical induction. It has been proved that the control strategy used in this article is an effective way for the realization of the synchronization. Besides, the numerical simulation has verified the main results of this article.

However, in reality world, the structure of the whole NNs could be changed as time goes by, it could follow certain rules, such as Markov switching. Therefore, this kind of issues could be further considered in our future works.
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