Abstract—A number of iterative learning control algorithms have been developed in a stochastic setting in recent years. The results currently available are in the form of algorithm derivation and the derivation of fundamental systems theoretic properties. This paper gives the results of an optimal iteration-varying stochastic ILC algorithm on a gantry robot system, which confirms that this algorithm is capable of delivering good performance in the experimental domain and can outperform a heuristic filter in the later learning stage when noise starts to dominate the signal.

I. INTRODUCTION

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive, or trial-to-trial, mode with the requirement that a reference trajectory $r(t)$ defined over a finite interval $0 \leq t \leq T$, where $T$ denotes the trial length, is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task to high precision, chemical batch processes or, more generally, the class of tracking systems. Since the original work [1], the general area of ILC has been the subject of intense research effort. An initial source for the literature here are the survey papers [2], [3]. Application areas include robotics, automated manufacturing plants and food processing.

A standard assumption in much of the current ILC literature is that the input signals are free of noise, although in many physical implementations measurement signals are subject to random noise and the system may be affected by random disturbances. A number of specialized algorithms have been developed for systems with stochastic disturbances [4], [5], [6]. The learning algorithms developed in this previous work provide asymptotic noise and non-repeating disturbance insensitivity but use a trial-varying, also termed iteration-varying, learning filter. This is in contrast to the many cases where iteration-invariant filters are employed.

Recent work [7] has developed algorithms for iteration-invariant learning filters for mixed deterministic and stochastic disturbances for single-input single-output (SISO) systems with linear time-invariant dynamics, but as yet there has been no experimental verification. This paper gives substantial new results in this area which is an essential precursor to industrial take-up.

II. BACKGROUND

The work in [7] is frequency domain based and it is assumed that the plant dynamics are modeled in discrete-time by

$$e_k(t) = -G(z)u_k(t) + d(t) + w_k(t)$$

where $t = 0, 1, \cdots$ is the time index, $k = 0, 1, \cdots$ is the iteration index, $z$ is the forward time-shift operator $zv(t) = x(t+1)$, $z^{-1}$ is the backward time-shift operator $u$ is the control input, $e$ is the error, $d$ is a deterministic signal, $w$ is a stationary random disturbance, and $G$ is a stable system. Moreover, $d(t)$ captures iteration-invariant disturbances [8] and initial conditions [9].

Consider also the application of a standard first order iteration-varying ILC law of the form

$$u_{k+1}(t) = Q(z)[u_k(t) + L(z)\hat{e}_k(t)]$$

where $\hat{e}_k(t)$ is the noise corrupted error measurement modeled as

$$\hat{e}_k(t) = e_k(t) + v_k(t)$$

and $v_k(t)$ is stationary random noise. Figure 1 shows how the learning update loop, learning filter $L_k(z)$ and robust filter $Q(z)$ are combined to form the overall control scheme, with the following assumptions: i) $u_0(t) = 0$, ii) $|d(t)| < M$, iii) $E[w_{j_1}(t_1)w_{j_2}(t_2)] = 0$, for all $j_1, j_2, t_1, t_2$, iv) $E[w_{j_1}(t_1)w_{j_2}(t_2)] = 0$, $E[v_{j_1}(t_1)v_{j_2}(t_2)] = 0$, $E[w_{j_1}(t_1)d(t_2)] = 0$, $E[v_{j_1}(t_1)d(t_2)] = 0$, for all $j_1 \neq j_2$ and all $t_1, t_2$, and v) $G(z)$ is a rational function with relative degree 0. If the relative degree is greater than zero then the analysis is easily modified as detailed in Section II of [7]. Finally, the spectrum of a signal, say $r(\gamma)$, as

$$\Phi_r(\omega) = \sum_{\infty} R_r(\tau)e^{-j\omega\tau}$$

where $R_r(\tau)$ is the autocorrelation function.

![Fig. 1. Diagram of the ILC control structure](image-url)
In order to obtain the power spectrum of the asymptotic error together with sufficient conditions for its convergence and the iteration-domain convergence rate, the first step is to use the equations describing the signal flow in Figure 1 to write

\[ e_j(t) = Q(z)[1 - L(z)G(z)]e_{j-1}(z) \]
\[ + (1 - Q(z))d(t)w_j(t) - Q(z)w_{j-1}(t) \]
\[ - Q(z)L(z)P(z)w_{j-1}(t) \] (4)

It is not possible to find the power spectrum of \( e_j \) from the recursive solution of (4) since \( e_{j-1} \) and \( w_{j-1} \) are correlated. Instead, it can be shown that the non-recursive solution of (4) is

\[ e_j(t) = X_j(z)d(t) \]
\[ + \sum_{i=0}^{j-1} Y_i(z)(w_{j-1-i}(t) + v_{j-1-i}(t)) + w_j(t) \] (5)

where

\[ X_j(z) = [Q(z)(1 - L(z)G(z))]^j \]
\[ + \sum_{i=0}^{j-1} [Q(z)(1 - L(z)G(z))]^i(1 - Q(z)) ] \]
\[ Y_i(z) = -[G(z)(1 - L(z)G(z)][Q(z)L(z)G(z)] \]

Using (4), the power spectrum of the error on iteration \( j \) is

\[ \Phi_{e_j} = |X_j(e^{j\omega})|^2 \Phi_d(\omega) \]
\[ + \sum_{i=0}^{j-1} |Y_i(e^{j\omega})|^2 (\Phi_w(\omega) + \Phi_v(\omega)) + \Phi_v(\omega) \] (6)

The following results are proved in [7].

**Theorem 1:** If

\[ \max_{\omega \in [\pi, \pi]} |Q(e^{j\omega})(1 - L(e^{j\omega})G(e^{j\omega}))| < 1 \]

the error spectrum converges and

\[ \Phi_{e\infty}(\omega) := \lim_{j \to \infty} \Phi_{e_j}(\omega) \]

exists and is given by

\[ \Phi_{e\infty}(\omega) = \frac{1}{W} \left| (1 - Q(e^{j\omega}))^2 (\Phi_d(\omega)) + \Phi_w(\omega) \right| \]
\[ + \frac{1}{W} \left| (Q(e^{j\omega})L(e^{j\omega})G(e^{j\omega}))^2 (\Phi_w(\omega) + \Phi_v(\omega)) \right| \] (7)

where

\[ W = |1 - Q(e^{j\omega})(1 - L(e^{j\omega})G(e^{j\omega}))|^2 \]

The necessary condition for convergence of the power spectrum here is the familiar frequency domain stability condition (see, for example, [9] for ILC but can also be shown to be related to the convergence rate of the spectrum.

Consider now the class of model-inversion learning functions

\[ L(e^{j\omega}) = \eta(\omega)G^{-1}(e^{j\omega}) \] (8)

where \( \eta \) is the real valued inversion gain. Also write \( Q(e^{j\omega}) \) in Euler form

\[ Q(e^{j\omega}) = \zeta(\omega)e^{j\psi(\omega)} \]

Then on substitution in (7), the problem is to find the best filter design, that is, \( \eta^*(\omega) \), \( \zeta^*(\omega) \) and \( \psi^*(\omega) \) that minimizes the asymptotic power spectrum. Further analysis, however, shows that as the minimum asymptotic power spectrum is approached, the convergence rate approaches unity. Hence it is necessary to find a trade-off between asymptotic performance and convergence rate.

The optimal design problem with model-inversion learning (8) and a maximum desired convergence rate \( \gamma \) find \( \eta^*(\omega) \), \( \zeta^*(\omega) \) and \( \psi^*(\omega) \) that solve

\[ \min_{\eta, \zeta, \psi} \gamma : \gamma \leq \gamma < 1 \] (9)

where

\[ \gamma = \max_{\omega \in [\pi, \pi]} |Q(e^{j\omega})(1 - L(e^{j\omega})G(e^{j\omega}))|^2 \] (10)

The solution of this problem is given as Theorem 2 in [7] (together with some extensions).

**A. Filter Construction**

The optimal \( Q \) and \( L \) filters of the previous section to exactly meet the optimal specifications in Theorem 2 of [7]. Moreover, in some applications it may not be cost-effective to develop accurate noise and disturbance spectra for optimally shaping the learning and \( Q \)-filters. An alternative in such cases is to use simple design guidelines based on an approximate, or assumed, deterministic-stochastic ratio (DSR) defined as \( \frac{\Phi_d(\omega)}{\Phi_w(\omega) + \Phi_v(\omega)} \).

The rationale behind the DSR is that at frequencies where the stochastic noise is very small, there is no penalty to fast unfiltered learning and hence for large DSR set \( \zeta^*(\omega) = 1 \) and \( \eta^*(\omega) = 1 \). At frequencies where the deterministic error is very small, there is little advantage to learning and hence set \( \zeta^*(\omega) = 0 \) or \( \eta^*(\omega) = 0 \). These guidelines are frequency dependent and may shape \( \zeta(\omega) \) and \( \eta(\omega) \) as the DSR changes in different frequency bands.

**III. EXPERIMENTAL VERIFICATION**

To progress beyond theory, supporting experimental evidence is required and in the remainder of this paper results for the filter design method of Section II-A are given. These have been obtained by implementing the algorithm of the previous section on a gantry robot, see Figure 2), which performs a “pick and place” task and is similar to systems which can be found in industry. Such industrial processes include food canning, bottle filling or automotive assembly, all of which require accurate tracking control with a minimum level of error in order to maximize production rates and minimize loss of product due to faulty manufacture.

The gantry robot is constructed from two types of linear motion device. The \( X \)-axis comprises the lowest horizontal section, and consists of one brushless linear dc motor and
frequency response obtained for the X-axis is shown in Figure 3 (the responses of the other axes are similar).

The X-axis dynamics are approximated by the 7th order transfer-function

\[
G_X(z) = \frac{0.00051745(z + 0.5823)(z - 0.3014)}{(z - 1)(z^2 - 0.07057z + 0.009459)} \\
\frac{(z^2 - 0.09718z + 0.008969)(z^2 - 0.2046z + 0.7846)}{(z + 0.3149z + 0.1024)(z^2 - 0.7757z + 0.5403)}
\]

(11)

where \( G^{-1}(z) \) will be implemented using the matrix \( G \) defined using the Markov parameters as

\[
G = 
\begin{bmatrix}
  CB & 0 & 0 & \cdots & 0 \\
  CAB & CB & 0 & \cdots & 0 \\
  CA^2B & CAB & CB & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  C A^{N-1}B & CA^{N-2}B & \cdots & \cdots & CB
\end{bmatrix}
\]

(12)

The required state-space model for the X-axis is

\[
A_X = 
\begin{bmatrix}
  2.41 & -0.86 & 0.85 & -0.59 & 0.30 & -0.19 & 0.32 \\
  4.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1.00 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0.50 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0.25 & 0 & 0
\end{bmatrix}
\]

\[
B_X = 
\begin{bmatrix}
  0.0313 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0.0095 & -0.0023 & 0.0048 & -0.0027 & 0.0029 & 0.0011 & 0.0029
\end{bmatrix}^T
\]

\[
C_X = 
\begin{bmatrix}
  0 & 0.09718 & 0.008969 & -0.09718 & 0.008969 & 0 & 0
\end{bmatrix}
\]

The trial duration for the “pick and place” operation is 2 seconds, which is equivalent to 30 units per minute (UPM). Figure 4 shows the reference trajectory. The stoppage time between each trial is to compute the control vector for the subsequent trial. The gantry axes are homed to a predefined point before each iteration begins with an accuracy of ±30 microns in order to minimize the effects of initial state error. A sampling time of \( T_s = 0.01 \)s has been used in all tests. Here only X-axis has been used for initial investigation, for which the reference trajectory is given in Figure 5. This facility has been widely used to test other (deterministic) ILC control algorithms, see, for example, [10], [11].

To obtain the disturbance model from the experiment rig, a number of zero input signals have been fed into the system and the output and error signals are measured. When input applied to (1) is zero, \( e_k(t) = d(t) + w_k(t) \) and when a number of zero input signals are used

\[
\sum_{k=1}^{N} e_k(t) = \sum_{k=1}^{N} d(t) + \sum_{k=1}^{N} w_k(t)
\]

where \( N \to \infty \), \( \sum_{k=1}^{N} w_k(t) = 0 \), and hence

\[
\sum_{k=1}^{N} e_k(t) \approx N d(t)
\]

\[
d(t) \approx \frac{1}{N} \sum_{k=1}^{N} e_k(t)
\]

(13)

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From the resulting measurements, linear approximations of the transfer function for each axis were determined and then refined using a non-linear optimization technique. The
Hence on each iteration

\[ w_k(t) \approx e_k(t) - d(t); \quad (14) \]

and the spectra can be computed using

\[ \Phi_w(\omega) \approx \frac{1}{N} \sum_{k=1}^{N} |FFT[w_k(t)]|^2 \quad (15) \]

Hence a transfer function, or even a constant for simplicity, can be fitted to \( \Phi_d(\omega) \) and \( \Phi_w(\omega) \). Also in implementation the random noise term \( \Phi_e(\omega) \) can be assumed to be zero or a very small constant offset.

Figure 6 gives the Bode gain plots of \( \Phi_d(\omega) \) and \( \Phi_w(\omega) \) obtained by application of the procedure described above. The filters that approximate the optimal \( \eta^* \) and \( \psi^* \) must have zero-phase and this is emulated here by applying the MATLAB \textit{filtfilt} technique to a fourth order low-pass filter to form the iteration-varying optimal filter

\[ H(z) = \frac{0.0002 + 0.0007z^{-1} + 0.0011z^{-2}}{1 - 3.5328z^{-1} + 4.7819z^{-2} - 2.9328z^{-3} + 0.6868z^{-4}} \quad (16) \]

Figure 7 shows the Bode gain plots for the learning filters on trial 0 (the initial filter), 5, 10, 15 and 20. The heuristic filter [12]:

\[ L_4(z) = \frac{1}{k+1} \cdot G^{-1}(z) \quad (17) \]

has similar gain at low frequency but at higher frequency the optimal filters can minimize the effect of amplifying the noise.

A. Experimental Results

Experiments have been carried out on the gantry robot using both the optimal and heuristic filters. In the first set of tests, the ILC algorithms were applied directly to the robot and in a further set they were applied to the closed-loop system resulting from application of a Proportional-plus Integral plus Derivative (PID) controller, as often arises in some applications, that is, the need to pre-stabilize the plant model or deal with undesired along the iteration dynamics. Figure 8 shows the mean squared error of both optimal and heuristic filters for the first set of experiments. As the iteration number increases, the tracking error is reduced and the noise starts to dominate the signal, and the performance of optimal filters is better. Figure 9 gives the results when the PID controller is used, In this case the noise starts to dominate from the very early stage of learning progress and the iteration-varying optimal filters again clearly outperform...
the heuristic filter. Figures 10 and 11 show the input, output

and error signals for each set for the filters designed in this paper and confirm that excellent performance is achieved.

IV. Conclusions

An ILC algorithm for systems with stochastic disturbances has been implemented on a multi-axis gantry robot and the performance assessed. The performance of the ILC controller
has been examined for the cases when a pre-conditioning PID feedback controller has been applied and when this is excluded and the ILC applied directly to the robot. In both cases, the performance obtained with the filters of this paper outperform heuristic filters previously proposed in the literature.

These results are for one case only but provide evidence to support further development of the filters considered in [7]. In due course it will also be necessary to compare performance against other model based stochastic based ILC algorithms. Aspects of this wide problem area are currently under investigation.

REFERENCES