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The original model-theoretic motivation for studying finite covers was to obtain a better understanding of the totally categorical structures (and later, the class of smoothly approximated structures [13]). At a naïve level, it is not hard to see that this class is closed under taking finite (and more generally, affine) covers. On a much deeper level, we have the results of Cherlin, Harrington, Lachlan ([12]), Zil'ber ([49]) and Hrushovski ([34]) describing how these structures can essentially be built up from basic, well-understood objects (Grassmannians of strictly minimal sets) by taking a sequence of covers (finite or affine).

However, it has emerged that there is a fairly rich theory of finite covers whose natural level of generality is far beyond this original context, and it is this which we outline in these notes. As our main concern is with $\aleph_0$-categorical structures, most of the results and methods are phrased in terms of permutation groups (the automorphism groups of the structures), which we usually think of as topological groups.

These notes (which are not intended for publication) are a condensed and truncated version of [26], which is a much more exhaustive survey of the area. I thank Sasha Ivanov and Dugald Macpherson for allowing me to use [26] in this way.

Contents

1 Basic definitions and results

1.1 Permutation structures

If W is any set then the symmetric group $\mathrm{Sym}(W)$ on W can be considered as a topological group by taking as open sets arbitrary unions of cosets of pointwise stabilisers of finite subsets of W. In this topology, closed subgroups are precisely automorphism groups of first-order structures with domain W. In fact, if H is a subgroup of $\mathrm{Sym}(W)$ then the closure of H in $\mathrm{Sym}(W)$ is the set of elements of $\mathrm{Sym}(W)$ which, for each $n \in \mathbb{N}$, preserve each H-orbit on W^n. Thus we employ the following notation and terminology.

Definition 1.1.1 A *permutation structure* is a pair $\langle W; G\rangle$ where W is a non-empty set (the *domain*), and G is a closed subgroup of $\mathrm{Sym}(W)$ (the group of *automorphisms*). We shall usually write $G = \mathrm{Aut}(W)$ and refer simply to 'the permutation structure W.' If A is a subset of W and B a subset of W (or more generally of some set on which $\mathrm{Aut}(W)$ is acting in an obvious way), then $\mathrm{Aut}(A/B)$ denotes the permutations of A which extend to elements of $\mathrm{Aut}(W)$ fixing every element of B. We regard $\mathrm{Aut}(W)$ as a topological group with the subspace topology from $\mathrm{Sym}(W)$: a base of open neighbourhoods of the identity consists of subgroups $\mathrm{Aut}(W/X)$ for finite $X \subseteq W$. We shall write permutations on the left of the elements of W.

In practice, the permutation structures we consider are obtained by taking automorphism groups of first-order structures on W, and we often regard a first-order structure as a permutation structure without explicitly saying so (by taking for the group of automorphisms of the permutation structure the automorphism group of the first-order structure). Of course, it is most interesting to do this when some model- theoretic property guarantees that the first-order structure has a 'rich' automorphism group. The strongest such property is $\aleph_0$-categoricity, where the Ryll-Nardzewski theorem shows (for countable W) that the automorphism group is *oligomorphic*, that is, $\mathrm{Aut}(W)$ has finitely many orbits on W^n for all $n \in \mathbb{N}$.

A permutation structure $\langle W; G\rangle$ is *primitive* if the only G-invariant equivalence relations on W are equality and the universal relation (note that this implies that G must act transitively on W). A transitive permutation structure is primitive if and only if the stabiliser of a point of W is a maximal subgroup of G.

If W_1 and W_2 are sets of the same cardinality then any bijection $\phi : W_1 \to W_2$ induces an isomorphism $f_\phi : \mathrm{Sym}(W_1) \to \mathrm{Sym}(W_2)$. We say that permutation structures $\langle W_1; G_1\rangle$ and $\langle W_2; G_2\rangle$ are *isomorphic* if for some bijection ϕ we have $f_\phi(G_1) = G_2$. (As pointed out to us by Martin Ziegler this produces a slight conflict in terminology: the group of isomorphisms from a permutation structure $\langle W; G\rangle$ to itself is actually the normaliser in $\mathrm{Sym}(W)$ of G, so it might be more correct to refer to this as the 'automorphism group of the permutation structure,' rather than G.)

Two permutation structures are *bi-interpretable* if their automorphism groups are isomorphic as topological groups. If the permutation structures arise from countable $\aleph_0$-categorical structures there is a model-theoretic interpretation of this notion due to G. Ahlbrandt and M. Ziegler ([2]: see also Section 7 of [40]). The following useful observation is due to E. Hrushovski ([34]).

Lemma 1.1.2 *A permutation structure $\langle W; G\rangle$ such that G has finitely many orbits on W is bi-interpretable with a transitive permutation structure $\langle W_1; G_1\rangle$.*

Proof. Let $\bar{x}$ be a finite tuple of elements from W containing (at least) one element from each G-orbit. Let W_1 be the orbit under G of $\bar{x}$. We get a natural continuous, injective homomorphism $G \to \mathrm{Sym}(W_1)$, and it is easy to see that the image G_1 of this is closed in $\mathrm{Sym}(W_1)$. The inverse map $G_1 \to G$ is also continuous, and so we have the result. □

Related to this construction is the notion of a *Grassmannian* of a transitive permutation structure W. First recall that if W has the property that $\mathrm{Aut}(W/X)$ has finitely many finite orbits for all finite subsets X of W then we define the *algebraic closure* $\mathrm{acl}(X)$ of X to be the union of the finite $\mathrm{Aut}(W/X)$-orbits. This is a closure operation on the finite subsets of W. If A is a finite algebraically closed subset of W then the Grassmannian $\mathrm{Gr}(W;A)$ is the permutation structure having domain $W_A = \{gA : g \in \mathrm{Aut}(W)\}$ and automorphism group those permutations induced on this set by $\mathrm{Aut}(W)$. To see that this is a closed subgroup of $\mathrm{Sym}(W_1)$ observe that, as in Lemma 1.1.2, the group of permutations induced by $\mathrm{Aut}(W)$ on the orbit of an enumeration of A is closed, and there is an invariant finite-to-one map from this orbit to W_A, so what we want follows from Lemma 1.3.3. If $\mathrm{Aut}(W)$ acts faithfully on W_A then $\mathrm{Gr}(W;A)$ is bi-interpretable with W.

We shall frequently employ the following terminology. Suppose C_0 and C are permutation structures with the same domain, and $\mathrm{Aut}(C) \leq \mathrm{Aut}(C_0)$. Then we say that C_0 is a *reduct* of C, or C is an *expansion* of C_0. We use the adjective *proper* to indicate that $\mathrm{Aut}(C) < \mathrm{Aut}(C_0)$.

1.2 Finite covers

We first give the group-theoretic definition of *finite cover*.

Definition 1.2.1 If C, W are permutation structures, then a finite-to-one surjection $\pi : C \to W$ is a *finite cover* if its fibres form an $\mathrm{Aut}(C)$-invariant partition of C, and the induced map $\mu : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ given by $\mu(g)w = \pi(g\pi^{-1}(w))$ for $g \in \mathrm{Aut}(C)$ and $w \in W$ has image $\mathrm{Aut}(W)$. The *kernel* of the finite cover is $\ker \mu = \mathrm{Aut}(C/W)$.

This is what will be used in these notes, but to provide a reference-point for model theorists, we give a model-theoretic version.

Definition 1.2.2 Let C and W be first-order structures. A finite-to-one surjection $\pi : C \to W$ is a *finite cover* of W if there is a 0-definable equivalence relation E on C whose classes are the fibres of π, and any relation on W^n which is 0-definable in the 2-sorted structure (C, W, π) is already 0-definable in W.

Observe that a finite cover (in the sense of 1.2.2) $\pi : C \to W$ induces a homomorphism

$$\mu : \mathrm{Aut}(C) \to \mathrm{Aut}(W),$$

given by putting $\mu(g)(w) = \pi(g\pi^{-1}(w))$ for all $g \in \mathrm{Aut}(C)$ and $w \in W$. In fact, if W is countable $\aleph_0$-categorical, then 1.2.2 is equivalent to saying that the fibres of π are the classes of an $\mathrm{Aut}(C)$-invariant equivalence relation on C, and the map $\mathrm{Aut}(C) \to \mathrm{Aut}(W)$ induced by π has image $\mathrm{Aut}(W)$ (Lemma 1.3.3 below ensures that Definition 1.2.2 implies the surjectivity). We refer to μ as the *restriction* homomorphism.

Suppose that $\pi : C \to W$ is a finite cover. Then $\mathrm{Aut}(C)$ has a normal subgroup K, the *kernel* of the cover, defined by

$$K := \{g \in \mathrm{Aut}(C) : \pi(x) = \pi(gx) \text{ for all } x \in C\},$$

(so also the kernel of the restriction homomorphism $\mathrm{Aut}(C) \to \mathrm{Aut}(W)$). We have a short exact sequence

$$1 \to K \to \mathrm{Aut}(C) \xrightarrow{\mu} \mathrm{Aut}(W) \to 1.$$

The cover *splits* if K has a closed complement in $\mathrm{Aut}(C)$, that is, there is a closed group $H \leq \mathrm{Aut}(C)$ such that $KH = \mathrm{Aut}(C)$ and $K \cap H = 1$. Equivalently, C is a reduct of a cover of W with trivial kernel (namely, a structure with automorphism group H).

For each $a \in W$ let $C(a)$ denote the fibre above a, that is $\{x \in C : \pi(x) = a\}$. We also define, for any $a \in W$, the *fibre group* of the cover at a as the permutation group induced by $\mathrm{Aut}(C)$ on $C(a)$. The *binding group* at a is a normal subgroup of the fibre group, and is the permutation group induced on a fibre $C(a)$ by the kernel K. Clearly, if $\mathrm{Aut}(W)$ acts transitively on W then all of the fibre groups are isomorphic as permutation groups, as are the binding groups. We refer to these as the fibre and binding groups of the cover. If these are unequal, we say that the cover is *twisted.*

We mention some special kinds of covers. We say that $\pi : C \to W$ is *free* if $\mathrm{Aut}(C/W) = \Pi_{w \in W} \mathrm{Aut}(C(w)/W)$, that is, the kernel is the full direct product of the binding groups (so as big as possible). At the other extreme, the cover is *trivial* if its kernel $\mathrm{Aut}(C/W)$ is the trivial group (this differs from the terminology in [3] and [4] where 'trivial' means 'split'). A *principal* cover $\pi : C \to W$ is a free finite cover where the fibre and binding groups at each point are equal. So the kernel of a principal cover is the direct product of all the fibre groups. A finite cover is called *superlinked* if its kernel is finite.

If C, C' are permutation structures with the same domain and $\pi : C \to W$ and $\pi' : C' \to W$ are finite covers with $\pi(c) = \pi'(c)$ for all $c \in C = C'$ then we say that π' is a *covering expansion* of π if $\mathrm{Aut}(C') \leq \mathrm{Aut}(C)$.

We say that finite covers $\pi_1 : C_1 \to W$ and $\pi_2 : C_2 \to W$ are *isomorphic* if there exists a bijection $\phi : C_1 \to C_2$ which sends the set of fibres of π_1 to the set of fibres of π_2 and such that the induced map $f_\phi : \mathrm{Sym}(C_1) \to \mathrm{Sym}(C_2)$ (as in Section 1.1) sends $\mathrm{Aut}(C_1)$ to $\mathrm{Aut}(C_2)$. If additionally $\phi(\pi_1^{-1}(w)) = \pi_2^{-1}(w)$ for all $w \in W$ then we say that π_1 and π_2 are isomorphic *over* W.

1.3 Topological arguments

We first record a triviality.

Lemma 1.3.1 *Let $\pi : C \to W$ be a finite cover. Then the restrction map $\mu : \mathrm{Aut}(C) \to \mathrm{Aut}(W)$ is continuous.*

Proof. A basic open set in $\mathrm{Aut}(W)$ is of the form $g\,\mathrm{Aut}(W/X)$ for some $g \in \mathrm{Aut}(W)$ and finite subset X of W. The preimage under μ of this contains $\hat{g}\,\mathrm{Aut}(C/\pi^{-1}(X))$ (where $\mu(\hat{g}) = g$) which is an open neighbourhood of $\hat{g}$. □

Lemma 1.3.2 *Let $\pi : C \to W$ be a finite cover with kernel K. Then*

(a) K is closed,

(b) K is compact.

Proof. (a) This follows from (1.3.1) as K is the kernel of a continuous homomorphism.

(b) Since the orbits of K are finite, K is a subgroup of a direct product of finite groups, and moreover the topology on K coincides with the relative topology on the direct product with the product topology. By Tychonoff's Theorem the latter is compact, so by (a), K is also compact. □

The next result is crucial to our use of topological arguments. In its full generality it appears as Lemma 1.1 of [21]. It was proved in [25], assuming C countable.

Lemma 1.3.3 *Let C be a permutation structure, W a set, and $\pi : C \to W$ be a finite-to-one surjection whose fibres form an $\mathrm{Aut}(C)$-invariant partition of C. Then*

(a) *The restriction map $\mu : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ (given by $\mu(g)(w) = \pi(g\pi^{-1}(w))$) for all $g \in \mathrm{Aut}(C)$ and $w \in W$) maps closed subgroups of $\mathrm{Aut}(C)$ to closed subgroups of $\mathrm{Sym}(W)$.*

(b) *If $\pi : C \to W$ is a finite cover, then the restriction map μ sends open subgroups of $\mathrm{Aut}(C)$ to open subgroups of $\mathrm{Aut}(W)$, so is an open map.*

Proof. We prove the results assuming C is countable.

(a) Let $(g_i : i \in \omega)$ be a sequence of permutations of C such that $(g_i \mid W : i \in \omega)$ converges to $h \in \mathrm{Aut}(W)$. By continuity of μ (see Lemma 1.3.1), it suffices to show that $(g_i : i \in \omega)$ has a convergent subsequence. Put $W := (w_i : i \in \omega)$, and enumerate C by enumerating $C(w_1), C(w_2), \ldots$

(where $C(w_i) := \mu^{-1}(w_i)$). By thinning out the g_i, we may assume that $g_i(w_j) = g_i'(w_j)$ and $g_i^{-1}(w_j) = g_{i'}^{-1}(w_j)$ whenever $i, i' \geq j$, so in particular $g_i, g_{i'} : C(w_j) \to C(hw_j)$ and $g_i, g_i^{-1} : C(w_j) \to C(h^{-1}w_j)$. It follows by the pigeon-hole principle that for each j there is infinite $I \subseteq \omega$ such that $g_i \mid C(w_j) = g_{i'} \mid C(w_j)$ and $g_i^{-1} \mid C(w_j) = g_{i'}^{-1} \mid C(w_j)$ whenever $i, i' \in I$. Apply this repeatedly to obtain the subsequence.

(b) By a result of Evans [19], the open subgroups of $\mathrm{Aut}(C)$ and $\mathrm{Aut}(W)$ are precisely the closed subgroups of countable index, so the result follows from (a). □

Corollary 1.3.4 *Let $\pi : C \to W$ be a finite cover with restriction map μ and kernel K. Then the natural group isomorphism $\mu_K : \mathrm{Aut}(M)/K \to \mathrm{Aut}(N)$ is a homeomorphism (where $\mathrm{Aut}(M)/K$ is given the quotient topology).*

Proof. This follows from Lemmas 1.3.1 and 1.3.3. □

1.4 Minimal covers

Definition 1.4.1 Suppose that $\pi : C \to W$ is a finite cover. Then we say that π is *minimal* if any proper expansion of C induces new structure on W. Equivalently, if $\mu : \mathrm{Aut}(C) \to \mathrm{Aut}(W)$ is the restriction map, π is minimal if, for every proper closed subgroup G_1 of $\mathrm{Aut}(C)$, $\mu(G_1) \neq \mathrm{Aut}(W)$.

The following group-theoretic result of Cossey, Kegel and Kovács [14] guarantees that an arbitrary finite cover can be expanded to a minimal one.

Proposition 1.4.2 *Let Γ and Σ be Hausdorff topological groups, and let $\mu : \Gamma \to \Sigma$ be a continuous epimorphism with compact kernel K. Then there is a closed subgroup $G \leq \Gamma$ such that $\mu(G) = \Sigma$, and for every proper closed subgroup H of G we have $\mu(H) < \Sigma$.*

Proof. Let $\mathcal{F}$ be the set of closed subgroups H of Γ such that $\mu(H) = \Sigma$. By Zorn's Lemma, it suffices to show that if $(H_i : i \in I)$ is a chain (under inclusion) of members of $\mathcal{F}$ then $H := \bigcap(H_i : i \in I)$ is in $\mathcal{F}$. Clearly H is closed. Let $g \in \Sigma$. Then there is $h_0 \in \Gamma$ such that $\mu(h_0) = g$. Now $\{h \in \Gamma : \mu(h) = g\}$ is just the compact set h_0K. Each set $H_i \cap h_0K$ is non-empty, so by compactness $H \cap h_0K$ is non-empty, and it follows that $H \in \mathcal{F}$. □

Corollary 1.4.3 *Let W be a permutation structure and let $\pi : C \to W$ be a finite cover. Then there is an expansion of C which is a minimal cover of W.*

Proof. This follows directly from (1.4.2) and (1.3.2). □

Corollary 1.4.4 *Let W be a permutation structure. Then the following are equivalent:*

(a) every finite cover of W splits;

(b) every minimal finite cover of W is trivial.

Proof. Suppose that (a) holds, and let $\pi : C \to W$ be a minimal finite cover with restriction map μ. By (a), its kernel K has a closed complement G. By minimality, $G = \mathrm{Aut}(C)$, so $K = 1$.

Conversely, assume (b), and let $\pi : C \to W$ be a finite cover. Then by Corollary 1.4.3 and (b), there is a minimal cover expanding C with automorphism group G and trivial kernel. Then G is a closed complement to the kernel of π. □

We shall state our next result in the broader language of topological groups. So we shall say that a continuous epimorphism of Hausdorff topological groups $\phi : G \to H$ is a *Frattini cover* if for every proper closed subgroup G_1 of G we have $\phi(G_1) \neq H$. (The reason for the terminology is that if G and H are profinite then ϕ is Frattini if and only if $\ker(\phi)$ is contained in the Frattini subgroup of G, that is, the intersection of the maximal open subgroups of G – see Section 20.6 of [27]). A version of the 'Frattini argument' for finite groups yields the following. It uses results from Section 20.10 of [27], where the Sylow theory for profinite groups is presented. (If p is a prime, a closed subgroup of a profinite group is a Sylow p-subgroup if the index of every open subgroup containing it is coprime to p, and no closed subgroup properly contained in it has this property.)

Lemma 1.4.5 ([22]) *Suppose that $\phi : G \to H$ is a Frattini cover of Hausdorff topological groups such that $K := \ker\phi$ is profinite. Then K is pronilpotent (that is, an inverse limit of nilpotent groups).*

Proof. It suffices to show that for each prime p the group K has a unique Sylow p-subgroup, for then each Sylow subgroup is normal in K, so the Sylow subgroups commute. Let P be a Sylow p-subgroup of K. Then, as P is closed, so is $N_G(P)$. Let $g \in G$. Then P^g is a Sylow p-subgroup of K, so by Proposition 20.43 of [27] there is $k \in K$ such that $P^g = P^k$. Hence $gk^{-1} \in N_G(P)$, so we obtain $G = KN_G(P)$. As ϕ is a Frattini cover and $N_G(P)$ is closed, it follows that $N_G(P) = G$, that is, P is normal in G. Now K is topologically generated by its Sylow subgroups (by 20.43(d) of [27]) and is a commuting product of them, and as each of the Sylow subgroups is pro-nilpotent, so is K. $\square$

Remark. If the Frattini cover arises from a minimal finite cover $\pi : C \to W$ with fibres of bounded finite size then the kernel is nilpotent, as it is also contained in a direct product of finite groups. This gives:

Corollary 1.4.6 Suppose W is a permutation structure with $\mathrm{Aut}(W)$ having finitely many orbits on W, and let $\pi : C \to W$ be a finite cover. Then there is a covering expansion of π with nilpotent kernel. $\square$

We omit the proof of the following (it uses the above Corollary). It enables us for some purposes just to work with covers with abelian kernel.

Corollary 1.4.7 ([22]) *Let W be a permutation structure. Then the following are equivalent:*

(a) every finite cover of W splits;

(b) every finite cover of W with elementary abelian kernel splits. $\square$

1.5 Free covers

Suppose $\pi : C \longrightarrow W$ is a finite cover. Then we have the following data:

- the base structure W
- for every $w \in W$, the fibre group $F(w) = \mathrm{Aut}(C(w)/w)$
- for every $w \in W$, the binding group $B(w) = \mathrm{Aut}(C(w)/W)$.

Here, $B(w)$ is a normal subgroup of $F(w)$, and these should both be regarded as permutation groups on the fibre $C(w) = \pi^{-1}(w)$.

First, we indicate what is needed to ensure that there is *some* finite cover with the given data.

Lemma 1.5.1 *Suppose $\pi : C \longrightarrow W$ is a finite cover. Then, for every $w \in W$, there is a continuous epimorphism $\chi_w : \mathrm{Aut}(W/w) \to F(w)/B(w)$.*

Proof. Let $g \in \mathrm{Aut}(W/w)$. Then there exists $h \in \mathrm{Aut}(C/w)$ which extends g. Suppose h' also extends g. Then $h^{-1}h' \in \mathrm{Aut}(C/W)$ and so $(h|C(w))B(w) = (h'|C(w))B(w)$. So if we define $\chi_w(g) = (h|C(w))B(w)$, we get a well-defined homomorphism $\chi_w : \mathrm{Aut}(W/w) \to F(w)/B(w)$, which is clearly onto. To see that χ_w is continuous, note that its kernel consists of those $g \in \mathrm{Aut}(W/w)$ which extend to an element of $Aut(C)$ inducing an element of $B(w)$ on $C(w)$. Thus $\ker\chi_w$ is the image (under the restriction map $\mathrm{Aut}(C) \to \mathrm{Aut}(W)$) of an open subgroup of $\mathrm{Aut}(C)$. By 1.3.3, this implies that $\ker\chi_w$ is an open subgroup of $\mathrm{Aut}(W/w)$. As $F(w)/B(w)$ is finite, this means that χ_w is continuous. $\square$

We refer to the epimorhisms χ_w as the *canonical homomorphisms* of the cover, and together with the fibre and binding groups, these form the *canonical data* of the cover.

Recall that a finite cover $\pi : C \longrightarrow W$ is *free* if (with the above notation)

$$\mathrm{Aut}(C/W) = \prod_{w \in W} B(w).$$

We summarise the main facts about free finite covers in the following. Full proofs can be found in [26].

Theorem 1.5.2 *Let W be a transitive permutation structure and $w \in W$.*

(i) Suppose F is a permutation group on a finite set X and B is a normal subgroup of F such that there exists a continuous surjection $\chi : \mathrm{Aut}(W/w) \to F/B$. Then there is a free finite cover π_0 of W with canonical data at w equal to F, B, χ. It is unique (up to isomorphism over W). If F splits over B then π_0 is split.

(ii) Suppose $\pi : C \to W$ is a finite cover. Then there exists a free finite cover $\pi_0 : C_0 \to W$ such that π is a covering expansion of π_0 and the canonical data are the same in π and π_0. $\square$

The following is a variation on Theorem 4.5 of [20].

Lemma 1.5.3 *Let W be a transitive permutation structure. Let $w \in W$ and let $\mathrm{Aut}^\circ(W/w)$ be the intersection of the closed subgroups of finite index in $\mathrm{Aut}(W/w)$. Suppose that $G = \mathrm{Aut}(W/w)/\mathrm{Aut}^\circ(W/w)$ is finite and non-trivial. Then there exists a non-split free finite cover of W.*

Proof. We need the following fact from finite group theory: there exists a non-split finite extension of G

$$1 \to B \to F \to G \to 1$$

(in fact, B can be taken to be elementary abelian). Now construct a free finite cover $\sigma : M \to W$ using 1.5.2, where the fibre group is F (in its regular representation) and binding group B and the kernels of the canonical homomorphisms are the groups $\mathrm{Aut}^\circ(W/w)$. Suppose, for a contradiction, that H is a closed complement to $K = \mathrm{Aut}(M/W)$. Thus $\mathrm{Aut}(M) = KH$, and $\mathrm{Aut}(M/w) = KH_w$, where H_w denotes the stabiliser in H of w. Restricting this equation to $M(w)$ we get that $F = BT$ where T is the restriction of H_w to $M(w)$. Thus $|T| \geq |F/B| = |G|$. But T is a finite, continuous homomorphic image of H_w, and the restriction map gives an isomorphism from H_w to $\mathrm{Aut}(W/w)$, so $|T| \leq |G|$. Thus $|T| = |G|$ and $T \cap K$ is the identity subgroup. So F splits over B, a contradiction. $\square$

1.6 An overview

So far, apart from the results of Ahlbrandt and Ziegler ([3, 4]), most of the successes have been with finite rather than affine covers, and for the remainder of these notes we will be concerned mainly with finite covers of countable $\aleph_0$-categorical structures (although we often present results in more generality). The material is, for the most part, taken from the papers of Ahlbrandt and Ziegler ([3, 4]), Evans ([20, 21, 22]), Evans and Hrushovski ([25]), Hodges and Pillay ([32]) and Ivanov ([36, 37]).

The main problem is:

> **The Cover Problem:** For a given $\aleph_0$-categorical structure W, describe its finite covers.

We are usually satisfied with only a partial solution to this for any particular W. Indeed, the only W for which a complete solution is known are the highly homogeneous structures (results due to Ivanov [37] and Ziegler [47]). The successes of the the theory have been in proving splitting results (showing that for certain W all finite covers split), and describing finite covers with finite or abelian kernel.

The results which we present can be divided into three types.

1. Constructions: describing abstract covering constructions which generalise familiar examples.
2. Reduction theorems: reducing the analysis of finite covers to that of simpler kinds.
3. Classification results: classifying finite covers of certain familiar $\aleph_0$-categorical structures.

The constructions show how non-split finite covers can arise from non-split finite extensions of finite homomorphic images of point stabilisers (Lemma 1.5.3) and, more subtly, from combinatorial properties of 0-definable binary and ternary relations on W (see Section 4). In these latter cases the kernels of the covers are finite.

The main reduction result is that any finite cover has a minimal covering expansion, the kernel of which is nilpotent (Corollary 1.4.3 and Lemma 1.4.5). It then follows that the splitting problem reduces to consideration of finite covers with abelian kernel (Corollary 1.4.7). For structures with trivial algebraic closure the situation is more straightforward, and one can reduce to consideration of finite covers with finite kernels (3.4.1). There is a well-developed theory of finite covers with finite kernels (Sections 3 and 4.2).

To analyse finite covers with abelian kernel (Section 2) we use cohomological methods first introduced into this subject by Ahlbrandt and Ziegler ([4]). These methods have been used (together with results from representation theory and cohomology of finite groups) to give very precise information about finite covers of Grassmannians of strictly minimal sets (Section 2.3), as well as information which is rather more qualitative. They can also be used to provide information about finite covers with finite kernels.

2 Finite covers with abelian kernels

2.1 Kernels

Lemma 2.1.1 *Suppose C, W are permutation structures and $\pi : C \to W$ is a split finite cover with kernel K. Let T be a closed complement to K in $\mathrm{Aut}(C)$. Then any closed subgroup H of K which is normalised by T is a kernel of some covering expansion of π.*

Proof. Let $\Gamma = \mathrm{Aut}(C)$. Then TH is a subgroup of Γ whose intersection with K is H. The lemma follows once we have shown that TH is actually a closed subgroup of Γ (because the covering expansion we want can be taken to have automorphism group TH). This is a general fact about topological groups. The map $\Gamma \times K \to \Gamma \times K$ given by $(g, k) \mapsto (gk, k)$ is a homeomorphism, and compactness of K implies that projection onto the first coordinate in $\Gamma \times K$ is a closed map. So the image of $T \times H$ under the composition of these maps is closed in Γ. □

Note that if K in the above is abelian then any subgroup of K which is normalised by T is actually normal in $\mathrm{Aut}(C)$. Combining this observation with Lemma 1.5.2 we obtain:

Lemma 2.1.2 *Let $\pi : C \to W$ be a free finite cover with abelian kernel in which the fibre groups split over the binding groups. Then a subgroup K of $\mathrm{Aut}(C/W)$ is the kernel of some covering expansion of π if and only if K is a closed normal subgroup of $\mathrm{Aut}(C)$.* □

Suppose $\pi_0 : C_0 \to W$ is a finite cover with abelian kernel K_0. So K_0 is a closed normal subgroup of $\Gamma_0 = \mathrm{Aut}(C_0)$ and we have the short exact sequence

$$1 \to K_0 \to \Gamma_0 \xrightarrow{\mu} G \to 1$$

where μ is restriction to W, and $G = \mathrm{Aut}(W)$. Recall (1.3.4) that $\Gamma_0/K_0 \cong G$ as topological groups. Now consider Γ_0 acting on K_0 by conjugation. As K_0 is abelian, K_0 is in the kernel of this action, and so we get an action of $G = \Gamma_0/K_0$ on K_0. From now on we shall write this additively, with the G-action on the left. Thus $gk = hkh^{-1}$, for $g \in G$, $k \in K_0$ and any $h \in \mu^{-1}(g)$. We have the following basic fact.

Lemma 2.1.3 *With this notation K_0 is a topological G-module.*

Proof. Recall that the restriction map μ is an open mapping (see Lemma 1.3.3). Give $G \times K_0$ the product topology. What is being claimed is that the G-action $\alpha : G \times K_0 \longrightarrow K_0$ is continuous. Let $\Gamma = \mathrm{Aut}(M_0)$ and consider the map $\beta : \Gamma \times K_0 \longrightarrow K_0$ given by conjugation. This is continuous, and if $O \subseteq K_0$ is open, then $\alpha^{-1}(O) = (\mu \times 1)\beta^{-1}(O)$, where $\mu \times 1 : \Gamma \times K_0 \longrightarrow G \times K_0$ is the obvious map. This is open, so α is continuous, as required. □

Putting these together, we get the following, which is a slight generalisation of a result of Ahlbrandt and Ziegler (Lemma 2.1 of [4]):

Theorem 2.1.4 *Suppose W is a transitive permutation structure with automorphism group G and $\pi_0 : C_0 \longrightarrow W$ is a free finite cover of W with abelian kernel K_0. Suppose the fibre groups in π_0 split over the binding groups (for example, if they are equal to them). Regard K_0 as a topological G-module. Then a subgroup K of K_0 is the kernel of some covering expansion of π_0 if and only if it is a closed G-submodule of K_0.* □

We shall mainly use this in the case where the fibre and binding groups are cyclic of order p, for some prime p. In this case, we can identify K_0 with the G-module F_p^W of function from W into F_p, the field of integers modulo p (–the G action is given by $(gf)(w) = f(g^{-1}w)$, for $f \in F_p^W$, $g \in G$, and $w \in W$). So we are interested in the closed G-invariant subspaces of F_p^W. These can sometimes more easily be described by making use of a simple instance of Pontriagin duality (for full details see [26]).

Definition 2.1.5 Let F_pW be the vector space of formal linear combinations of elements of W, and regard this as a G-module in the natural way. Let X be a subspace of F_pW and define its *annihilator* in F_p^W to be

$$X^- = \{f \in F_p^W : \Sigma_{w\in W} a_w f(w) = 0 \text{ for all } \Sigma_{w\in W} a_w w \in X\}.$$

Note that $X^- \leq Y^-$ if and only if $Y \leq X$.

Theorem 2.1.6 *The closed G-invariant subspaces of F_p^W are precisely the annihilators X^- of G-invariant subspaces X of F_pW.* □

In summary, to determine kernels of finite covers of W where the fibre and binding groups are of prime order p, it is enough to determine the G-submodules of the permutation module F_pW.

2.2 Derivations

The first uses of cohomology groups to classify covers of $\aleph_0$-categorical structures are by Martin in [45] and Ahlbrandt and Ziegler in [4]. In this section, we follow rather closely the approach of [4] as modified by Hodges and Pillay in [32]. (Hodges and Pillay work in a wider context of 'symmetric extensions'. This sometimes means that they restrict attention to countable structures. As we do not wish to do this, we will work throughout with finite covers, where countablility of our structures is not necessary.)

Recall that if G is a group and M is a G-module, then a *derivation* from G to M is a map $d : G \longrightarrow M$ which satisfies $d(gh) = d(g) + gd(h)$ for all $g, h \in G$. An *inner* derivation is a derivation of the form d_a (for $a \in M$) where $d_a(g) = ga - a$ for all $g \in G$. The set of all such derivations forms an abelian group, and the inner derivations form a subgroup. The quotient group is denoted by $H^1(G, M)$, and is referred to as the first cohomology group of G on M. If M is a topological G-module then the continuous derivations form a subgroup of the group of all derivations, and this clearly contains all the inner derivations. We denote the quotient group here by $H^1_c(G, M)$.

Suppose now that $\pi_0 : C_0 \rightarrow W$ is a finite cover of the permutation structure W, and suppose from now on that the kernel $K_0 = \mathrm{Aut}(C_0/W)$ is abelian. Then conjugation in $\mathrm{Aut}(C)$ gives

K_0 the structure of a topological Aut(W)-module (2.1.3). Let $\mu : \mathrm{Aut}(C_0) \rightarrow \mathrm{Aut}(W)$ be the restriction homomorphism. Suppose K is a G-invariant subgroup of K_0 such that there exists $H_0 \leq \mathrm{Aut}(C_0)$ with $H_0 \cap K_0 = K$ and $\mu(H_0) = G$. Then we have ([32] Propositions 16 and 17):

Proposition 2.2.1 *There is a one-to-one correspondence between $H^1(G, K_0/K)$ and $Aut(C_0)$-conjugacy classes of subgroups H of $Aut(C_0)$ satisfying $H \cap K_0 = K$ and $\mu(H) = G$.*

It is worth noting how the correspondence in the above is obtained. Note that $G \cong H_0/H_0 \cap K_0 \leq Aut(C_0)/K$. So there is an embedding $\sigma_0 : G \longrightarrow Aut(C_0)/K$ given by $\sigma_0(g) = (\mu^{-1}(g) \cap H_0)K$. Now take any H as in the proposition. We obtain similarly a map $\sigma_H : G \longrightarrow K_0/K$. Then the map given by $d_H(g) = \sigma_H(g)\sigma_0(g)^{-1}$ has image in K_0/K and is a derivation.

Suppose now that K is a closed subgroup of K_0, and H_0 is a closed subgroup of $Aut(C_0)$ with $\mu(H) = G$ and $H_0 \cap K_0 = K$. Then by Lemma 1.3.3 σ_0 is continuous. We have:

Corollary 2.2.2 *There is a one-to-one correspondence between $H^1_c(G, K_0/K)$ and conjugacy classes of closed subgroups H of $Aut(C_0)$ which satisfy $\mu(H) = G$ and $H \cap K_0 = K$.*

Proof. This is proved under the assumption that C_0 is countable in Corollary 18 of [32]. The context in which Hodges and Pillay work (symmetric extensions) is more general than ours, and the countability assuption is required to invoke their Lemma 6 in the proof of Theorem 11 of [32]. However, we can substitute Lemma 1.3.3 in place of Lemma 6 of [32], and the proofs of Theorem 11 and Corollary 18 of [32] go through without further modification. □

In practice, $\pi_0 : C_0 \rightarrow W$ will be a free finite cover and we will be interested in classifying covering expansions of this which have as kernel some particular G-invariant closed subgroup K of K_0. Corollary 2.2.2 indicates that to do this we should compute the cohomology group $H^1_c(G, K_0/K)$. The hope is that this will be trivial (or at least small).

Corollary 2.2.3 *Suppose $\pi_0 : C_0 \rightarrow W$ is a split finite cover with abelian kernel K_0 and K is a closed* Aut(W)*-invariant subgroup of K_0. If $H^1_c(\mathrm{Aut}(W), K_0/K) = \{0\}$, then there is a covering expansion of π_0 with kernel K. It is unique (up to isomorphism over W) and split.*

Proof. Existence of a split covering expansion follows from (2.1.1). The uniqueness follows from (2.2.2): the automorphism groups of any two covering expansions of π_0 with kernel K are conjugate in $\mathrm{Aut}(C_0)$. □

We now give some modifications of standard results from cohomology of discrete groups results which will assist in the computation of the groups H^1 and H^1_c. Proofs can be found in Section 7.1 of [26].

Definition 2.2.4 If G is a group and M a G-module then we define the *zeroth cohomology group* $H^0(G, M)$ to be the elements of M fixed by all elements of G. Note that if G is a topological group and M a topological G-module, then this is a closed subgroup of M.

Lemma 2.2.5 *(*'The long exact sequence'*) Suppose G is a group and*

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$$

is an exact sequence of G-modules. Then there is an exact sequence of abelian groups:

$$0 \rightarrow H^0(G, K) \rightarrow H^0(G, M) \rightarrow H^0(G, N) \rightarrow$$

$$\rightarrow H^1(G, K) \rightarrow H^1(G, M) \rightarrow H^1(G, N).$$

If, moreover, G is a topological group and the the short exact sequence is a sequence of topological G-modules in which the homomorphisms are continuous open maps, then there is a long exact sequence as above in which the H^1 terms are replaced by H^1_c. □

Remarks 2.2.6 1. In our context, the modules which arise are closed subgroups of kernels of finite covers, so are profinite, and a continuous map between profinite groups is automatically an open map.

2. Our main use of the long exact sequence will be to effect a trick known as 'dimension shifting': we shift a problem about computing H^1 to one about computing H^0. The procedure, roughly, is this. We want to compute $H^1(G, K)$ for some G-module K. Suppose we can embed K in a module M for which we know $H^1(G, M)$. Then by the long exact sequence, if we can compute H^0 of K, M and M/K, we can read off $H^1(G, K)$.

All this relies on having a good supply of G-modules whose cohomology we know about. In our context, the appropriate modules are kernels of free finite covers. In the group theoretic terminology (at least if the base W of the cover is transitive) these modules are *coinduced* from a finite module for the stabiliser of a point: the relevant module is the binding group at that point. The next lemma is then seen as a special case of Shapiro's lemma in group cohomology (cf. [6], Proposition III.6.2), and it tells us how to compute the cohomology of the coinduced modules.

Lemma 2.2.7 (Shapiro's lemma) *Let W be a transitive permutation structure and $G = Aut(W)$. Suppose $\pi : C \longrightarrow W$ is a free finite cover with abelian kernel K. Let $w \in W$, $H = Aut(W/w)$ and $A = Aut(C(w)/W)$. Then A is naturally an H-module (via the canonical homomorphism χ_w), and for $i = 0, 1$ we have*

$$H^i(G, K) = H^i(H, A).$$

Moreover $H^1_c(G, K) = H^1_c(H, A)$. □

Computation of the groups $H^i_c(H, A)$ is, in the G-finite case, a problem about finite groups:

Lemma 2.2.8 *Suppose Σ is a topological group and A a finite topological H-module. Suppose further that Σ° is of finite index in Σ. Then*

(i) $H^0(\Sigma, A) = H^0(\Sigma/\Sigma^\circ, A)$;

(ii) $H^1_c(\Sigma, A) = H^1(\Sigma/\Sigma^\circ, A)$.

In particular, these groups are finite. □

2.3 Grassmannians of modular strictly minimal sets

In this section we give some results classifying finite covers with fibre groups cyclic of prime order p of a projective space over the field with p elements, and of the Grassmannian of k-sets for a disintegrated set. First we describe the possible kernels.

Example 2.3.1 The following results are due to Ahlbrandt and Ziegler, and are to be found in the paper [3]. However, our presentation of the results is slightly different.

Let $V = V(\aleph_0, 2)$ be a countably infinite dimensional vector space over the field F_2 with 2 elements, and $G = GL(\aleph_0, 2)$ its automorphism group. Then G is transitive on W, the non-zero vectors in V. We consider finite covers of W where the fibre and binding groups are cyclic of order 2. (Ahlbrandt and Ziegler also consider the more difficult case of affine covers of W where the structure groups are isomorphic to $V/\langle w\rangle$, for $w \in W$). So, we wish to determine the closed, G-invariant subgroups of F_2^W (Theorem 2.1.4). By duality (Theorem 2.1.6), this is equivalent to determining the G-invariant subgroups of the permutation module F_2W. We now describe these.

For any vector space X of dimension $n \leq \omega$ over a finite field F_q, and finite k with $0 \leq k \leq n$, let $[[X]]^k$ be the set of k-dimensional subspaces of X, considered as a permutation structure with $GL(X)$ acting. If $l \leq k$ there is a natural homomorphism of $GL(X)$-modules

$$\beta_{k,l} : F_q[[X]]^k \longrightarrow F_q[[X]]^l$$

given by

$$\beta_{k,l}(w) = \Sigma_{w' \in [[w]]^l} w'$$

for $w \in [[X]]^k$. (Of course, this can also be defined for permutation modules over a different field.) It is easy to show that if $l \leq k' \leq k$ then there is a non-zero $t \in F_q$ such that $\beta_{k,l} = t\beta_{k',l}\beta_{k,k'}$ and so $im(\beta_{k,l}) \leq im(\beta_{k',l})$. So as submodules of $F_q[[X]]^1$ we have $im(\beta_{k,1})$ for $1 \leq k \leq n$, and the intersections of these with $ker(\beta_{1,0})$. The main result of Part 1 of [3] is that for finite n and $q = 2$, these are the only $GL(X)$-submodules of $F_2[[X]]^1$. It is then a straightforward matter to deduce that the same holds for the infinite dimensional case: any element of the permutation module has its support contained in a finite dimensional subspace, and so we can read off from the finite case that it must generate some $im(\beta_{k,1})$ or its intersection with $ker(\beta_{1,0})$. Now, we can identify W with the set of 1-dimensional subspaces of V, and so we know all the G-invariant subgroups of F_2W.

From the duality, we now get the following:

Theorem 2.3.2 (Ahlbrandt-Ziegler, [3]) *The closed, G-invariant subgroups of F_2^W consist of the submodules*

$$Pol_k = \{f \in F_2^W : \Sigma_{x \in w \setminus \{0\}} f(x) = 0 \, \forall w \in [[V]]^{k+1}\}$$

and the sum of these with the one dimensional submodule of constant functions. □

With the benefit of hindsight, the above submodules of $F_2[[X]]^1$ are recognisable as 'well-known' objects from the theory of error-correcting codes: they are the Reed-Muller codes. On closer examination of the coding-theoretic literature (notably [1] and [15]), Darren Gray discovered that the above result is a special case of a result of Delsarte (see Theorem 8 of [15]): for arbitrary prime p, the $GL(n,p)$-invariant subgroups of $F_p[[V(n,p)]]^1$ are given by the images of the $\beta_{k,1}$ maps, and their intersections with $ker(\beta_{1,0})$. There is, of course a corresponding result for the infinite dimensional case, and a dual version. So Theorem 2.3.2 holds for all primes, not just for the prime 2.

Example 2.3.3 We report some work of D. Gray ([28]), which is (in spirit) similar to that of the previous example, and deals with the case where $W = [D]^k$ is the permutation structure of k-sets from a disintegrated set D (– so the automorphism group here is $G = \mathrm{Sym}(D)$). Let F be any field, and consider the FG-permutation modules $F[D]^k$, for $k < \omega$. Again, for $k \geq l$, there are natural FG-module homomorphisms

$$\beta_{k,l} : F[D]^k \longrightarrow F[D]^l$$

given by

$$\beta_{k,l}(w) = \Sigma_{w' \in [w]^l} w'$$

for $w \in [D]^k$.

Theorem 2.3.4 (D. Gray, [28]) *If D is infinite, the FG-submodules of $F[D]^k$ are given by intersections of kernels $ker(\beta_{k,l})$, where $0 \leq l \leq k$.* □

The proof uses the representation theory of the finite symmetric groups, as developed in the book of G. D. James [39]. In fact, there is an effective algorithm for determining the complete submodule lattices in the above (they depend only on k and the characteristic p of F, and the algorithm involves only the checking of whether certain binomial coefficients are divisible by p). For our purposes, the main consequence of Gray's results is (by Theorem 2.1.6) a determination of the possible kernels of a finite cover of $W = [D]^k$, with fibre and binding groups cyclic of order p.

We now move on to describing the finite covers with the above kernels, using the cohomological machinery.

First, we return to Example 2.3.1. So, let $V = V(\aleph_0, 2)$ be a countably infinite dimensional vector space over the field with 2 elements, and $G = GL(\aleph_0, 2)$ its automorphism group. Let $W = V \setminus \{0\}$. We consider finite covers of W where the fibre and binding groups are cyclic of order 2. Let $\pi_0 : C_0 \rightarrow W$ be the free cover of W with fibre and binding groups cyclic of order 2 (and each fibre of size 2). Let K_0 be the kernel of this. In 2.3.1 we described the closed G-invariant subgroups K of K_0: a result which was deduced from the parallel situation of finite-dimensional V.

Ahlbrandt and Ziegler show that

Theorem 2.3.5 *For each possible kernel K we have $H^1_c(G, K_0/K) = \{0\}$.*

Applying (2.2.3) we get:

Corollary 2.3.6 *All finite covers of W with fibres of size 2 split. Any such finite cover is determined (up to isomorphism over W) by its kernel, and the possibilities for the kernels are given in Theorem 2.3.2.* □

Ahlbrandt and Ziegler deduce Theorem 2.3.5 from known results about the vanishing of the first cohomology groups of the finite general linear groups $GL(n, 2)$ acting on certain natural modules (duals of exterior powers of $V(n, 2)$ (if $n > 5$)), together with results on envelopes in totally categorical structures. We shall describe the use of the finite group theoretic results, but avoid mentioning envelopes, substituting instead the following, slightly *ad hoc* result, taken from [23].

Lemma 2.3.7 *Let Γ be a Hausdorff topological group and M a compact topological Γ-module. Suppose there exists $(G_i : i < \omega)$, an increasing chain of subgroups of Γ such that $G = \bigcup_{i<\omega} G_i$ is dense in Γ. Suppose also that for each i we have an open, G_i-invariant subgroup M_i of M, and that $M_{i+1} \leq M_i$ for all $i < \omega$ and $\bigcap_{i<\omega} M_i = \{0\}$. Suppose further that for all i, any continuous derivation from G_i to M/M_i is inner. Then any continuous derivation $d : \Gamma \longrightarrow M$ is inner.*

Proof. Note first that if two continuous derivations $\Gamma \longrightarrow M$ agree on a dense subgroup, then they must be equal. So (as inner derivations are continuous) it will suffice to prove that $\delta = d|G$ is inner. The hypotheses imply that M is metrizable, with a metric θ such that the diameters of the M_i tend to zero.

For every $i < \omega$ there exists $a_i \in M$ such that for all $g \in G_i$ we have

$$\delta(g) + M_i = ga_i - a_i + M_i.$$

By compactness of M we may assume that the a_i converge to some $a \in M$. Let d_a denote the inner derivation obtained from a. Thus, for $g \in G_i$, for every $j > i$ there exists $m_j \in M_j$ such that

$$\theta(\delta(g), d_a(g)) = \theta(ga_j - a_j + m_j, ga - a).$$

Now, the m_j tend to 0 as j tends to infinity, and so (by continuity of the Γ-action) $\theta(\delta(g), d_a(g))$ can be arbitrarily small. So $\delta(g) = d_a(g)$. But this holds for all i, and so we conclude that $d = d_a$, as required. □

The following is easy, but useful.

Lemma 2.3.8 *Let Γ be a topological group and M a continuous Γ-module. Let N be a closed submodule of M and suppose that $H^1_c(\Gamma, M/N)$ and $H^1_c(\Gamma, N)$ are trivial. Then $H^1_c(\Gamma, M)$ is trivial.* □

Proof of 2.3.5. We use the lemma with $\Gamma = GL(V)$ and $M = K_0/K$. Remember that $K_0 = F_2^W$ and K is a closed, Γ-invariant subgroup of K_0. Let $(V_i : i < \omega)$ be an increasing chain of finite dimensional subspaces of V (of dimension at least 6) with union the whole of V. Let T_i be a complement to V_i in V, and choose these so that $T_i \geq T_{i+1}$ for all i. Let

$$G_i = \{g \in \Gamma : gV_i = V_i \text{ and } gx = x \ \forall x \in T_i\}.$$

Then the G_i form an increasing chain whose union is dense in Γ. Let K_i be those functions in K_0 which are zero on V_i. Thus, K_0/K_i is isomorphic to $F^{V_i \setminus \{0\}}$. Let $M_i = (K + K_i)/K$. Then

$$M/M_i = (K_0/K)/(K + K_i/K) \cong K_0/(K + K_i) \cong (K_0/K_i)/(K + K_i/K_i)$$

and all these isomorphisms hold as isomorphisms of G_i-modules. But now we claim that Theorem 4.1 of [4], and the description of the possibilities for K given in Theorem 3.1 (*ibid.*) show that any derivation from G_i to M/M_i is inner. Put more explicitly, what we want to show in order to apply our lemma, is that $H^1(G_i, M/M_i) = \{0\}$. Fix i, and to ease notation write $X = V_i$. Now, G_i can be identified with the finite general linear group $GL(X)$, and M/M_i is a quotient module of the $GL(X)$-module $F_2^{X\setminus\{0\}}$. The finite-dimensional version of the results in 2.3.1 tell us precisely what are the possibilities for M/M_i. Moreover, the composition factors of M/M_i are well-known $GL(X)$-modules: they are (duals of) exterior powers of X (see lemma 4.3 of [4]). Results of G. B. Bell ([5]) show that the the first cohomology of $GL(X)$ on these modules is trivial, and so (for example, by 2.3.8) $H^1(GL(X), M/M_i) = \{0\}$. The lemma is now applicable, and this finishes the proof of 2.3.5. □

The following result is proved using similar methods, but making use of the description of kernels given in 2.3.3 in place of 2.3.1.

Theorem 2.3.9 ([23]) *Let $W = [D]^k$ be the Grassmannian of k-sets from a countably infinite disintegrated set D. Then any finite cover $\pi : C \longrightarrow W$ with fibre groups cyclic of prime order is split. Together with the results in 2.3.3, this gives a complete classification of all such covers.* □

We remark that (with W as above) it follows that any finite cover $\pi : C \longrightarrow W$ with fibre group of odd order is split. By 1.4.7, it suffices to prove this for the case where the kernel is an elementary abelian p-group (for all odd primes p). By 1.5.1, and the fact that any non-trivial finite homomorphic image of $\mathrm{Aut}(W/w)$ has even order, the fibre and binding groups of π are equal. It is then easy to reduce to the case where the binding groups are actually Z_p. This is then handled by our theorem.

3 Finite covers with finite kernels

This section is taken from [22].

3.1 Exact sequences

Throughout, W will be an irreducible, transitive permutation structure. We are interested in the minimal (alternatively, irreducible) finite covers $\pi : C \to W$ which have finite kernel. Note that as the centraliser in $\mathrm{Aut}(C)$ of the kernel is a closed subgroup of finite index, it follows that the kernel is central in $\mathrm{Aut}(C)$: in particular, it is abelian.

Let $G = \mathrm{Aut}(W)$ and let A be a finite abelian group. We wish to classify finite covers $\pi : C \to W$ where the kernel K is isomorphic to A and central in $\mathrm{Aut}(C)$. Such a cover is a covering expansion of a free finite $\pi_0 : C_0 \to W$ with the same fibre and binding groups and canonical homomorphisms.

Let K_0 denote the kernel of π_0. Let A^W be the module of functions $f : W \to A$ with the product topology and G-action $(g.f)(w) = f(gw)$ (for $g \in G$), and let $\Delta(A)$ consists of the constant functions $W \to A$. Then, K_0/K and $A^W/\Delta(A)$ are isomorphic as topological G-modules. In this section we give an explicit method for computing $H^1_c(G, A^W/\Delta(A))$. By Corollary 2.2.2 this gives a description of all the covering expansions of C_0 with kernel K (although it should be stressed that this is predicated on the existence of some covering expansion with kernel K).

For $n \in \mathbb{N}$ let $W^{(n)}$ denote the set of n-tuples of distinct elements of W. Let $M_n = A^{W^{(n)}}$ be the set of functions $f : W^{(n)} \to A$ considered as a topological G-module (as for A^W). Define $\Delta : A \to A^W$ so that $\Delta(a)$ is the constant function with image a, for $a \in A$. Clearly this is a continuous G-module homomorphism, if we regard A as a trivial G-module. Define the map $d_n : M_n \to M_{n+1}$ by

$$(d_n f)(w_1, \ldots, w_{n+1}) = \sum_{i=1}^{n+1} (-1)^i f(w_1, \ldots, \hat{w}_i, \ldots, w_{n+1})$$

where $f \in M_n$, $(w_1, \ldots, w_{n+1}) \in W^{(n+1)}$, and the $\hat{w}_i$ denotes that the i-th term is to be deleted. Then d_n is a continuous G-module homomorphism. The following is well-known.

Lemma 3.1.1 *The sequence*

$$0 \to A \xrightarrow{\Delta} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \xrightarrow{d_3} \cdots$$

is exact. □

From now on assume that G has t orbits on $W^{(3)}$ for some finite t. Then the number s of G-orbits on $W^{(2)}$ is also finite. Let (x_i, y_i) (for $i = 1, \ldots, s$) be representatives of these orbits. Suppose further that the group

$$X_i = \mathrm{Aut}(W/x_i, y_i)/(Aut(W/x_i, y_i))^\circ$$

is finite for $i = 1, \ldots, s$. (Note that all of these hypotheses are satisfied if W is a G-finite oligomorphic permutation structure.) Below, $\mathrm{Hom}(X_i, A)$ denotes the group of all homomorphisms from X_i to A.

Lemma 3.1.2 *There is an exact sequence of abelian groups*

$$0 \to A^s \to H^0(G, M_2/d_1(M_1)) \to H^1_c(G, A^W/\Delta(A)) \to \bigoplus_{i=1}^{s} \mathrm{Hom}(X_i, A).$$

Proof. By Lemma 3.1.1 we have a short exact sequence

$$0 \to A^W/\Delta(A) \to M_2 \to M_2/d_2(M_1) \to 0.$$

The maps in this sequence are continuous, open, G-module homomorphisms. The lemma follows from the long exact sequence (Lemma 2.2.5) and the following observations.

Claim 1. $H^0(G, A^W/\Delta(A)) = 0$. Indeed, suppose $f + \Delta(A)$ is fixed by G. Then $\langle f, \Delta(A) \rangle$ is a finite submodule of A^W. As G is irreducible, G acts trivially on this, so f is fixed by G. But the only functions in A^W fixed by G are the constant functions, so $f \in \Delta(A)$.

Claim 2. $H^0(G, M_2) \cong A^s$. A function in M_2 is fixed by G if and only if it is constant on each G-orbit on W^2. The claim then follows immediately.

Claim 3. $H^1_c(G, M_2) \cong \bigoplus_{i=1}^{s} \mathrm{Hom}(X_i, A)$.
Let $O_1, \ldots, O_s$ be the G-orbits on $W^{(2)}$. Then

$$H^1_c(G, M_2) \cong \bigoplus_{i=1}^{s} H^1_c(G, A^{O_i}).$$

By Shapiro's lemma (Lemma 2.2.7),

$$H^1_c(G, A^{O_i}) \cong H^1_c(\mathrm{Aut}(W/x_i, y_i), A).$$

By Lemma 2.2.8, this latter group is isomorphic to $H^1(X_i, A)$, and as A here is a trivial module, this is the same as $\mathrm{Hom}(X_i, A)$. □

Corollary 3.1.3 *The group $H^1_c(G, A^W/\Delta(A))$ is finite. In fact*

$$|H^1_c(G, A^W/\Delta(A))| \leq |A|^{t-s} \prod_{i=1}^{s} |\mathrm{Hom}(X_i, A)|.$$

Proof. This follows immediately from Lemma 3.1.2 once we show that the size of $H^0(G, M_2/d_1(M_1))$ is bounded by $|A|^t$. But by the exact sequence in Lemma 3.1.1, $M_2/d_1(M_1)$ is isomorphic to a submodule of M_3, and the submodule of fixed points of G on M_3 is isomorphic to A^t (as in the proof of Claim 2 in the above). □

3.2 Computation

We can be more precise about the group $H^0(G, M_2/d_1(M_1))$ by using more of the exact sequence in Lemma 3.1.1. By the exactness, we have the isomorphisms

$$M_2/d_1(M_1) \cong d_2(M_2) \cong \ker(d_3).$$

Thus, we want to know the fixed points of G on the kernel of d_3. This is the same as the kernel of d_3 restricted to the fixed points of G on M_3. We outline an algorithm for computing this.

List the G-orbits on $W^{(3)}$ as $P_1, \ldots, P_t$ and let $(Q_i : i \in I)$ be the G-orbits on $W^{(4)}$. Define an $|I| \times t$ matrix of integers M as follows. For $i \in I$ choose $(w_1, w_2, w_3, w_4) \in Q_i$. For $j = 1, \ldots, t$ let

$$m_{ij} = \sum\{(-1)^k : (w_1, \ldots, \hat{w}_k, \ldots, w_4) \in P_j\}.$$

This depends only on i and j. Note that if $f \in M_3$ is fixed by G and takes the value a_j on P_j then

$$(d_3 f)(w_1, \ldots, w_4) = \sum_{j=1}^{t} m_{ij} a_j.$$

Let $e_1, \ldots, e_t$ be the invariants of the integer matrix M (so these are non-negative integers, and e_j divides e_{j+1} for $j = 1, \ldots, t-1$). Then it follows that the kernel of d_3 restricted to the G-fixed points in M_3 is isomorphic to

$$\bigoplus_{j=1}^{t}\{a \in A : e_j a = 0\}.$$

We summarise this as:

Corollary 3.2.1 *Let $e_1, \ldots, e_t$ be the invariants of the matrix M as above. Then:*
(i) Writing $U = \bigoplus_{j=1}^{t} \mathbb{Z}/e_j\mathbb{Z}$, we have

$$H^0(G, M_2/d_1(M_1)) = A \otimes_{\mathbb{Z}} U.$$

(ii) Suppose stabilisers of pairs of points in W are irreducible, let $m \in \mathbb{N}$ and let A be cyclic of order m. Then

$$|H^1_c(G, A^W/\Delta(A))| = \frac{\prod_{j=1}^{t}(e_j, m)}{m^s}.$$

(where (m, n) denotes the highest common factor of m and n).

Proof. (i) follows from the above remarks. Then (ii) follows from (i) and Lemma 3.1.2. □

The following lemma is straightforward, but useful.

Lemma 3.2.2 *Let $1 \leq n \leq t$. Suppose it is possible to label the G-orbits on $W^{(3)}$ as $P_1, \ldots, P_t$ so that if $j > n$ and $(x_2, x_3, x_4) \in P_j$ then there exists $x_1 \in W$ such that $(x_1, x_2, x_3), (x_1, x_2, x_4)$ and (x_1, x_3, x_4) lie in $P_1, \ldots, P_{j-1}$. Then at least $t - n$ of the invariants of M are equal to 1.* □

Example 3.2.3 Suppose W is a binary homogenous structure with a 2-type Q which has the property that for all $(x, y, z) \in W^{(3)}$ there exists $w \in W$ such that $(w, x), (w, y), (w, z) \in Q$. Then the hypotheses of the above lemma hold with $n = s$, so (at least) $t - s$ of the invariants are equal to 1. So the rank of the group U in Corollary 3.2.1 is at most s, and if stabilisers of pairs of points in W are irreducible then the cohomology groups $H^1_c(G, A^W/\Delta(A))$ are trivial for all finite abelian groups A.

3.3 Bounding the rank of the kernel

The bound on the size of the cohomology group in Corollary 3.1.3 can be used to prove the following. The proof gives an explicit bound for r, as can be seen from the two lemmas from which it is deduced.

Theorem 3.3.1 ([22]) *Let W be a transitive irreducible, permutation structure with automorphism group $G = Aut(W)$. Suppose that G has finitely many orbits on triples from W, and that for all $x, y \in W$, each of $Aut(W/x)$ and $Aut(W/x, y)$ has a smallest closed subgroup of finite index. Then there is a natural number r such that if $\pi : C \longrightarrow W$ is a minimal finite cover with finite kernel K, then K can be generated by r elements.* □

The result follows from the next two lemmas, and for these the hypotheses of Theorem 3.3.1 which relate to W will be in force.

Lemma 3.3.2 *Suppose there is a non-trivial, minimal finite cover $\pi : C \longrightarrow W$ with finite kernel K of rank r. Then for some prime p there is a minimal finite cover of W whose kernel is an elementary abelian p-group of rank r.* □

For a prime p, let F_p denote the cyclic group of order p (and also think of this as the field with p elements). We use the notation of the previous section.

Lemma 3.3.3 *There exists a natural number n (depending only on W) with the following property. Suppose that $H^1_c(G, F_p^W/\Delta(F_p))$ is finite, of cardinality p^k. Let $\pi : C \longrightarrow W$ be an irreducible finite cover whose kernel is a finite elementary abelian p-group of rank r. Then $r \leq k + n$.* □

We clarify what n is here. By assumption, there is a number m such that any continuous finite image of the stabiliser of a point in W has size at most m. By (2.1) of [14] there exists an integer n such that if T is a finite group of size at most m and $\phi : S \longrightarrow T$ is a Frattini cover with kernel Z, then Z has rank at most n. In particular, if point stabilisers are irreducible, then $n = 0$.

More generally, we have the following corollary:

Theorem 3.3.4 ([22]) *Let W be a G-finite oligomorphic permutation structure. Then there exists a natural number r such that if $\pi : C \rightarrow W$ is a minimal finite cover with finite kernel K, then K has rank at most r.* □

Remark 3.3.5 The above results (3.3.1 and 3.3.4) have also been proved independently by Hrushovski. The original proof of (3.3.1) appears in [26]. The use of the exact sequence (3.1.2) simplifies the proof and is suggested by Hrushovski's proof.

3.4 Examples

We use the machinery just developed to analyse the minimal finite covers of some of the countable primitive homogeneous graphs and digraphs. There is an explicit classification of these given by Lachlan and Woodrow ([43]), and Cherlin ([11]). All of the graphs and directed graphs have trivial algebraic closure and weak elimination of imaginaries (see [20] for a proof). In particular all pointwise stabilisers of finite sets are irreducible. We have the following reduction theorem.

Theorem 3.4.1 ([20], Lemma 2.5) *Suppose that W is an irreducible primitive permutation structure with trivial algebraic closure, and that each point stabiliser* $\mathrm{Aut}(W/w)$ *is irreducible. If every finite cover of W with finite kernel splits, then every finite cover of W splits.* □

Different proofs of parts of the following have been given in [20], [37], [47].

Theorem 3.4.2 *Suppose W is a countable, primitive homogeneous graph or directed graph not equal to the dense local order, myopic local order or local partial order. Then any finite cover of W splits.*

Proof. By 3.4.1 and the known facts about W this reduces to the superlinked case. For this, it is enough to show that each cohomology group $H^1_c(\mathrm{Aut}(W), W^A/\Delta(A))$ is trivial (for all finite abelian groups A). This follows as in Example 3.2.3, by finding an appropriate 2-type Q in each case. □

The three exceptions in the theorem are genuine exceptions ([20, 24]). The following (together with Section 4.2) deals with the dense local order (and possibly with the other two cases).

Theorem 3.4.3 ([22]) *Let W be an irreducible, primitive permutation structure such that:*

- *the stabiliser of any point in W is irreducible;*
- *algebraic closure in W is trivial;*
- *if A is a cyclic finite abelian group then*

$$H^1_c(\mathrm{Aut}(W), A^W/\Delta(A)) \cong A.$$

Then a minimal finite cover of W has finite kernel.

4 Some non-split finite covers

Examples of split finite covers can be manufactured using Lemma 2.1.1. Non-split free covers are produced from non-split extensions of finite groups in Lemma 1.5.3. In this section we give three more examples of non-split finite covers.

4.1 Vector spaces

Let V be an infinite-dimensional vector space over a finite field F_q, let $V^* := V \setminus \{0\}$, and let PV be the corresponding projective space (which has as domain the set of 1-dimensional subspaces of V). We regard V^* as a structure with automorphism group $\mathrm{GL}(V)$ (the group of invertible linear transformations $V \to V$) and PV as a structure with automorphism group $\mathrm{PGL}(V)$ (the quotient of $\mathrm{GL}(V)$ by the central subgroup of linear transformations acting as scalars). The map $V^* \to PV$ given by $v \mapsto \langle v \rangle$ is a cover of PV. The kernel is just the centre of $\mathrm{GL}(V)$ (equal to the group of scalar transformations, isomorphic to the multiplicative group F_q^* of F_q), and the extension is non-split. The fibre group and binding group are both F_q^*.

4.2 Digraph coverings

We summarise some results from [20, 21]. This will provide us with examples of finite covers with finite kernels. The construction is very closely related to the topological notion of a covering space.

Definition 4.2.1 **(i)** A digraph on a set L is an irreflexive binary relation (usually denoted R) on L which is either a symmetric relation, or an antisymmetric relation. If (L, R) is a digraph and $a \in L$, put

$$a^+ := \{a' \in L : R(a, a')\}$$

and

$$a^- := \{a' \in L : R(a', a)\}.$$

A *path* in (L, R) is a sequence $x_0, \ldots, x_n$ such that for each $0 \le i \le n-1$, $R(x_i, x_{i+1})$ or $R(x_{i+1}, x_i)$ holds. The digraph is *connected* if any two vertices are linked by a path.

(ii) Let $\mathcal{S}$ denote the set of paths in L. We say that $p_1, p_2 \in \mathcal{S}$ are *elementarily homotopic* if one can be obtained from the other by one of the following operations (*elementary homotopies*):

(a) replace a consecutive triple abc with $R(a, b), R(b, c), R(a, c)$ by ac,

(b) replace a consecutive triple abc with $R(b, a), R(c, a), R(c, b)$ by ac,

(c) replace a consecutive triple aba by a.

Two paths are *homotopic* if one can be obtained from the other by a sequence of elementary homotopies. This is an equivalence relation on $\mathcal{S}$, and the homotopy class of p is denoted $[p]$. A connected digraph is *simply connected* if any two paths with the same endpoints are homotopic.

(iii) Let (A, R), (B, R') be two digraphs, both symmetric or both antisymmetric. A map $\sigma : A \to B$ is a *homomorphism* if, for every $b, b' \in \mathrm{Im}(\sigma)$, we have $R'(b, b')$ is and only if there are $a, a' \in A$ such that $R(a, a')$ and $\sigma(a) = b$, $\sigma(a') = b'$. A surjective homomorphism $\sigma : A \to B$ is a *covering* of digraphs if, for each $a \in A$, σ induces digraph isomorphisms $a^+ \to \sigma(a)^+$ and $a^- \to \sigma(a)^-$.

(iv) Let (L, R) be a connected digraph, fix $x_0 \in L$, let $\mathcal{P}$ denote the members of $\mathcal{S}$ which start at x_0, and U be the corresponding set of homotopy classes. Define $\sigma : U \to L$ by

$$\sigma([x_0 \dots x_r]) = x_r.$$

We can make U into a digraph with edge relation S (and σ into a covering) by putting $S([p_1], [p_2])$ if and only if $R(\sigma([p_1]), \sigma([p_2]))$ and $[p_2] = [p_1\sigma([p_2])]$. We call $\sigma : U \to L$ the *universal covering* of L.

It is shown in Lemma 5.3 of [21] that a universal covering $\sigma : U \to L$ is indeed a covering, that it has the expected universal property with respect to all coverings of L, and that it is simply connected and indeed is up to isomorphism the *unique* simply connected covering of L.

With $\sigma : U \to L$ as above, let $\Gamma(L, R)$ denote the subgroup of $\mathrm{Aut}\, L$ which maps fibres of σ to fibres of σ. The induced map $\Gamma(L, R) \to \mathrm{Aut}(L)$ is surjective and its kernel $\Delta(L, R)$ has the following characterisation. Fix $x_0 \in L$ as above, and let $p = [x_0 \dots x_r x_0] \in \sigma^{-1}(x_0)$. Define the *deck transformation* $d_p : U \to U$ by

$$d_p([x_0 x_1' \dots x_r']) = [x_0 \dots x_r x_0 x_1' \dots x_r'].$$

Then $\Delta(L, R)$ is the set of all deck transformations. Taking 'quotients' of σ by normal subgroups of finite index in Δ gives finite covers of L with finite kernels (see the 'converse' part of the statement of 1.14 in [21] for a precise formulation). We illustrate this with the following example, taken from [20].

Example 4.2.2 (The dense local order) Among the homogeneous directed graphs there is a tournament (that is, a digraph such that between any two vertices there is a directed edge) which can be described as follows. Let the vertex set of D be any countable dense subset of the unit circle, with no two antipodal points. Put $x \to y$ if it is faster to go anticlockwise from x to y than to go clockwise. For example, take D as having vertex set

$$\{e^{i\theta} : \theta \in \mathbb{Q} \cap [0, 2\pi)_{\mathbb{R}}\},$$

in the complex plane, with an edge $e^{i\theta_1} \to e^{i\theta_2}$ if and only if the angle at 0 subtended by the circular arc from $e^{i\theta_1}$ to $e^{i\theta_2}$ is less than π. All finite sufficiently nice covers of D are quotients of the universal covering $V \to D$, in a natural sense. The domain of V may be considered as the subset

$$2\pi\mathbb{Z} + (\mathbb{Q} \cap [0, 2\pi)_{\mathbb{R}})$$

of $\mathbb{R}$, and there is an arc $x \to y$ in V if and only if $0 < y - x < \pi$. Let $\tau : V \to D$ be the map $x \mapsto e^{ix}$. Observe that τ is surjective, and that $u \to w$ in D if and only if there are $x, y \in V$ such that $\tau(x) = u$, $\tau(y) = w$, and $x \to y$. Now it is easily seen that τ is simply connected.

For any $n \in \mathbb{N}$, let V_n be the quotient group $V/2\pi n\mathbb{Z}$ in the additive group $\mathbb{R}/2\pi n\mathbb{Z}$, with the natural map $\nu_n : V \to V_n$. The map τ factors through ν_n to give a finite-to-one map $\tau_n : V_n \to D$. Now, ν_n induces a digraph structure on V_n (we have $u \to w$ in V_n if and only if there are $x, y \in V$

with $x \to y$ and $\nu_n(x) = u$, $\nu_n(y) = w$) together with an equivalence relation (the fibres of τ_n). Then τ_n is a non-split finite cover of D, and its fibre groups and binding groups are isomorphic to Z_n.

Note that the group $\Delta(V, \to)$ is isomorphic to $\mathbb{Z}$. The kernel of V_n is the natural homomorphic image of this group.

We give a combinatorial condition which holds of many homogeneous structures and guarantees that minimal superlinked finite covers arise from digraph coverings. (This is done in slightly more generality in [21].)

If W is a permutation structure, A a finite subset of W, and $n \in \mathbb{N}$, then by an *n-type* of W over A we mean an $\mathrm{Aut}(W/A)$-orbit P on W^n (and if $A = \emptyset$ we refer to this simply as an n-type). For $\bar{x} \in W^n$ we write $P(\bar{x})$ to indicate that $\bar{x} \in P$.

Definition 4.2.3 Suppose that L is an irreducible transitive permutation structure with a 3-type P and 2-types Q, R. We say that (P, Q, R) is a *graphic triple of types* if the following hold.

1. $P(w, x, y)$ implies $Q(w, x), Q(w, y), R(x, y)$;
2. R (as a digraph relation) is connected;
3. either of the following holds:
   (a) if $R(x, y), R(x, z), R(y, z)$ then there is $w \in L$ such that $P(w, x, y)$, $P(w, y, z)$ and $P(w, x, z)$;
   (b) $P := \{(w, x, y) \in L^3 : R(w, x), R(w, y), R(x, y)\}$.

The following describes the finite covers of a structure L with a graphic triple (P, Q, R) in terms of digraph coverings of $(L; R)$.

Theorem 4.2.4 ([21], Theorem 1.13) *Let L be a transitive, irreducible permutation structure with a graphic triple of types (P, Q, R). Suppose further that $\mathrm{Aut}(W/x)$ and $\mathrm{Aut}(W/w, x)$ are irreducible, for $x, w \in L$ with $Q(w, x)$. Let $\tau : C \to L$ be a transitive, irreducible finite cover with finite kernel. Then there is an $\mathrm{Aut}(C)$-invariant digraph relation R' on C such that $\tau : (C, R') \to (L, R)$ is a covering of digraphs, and an automorphism g of the pure digraph (C, R') is an automorphism of C if and only if g maps τ-fibres to τ-fibres and induces an automorphism of L.*$\square$

We omit the proof here, but observe that R' is defined as follows:

> For $a, b \in C$, we have $R'(a, b)$ if and only if there is $w \in L$ such that $P(w, \tau(a), \tau(b))$ and a, b lie in the same $\mathrm{Aut}(C/C(w))$-orbit.

It is clear that the dense local order in Example 4.2.2 has a graphic triple of types, and so Example 4.2.2 gives all of its irreducible superlinked finite covers.

4.3 Two-graphs

The following material is due to Sasha Ivanov.

A *two-graph* T on a set X is a set of 3-subsets of X with the property that, for any 4-set $Y \subseteq X$, an even number of the 3-subsets of Y belong to T. Given any graph on X with edge set R, the set of triples carrying an odd number of edges of R is a two-graph on X. Any two-graph (X, T) arises in this way: let $x \in X$ and take as R the set of $\{y, z\}$ such that $\{x, y, z\} \in T$. The operation of *switching* a graph on X with respect to a partition of X into two parts replaces all edges between the parts with nonedges and all nonedges with edges, leaving edges and nonedges within each part unaltered. Any pair of graphs on X give the same two-graph if and only if they lie in the same switching class (that is, each can be obtained from the other by switching). See [8] for more details.

Any graph (X,R) determines a *double covering* (X^*,R^*) of the complete graph on X as follows. Let $X^* = \{x^+, x^- : x \in X\}$, where $(x^+,y^+),(x^-,y^-) \in R^*$ if and only if $(x,y) \in R$, and $(x^+,y^-),(x^-,y^+) \in R^*$ if and only if (x,y) is not in R (in [8] the double covering is considered under the complement of our R^*). Under R^* the transversals X^+ and X^- are copies of $W_0 = (X,R)$. Switching corresponds to interchanging the labels x^+ and x^- for some points x ([8]). So we can identify the set of switching operations with F_2^X/F_2 (that is, characteristic functions modulo 2 of subsets of X, quotiented out by the constant functions, to identify a set and its complement).

The triples from X inducing two triangles in the double covering form the two-graph T corresponding to R. It is easily seen that $M = (X^*,R^*)$ is a cover of $W = (X,T)$ under the natural map $\pi : X^* \to X$. Indeed, if $\alpha \in \mathrm{Aut}(W)$ then the graphs (X,R) and $(X,\alpha R)$ are in the same switching class so these exists $k \in F_2^X$ such that $k\alpha \in \mathrm{Aut}(M)$ (where, of course, we define $\alpha(x^+) = (\alpha(x))^+$ etc.). Moreover, the kernel of π is of order 2 (the non-identity element interchanges x^+ and x^- for every $x \in X$).

We can now produce several non-split covers. The most natural one is built in the following way.

Example 4.3.1 Let (X,R) be the countable, universal, homogeneous graph (the 'random graph'). Let $W = (X,T)$ be the corresponding two-graph, and let $\pi : M \to W$ be the double cover constructed as above, with $M = (X^*,R^*)$. We show that this does not split.

Let $G = \mathrm{Aut}(W)$. Let $x \in X$ and $Y = \{y \in X : (x,y) \in R\}$. Switching with respect to Y we get a graph on X with x as an isolated vertex and a copy of the random graph on $X \setminus \{x\}$. It follows that G is transitive on W and G_x, the stabiliser in G of x, is isomorphic to the automorphism group of the random graph (on $X \setminus \{x\}$). In particular, G acts 2-transitively on X and point stabilisers have no proper closed subgroups of finite index.

Suppose, for a contradiction, that π splits. Let H be a closed complement to $\mathrm{Aut}(M/W)$ in $\mathrm{Aut}(M)$. So H is the automorphism group of a trivial covering expansion of π. By Lemma 1.5.1 the fibre group of this covering expansion is trivial, and so H has two orbits on X^*, and acts 2-transitively on each of these. This implies that either (X^*,R^*) or its complement is bipartite. But neither of these is the case: it is easy to see that both the complete graph and the null graph on three vertices can be embedded in (X^*,R^*). This is the contradiction, and so π is non-split.

Remark 4.3.2 It is easy to use Corollary 3.2.1 to show that with W as in this example, the cohomology group $H^1_c(\mathrm{Aut}(W), Z_m^W/\Delta(Z_m))$ has size 1 or 2 depending on whether m is odd or even.

5 Problems

This section contains a selection of problems taken from [26].

Our first problem is open-ended.

Problem 5.1 Take a particular $\aleph_0$-categorical structure W and investigate its finite covers.

Here, it might be interesting to take as W a Grassmannian of a structure D for which there is good information about its finite covers (for example, a homogeneous graph or directed graph), and look at the covers with abelian kernel.

Problem 5.2 Suppose W is a countable, G-finite $\aleph_0$-categorical structure and $\pi_0 : C \to W$ is a finite cover. Let K be a closed subgroup of the kernel of π_0. Are there only finitely many isomorphism types of covering expansions of π_0 with kernel $K\Gamma$

By Proposition 2.2.2 the above has an affirmative answer in the case of π_0 having abelian kernel if the following is true in general.

Problem 5.3 Suppose W is a G-finite $\aleph_0$-categorical structure and $\pi_0 : C_0 \to W$ is a finite cover with abelian kernel K_0. Let π be a covering expansion of π_0 with kernel K. Is the cohomology group $H^1_c(\mathrm{Aut}(W), K_0/K)$ necessarily finiteΓ

It is shown in [22] that this has an affirmative answer if the following does (for definitions, see [22] or [26]):

Problem 5.4 Suppose W is a countable, G-finite $\aleph_0$-categorical structure. Does W have a nice enumerationΓ Does W have qdcc on finite coversΓ

Problem 5.5 In cases where Problem 5.3 is known to have an affirmative answer (for example, Grassmannians of vector spaces over finite fields ([22]), give explicit bounds on the size of the cohomology groups.

Also related to these is:

Problem 5.6 Is a finite cover of a countable, G-finite $\aleph_0$-categorical structure necessarily G-finiteΓ

Many of the results we have described have been directed at showing that certain finite covers must split. In fact, all of the non-split examples we have described have involved a non-split cover with finite kernel or have come from a non-split extension of a finite group (via the free cover construction, as in 1.5.3). So we pose the following:

Problem 5.7 Does there exist a primitive, irreducible $\aleph_0$-categorical structure W with irreducible one-point stabilisers, which has a minimal finite cover $\pi : C \longrightarrow W$ with infinite elementary abelian kernelΓ

It would be remarkable if there were no such example. There is an example due to Ivanov of a minimal finite cover of the random two-graph with infinite kernel where the fibre groups are cyclic of order 4.

The following is of course suggested by the work [2] of Ahlbrandt and Ziegler (and others) on quasi finite-axiomatisability of totally categorical structures.

Problem 5.8 Investigate finite axiomatisability of a finite cover $\pi : C \rightarrow W$ relative to an axiomatisation of W.

Finally a close examination of the argument at the end of [32] might be a starting place for the following.

Problem 5.9 Develop a theory of affine covers. In particular, examine the case of affine covers where the kernel is central in the automorphism group.

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