MONOCHROMATIC PATHS
AND QUASI-MONOCHROMATIC CYCLES
IN EDGE-COLOURED BIPARTITE TOURNAMENTS

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Abstract
We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike. A directed cycle is called quasi-monochromatic if with at most one exception all of its arcs are coloured alike.

A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

(i) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and

(ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic directed path.

In this paper it is proved that if $D$ is an $m$-coloured bipartite tournament such that: every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic, and $D$ has no induced particular 6-element bipartite tournament $T_6$, then $D$ has a kernel by monochromatic paths.

Keywords: kernel, kernel by monochromatic paths, bipartite tournament.

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1. Introduction

For general concepts we refer the reader to [1]. Let $D$ be a digraph, and let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of $D$, respectively. An arc $(u_1, u_2) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of $D$ (resp. symmetrical part of $D$) which is denoted by $\text{Asym}(D)$ (resp. $\text{Sym}(D)$) is the spanning subdigraph of $D$ whose arcs are the asymmetrical (resp. symmetrical) arcs of $D$. If $S$ is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by $S$ is the digraph having vertex set $S$, and whose arcs are those arcs of $D$ joining vertices of $S$.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$, that is an arc from $z$ to some vertex in $N$. A digraph $D$ is called a kernel-prefect digraph when every induced subdigraph of $D$ has a kernel. Sufficient conditions for the existence of kernels in a digraph have been investigated by several authors, Von Neumann and Morgenstern [14], Richardson [11], Duchet and Meyniel [3] and Galeana-Sánchez and Neumann-Lara [4]. The concept of kernel is very useful in applications. Clearly, the concept of kernel by monochromatic paths generalizes those of kernel.

A digraph $D$ is called a bipartite tournament if its set of vertices can be partitioned into two sets $V_1$ and $V_2$ such that: (i) every arc of $D$ has an endpoint in $V_1$ and the other endpoint in $V_2$, and (ii) for all $x_1 \in V_1$ and for all $x_2 \in V_2$, we have $|(\{x_1, x_2\}, (x_2, x_1)) \cap A(D)| = 1$. We will write $D = (V_1, V_2)$ to indicate the partition.

If $\mathcal{C} = (z_0, z_1, \ldots, z_n, z_0)$ is a directed cycle and if $z_i, z_j \in V(\mathcal{C})$ with $i \leq j$ we denote by $(z_i, \mathcal{C}, z_j)$ the $z_i z_j$-directed path contained in $\mathcal{C}$, and $\ell(z_i, \mathcal{C}, z_j)$ will denote its length; similarly $\ell(\mathcal{C})$ will denote the length of $\mathcal{C}$.

If $D$ is an $m$-coloured digraph, then the closure of $D$, denoted by $\mathcal{C}(D)$ is the $m$-coloured multidigraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, $A(\mathcal{C}(D)) = A(D) \cup \{(u, v) \text{ with colour } i \mid \text{there exists an } uv\text{-monochromatic directed path coloured } i \text{ contained in } D\}$.

Notice that for any digraph $D$, $\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D)$ and $D$ has a kernel by monochromatic paths if and only if $\mathcal{C}(D)$ has a kernel.

In [13] Sands et al. have proved that any 2-coloured digraph has a kernel by monochromatic paths; in particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let $T$ be a 3-coloured tournament such that every directed
cycle of length 3 is quasi-monochromatic; must $\mathcal{C}(T)$ have a kernel? (This question remains open.) In [12] Shen Minggang proved that if in the problem we ask that every transitive tournament of order 3 be quasi-monochromatic, the answer will be yes; and the result is best possible for $m$-coloured tournaments with $m \geq 5$. In 2004 [9] presented a 4-coloured tournament $T$ such that every directed cycle of order 3 is quasi-monochromatic; but $T$ has no kernel by monochromatic paths. The known sufficient conditions for the existence of kernel by monochromatic paths in $m$-coloured ($m \geq 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of certain subdigraphs. In [5] it was proved that if $T$ is an $m$-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then $\mathcal{C}(T)$ is kernel-perfect. A generalization of this result was obtained by Hahn, Ille and Woodrow in [10]; they proved that if $T$ is an $m$-coloured tournament such that every directed cycle of length $k$ is quasi-monochromatic and $T$ has no polychromatic directed cycles of length $\ell$, $\ell < k$, for some $k \geq 4$, then $T$ has a kernel by monochromatic paths. (A directed cycle is polychromatic if it uses at least three different colours in its arcs). Results similar to those in [12] and [5] were proved for the digraph obtained from a tournament by the deletion of a single arc, in [7] and [6], respectively. Kernels by monochromatic paths in bipartite tournaments were studied in [8]; where it is proved that if $T$ is a bipartite tournament such that every directed cycle of length 4 is monochromatic, then $T$ has a kernel by monochromatic paths.

We prove that if $T$ is a bipartite tournament such that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic and $T$ has no induced subtournament isomorphic to $\tilde{T}_6$, then $T$ has a kernel by monochromatic paths.

$\tilde{T}_6$ is the bipartite tournament defined as follows:

\begin{align*}
V(\tilde{T}_6) &= \{u, v, w, x, y, z\}, \\
A(\tilde{T}_6) &= \{(u, w), (v, w), (w, x), (w, z), (x, y), (y, u), (y, v), (z, y)\} \text{ with } \\
&\{(u, w), (w, x), (y, u), (z, y)\} \text{ coloured 1 and } \{(v, w), (w, z), (x, y), (y, v)\} \text{ coloured 2.} \ (\text{See Figure 1).}
\end{align*}

We will need the following result.

**Theorem 1.1** Duchet [2]. If $D$ is a digraph such that every directed cycle has at least one symmetrical arc, then $D$ is a kernel-perfect digraph.
2. The Main Result

The following lemmas will be useful in the proof of the main result.

**Lemma 2.1.** Let $D = (V_1, V_2)$ be a bipartite tournament and $C = (u_0, u_1, \ldots, u_n)$ a directed walk in $D$. For $\{i, j\} \subseteq \{0, 1, \ldots, n\}$, $(u_i, u_j) \in A(D)$ or $(u_j, u_i) \in A(D)$ if and only if $j - i \equiv 1 \pmod{2}$.

**Lemma 2.2.** For a bipartite tournament $D = (V_1, V_2)$, every closed directed walk of length at most 6 in $D$ is a directed cycle of $D$.

**Lemma 2.3.** Let $D$ be an $m$-coloured bipartite tournament such that every directed cycle of length 4 is quasi-monochromatic and every directed cycle of length 6 is monochromatic. If for $u, v \in V(D)$ there exists a $uv$-monochromatic directed path and there is no $vu$-monochromatic directed path (in $D$), then at least one of the following conditions hold:

(i) $(u, v) \in A(D)$,

(ii) there exists (in $D$) a $uv$-directed path of length 2,

(iii) there exists a $uv$-monochromatic directed path of length 4.

**Proof.** Let $D, u, v$ be as in the hypothesis. If there exists a $uv$-directed path of odd length, then it follows from Lemma 2.1 that $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Since there is no $vu$-monochromatic directed path in $D$, then $(u, v) \in A(D)$ and Lemma 2.3 holds. So, we will assume that every $uv$-directed path has even length. We proceed by induction on the length of a $uv$-monochromatic directed path.

Clearly Lemma 2.3 holds when there exists a $uv$-monochromatic directed path of length at most 4. Suppose that $T = (u = u_0, u_1, u_2, u_3, u_4, u_5,$
\( u_6 = v \) is a \( uv \)-monochromatic directed path of length 6. It follows from Lemma 2.1 that \((u, u_5) \in A(D)\) or \((u_5, u) \in A(D)\) and also \((u_1, v) \in A(D)\) or \((v, u_1) \in A(D)\). If \((u, u_5) \in A(D)\) or \((u_1, v) \in A(D)\) then we obtain a \( uv \)-directed path of length two, and we are done. So, we will assume that \((u_5, u) \in A(D)\) and \((v, u_1) \in A(D)\). Thus \((u = u_0, u_1, u_2, u_3, u_4, u_5, u_0 = u)\) is a directed cycle of length 6 which is monochromatic and has the same colour as \( T \). Also \((u_1, u_2, u_3, u_4, u_5, u_6 = v, u_1)\) is a directed cycle coloured as \( T \). Hence \((v = u_6, u_1, u_2, u_3, u_4, u_5, u_0 = u)\) is a \( vv \)-monochromatic directed path, a contradiction. Suppose that Lemma 2.3 holds when there exists a \( uv \)-monochromatic directed path of even length \( \ell \) with \( 6 \leq \ell \leq 2n \). Now assume that there exists a \( uv \)-monochromatic directed path say \( T = (u = u_0, u_1, \ldots, u_2(n+1) = v) \) with \( \ell(T) = 2(n + 1) \); we may assume w.l.o.g. that \( T \) is coloured 1.

From Lemma 2.1 we have that for each \( i \in \{0, 1, \ldots, 2(n+1) - 5\} \), \((u_{i+5}, u_i) \in A(D)\) or \((u_i, u_{i+5}) \in A(D)\). We will analyze two possible cases:

**Case a.** For each \( i \in \{0, 1, \ldots, 2(n+1) - 5\} \), \((u_{i+5}, u_i) \in A(D)\).
In this case \( C_6 = (u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}, u_{i+5}, u_i) \) is a directed cycle with \( \ell(C_6) = 6 \); so it is monochromatic and coloured 1 (as \((u_i, u_{i+1})\) is coloured 1). Let \( k \in \{1, 2, 3, 4, 5\} \) such that \( k \equiv 2(n + 1) \pmod{5} \), then \((v = u_2(n+1), u_2(n+1) - 5, u_2(n+1) - 10, \ldots, u_k) \cup (u_k, T, u_5) \cup (u_5, u_0)\) is a \( vv \)-monochromatic directed path in \( D \), a contradiction.

**Case b.** For some \( i \in \{0, 1, \ldots, 2(n+1) - 5\} \), \((u_i, u_{i+5}) \in A(D)\).
Notice that from Lemma 2.1, there exists an arc between \( u_1 \) and \( u_{2(n+1)} \) and also there exists an arc between \( u_0 \) and \( u_{2n+1} \). If \((u_1, u_{2(n+1)}) \in A(D)\) or \((u_0, u_{2n+1}) \in A(D)\), then we obtain a \( uv \)-directed path of length two, and we are done. So, we will assume that \((u_{2(n+1)}, u_1) \in A(D)\) and \((u_{2n+1}, u_0) \in A(D)\). Observe that: If for some \( i \in \{1, \ldots, 2(n+1) - 5\} \), \((u_{2(n+1)}, u_i) \in A(D)\) and the arcs \((u_{2(n+1)}, u_i)\) and \((u_{2n+1}, u_0)\) are coloured 1, then \((v = u_{2(n+1)}, u_i) \cup (u_i, T, u_{2n+1}) \cup (u_{2n+1}, u_0)\) is a \( vu \)-directed path coloured 1, contradicting the hypothesis. Hence we have:

(a) If for some \( i \in \{1, 2, \ldots, 2(n+1) - 5\} \) we have \((u_{2(n+1)}, u_i) \in A(D)\), then \((u_{2(n+1)}, u_i)\) is not coloured 1 or \((u_{2n+1}, u_0)\) is not coloured 1.

**Case b.1.** \((u_{2n+1}, u_0)\) is not coloured 1.
Recall that for some \( i \in \{0, 1, \ldots, 2(n+1) - 5\} \), \((u_i, u_{i+5}) \in A(D)\). Let \( \{i_0, j_0\} \subseteq \{0, 1, \ldots, 2(n+1)\} \) be such that \( j_0 - i_0 = \max\{j - i \mid \{i, j\} \subseteq \{0, 1, \ldots, 2(n+1)\} \} \).
whose length is less than $f$.

Hence $A$ is a directed cycle of length 6 (monochromatic and coloured 1). Thus $(u_{i_0-2}, u_{j_0+2}, u_{i_0-2}) \in A(D)$. Thus $(u_{i_0-2}, u_{i_0-1}, u_{i_0}, u_{j_0+1}, u_{j_0+2}, u_{i_0-2})$ is a directed cycle of length 6 and hence monochromatically coloured 1. Now $(u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, u_{2(n+1)} = v)$ is a $uv$-monochromatic directed path whose length is less than $\ell(T)$. Then the assertion of Lemma 2.3 follows from the inductive hypothesis.

Case b.1.2. $i_0 = 0$.

When $j_0 \leq 2n-3$, it follows from Lemma 2.1 and the choice of $\{i_0, j_0\}$ that $(u_{j_0+4}, u_0 = u_0) \in A(D)$. Thus $(u_0 = u_{i_0}, u_{j_0}, u_{j_0+1}, u_{j_0+2}, u_{j_0+3}, u_{j_0+4}, u_0)$ is a monochromatic directed cycle (it has length 6), coloured 1. Hence $(u_0, u_{j_0}) \cup (u_{j_0}, T, v)$ is a $uv$-monochromatic directed path whose length is less than $\ell(T)$ and the assertion follows from the inductive hypothesis.

When $j_0 \geq 2n-1$, we have $j_0 = 2n-1$ (recall $(u_{2n+1}, u_0) \in A(D), j_0 - i_0 \equiv 1 \pmod{2}, i_0 = 0$). So, $(u_0 = u_{i_0}, u_{j_0} = u_{2n-1}, u_{2n}, u_{2n+1}, u_0)$ is a directed cycle of length 4 which by hypothesis is quasi-monochromatic. Since $(u_{2n+1}, u_0)$ is not coloured 1, then $(u_{i_0}, u_{j_0})$ is coloured 1, and $(u = u_{i_0}, u_{j_0} = u_{2n-1}, u_{2n}, u_{2n+1}, u_{2n+2} = v)$ is a $uv$-monochromatic directed path coloured 1 of length 4.

Case b.1.3. $i_0 = 1$.

When $j_0 \leq 2n-2$, we have $(u_{j_0+4}, u_0 = u_1) \in A(D)$ (Lemma 2.1 and the choice of $\{i_0, j_0\}$). Thus $(u_1 = u_{j_0}, u_{j_0}, u_{j_0+1}, u_{j_0+2}, u_{j_0+3}, u_{j_0+4}, u_0)$ is a directed cycle of length 6 (monochromatic and coloured 1). Hence $(u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, v)$ is a $uv$-monochromatic directed path whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

When $j_0 \geq 2n$, we have $j_0 = 2n$ (as $j_0 - i_0 \equiv 1 \pmod{2}, i_0 = 1$ and $(u_{2n+2}, u_1) \in A(D)$). Hence $(u_1 = u_{i_0}, u_{j_0} = u_{2n}, u_{2n+1}, u_0, u_1)$ is a directed cycle of length 4, from the hypothesis it is quasi-monochromatic and $(u_{2n+1}, u_0)$ is not coloured 1, so $(u_{i_0}, u_{j_0})$ is coloured 1. Therefore $(u_0, u_1 = u_{i_0}, u_{j_0} = u_{2n}, u_{2n+1}, u_{2n+2} = v)$ is a $uv$-monochromatic directed path, coloured 1, of length 4.
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Case b.1.4. $j_0 = 2n + 1$.
When $i_0 \geq 4$, we have $(u_{2n+1} = u_{j_0}, u_{i_0-4}) \in A(D)$ ($j_0 - i_0 \equiv 1 (\mod 2)$, $j_0 - (i_0 - 4) \equiv 1 (\mod 2)$, and the choice of $(i_0, j_0)$). Therefore $(u_{i_0}, u_{j_0} = u_{2n+1}, u_{i_0-4}, u_{i_0-3}, u_{i_0-2}, u_{i_0-1}, u_{i_0})$ is a directed cycle of length 6 (and thus it is monochromatic) coloured 1. Thus $(u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, v)$ is a $uv$-directed path coloured 1, whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

When $i_0 \leq 2$, we have $i_0 = 2$ (as $j_0 - i_0 \equiv 1 (\mod 2)$, $j_0 = 2n + 1$, and $(u_{2n+1}, u_0) \in A(D)$). Hence $(u_2 = u_{i_0}, u_{j_0} = u_{2n+1}, u_0, u_1, u_2)$ is quasi-monochromatic (as it has length 4). Since $(u_{2n+1}, u_0)$ is not coloured 1, it follows that $(u_{i_0}, u_{j_0})$ is coloured 1. We conclude that $(u_0, u_1, u_2 = u_{i_0}, u_{j_0} = u_{2n+1}, u_{2n+2} = v)$ is a $uv$-directed path coloured 1 of length 4.

Case b.1.5. $j_0 = 2n + 2$.
When $i_0 \geq 5$, we have $(u_{2n+2} = u_{j_0}, u_{i_0-4}) \in A(D)$ (arguing as in b.1.4). Thus $(u_{i_0}, u_{j_0}, u_{i_0-4}, u_{i_0-3}, u_{i_0-2}, u_{i_0-1}, u_{i_0})$ is monochromatic (as it is a directed cycle of length 6). Hence $(u, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, u_{2n+2}, v)$ is a $uv$-monochromatic directed path with length less than $\ell(T)$; and the result follows from the inductive hypothesis.

When $i_0 \leq 3$, we have $i_0 = 3$ (as $j_0 - i_0 \equiv 1 (\mod 2)$ and $(u_{2n+2}, u_1) \in A(D)$). Hence $(u_3 = u_{i_0}, u_{j_0} = u_{2n+2}, u_1, u_2, u_3)$ is quasi-monochromatic. If $(u_{i_0}, u_{2n+2})$ is coloured 1, then $(u_0, u_1, u_2, u_3 = u_{i_0}, u_{2n+2} = v)$ is a $uv$-monochromatic directed path of length 4. So we will assume that $(u_{i_0}, u_{2n+2})$ is not coloured 1, and hence $(u_{2n+2}, u_1)$ is coloured 1.

If $(u_i, u_0) \in A(D)$ for some $i \in \{3, \ldots, 2n + 1\}$, then $(u_i, u_0)$ is not coloured 1 (otherwise $(v = u_{2n+2}, u_1) \cup (u_1, T, u_i) \cup (u_i, u_0)$ is a $uv$-monochromatic directed path, contradicting our hypothesis).

Now observe that $(u_0, u_5) \in A(D)$; otherwise $(u_5, u_0) \in A(D)$ and $(u_0, u_1, u_2, u_3, u_4, u_5, u_0)$ is monochromatic which implies $(u_5, u_0)$ is coloured 1, a contradiction.

Let $k_0 = \max\{i \in \{5, 6, \ldots, 2n - 1\} | (u_0, u_i) \in A(D)\}$. Then, we have $(u_0, u_{k_0}) \in A(D)$ and $(u_{k_0+2}, u_0) \in A(D)$; moreover $(u_{k_0+2}, u_0)$ is not coloured 1. Since $(u_0, u_{k_0}, u_{k_0+1}, u_{k_0+2}, u_0)$ is quasi-monochromatic and $(u_{k_0+2}, u_0)$ is not coloured 1, we have $(u_0, u_{k_0})$ is coloured 1. Thus $(u = u_0, u_{k_0}) \cup (u_{k_0}, T, u_{2n+1} = v)$ is a $uv$-monochromatic directed path whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

Case b.2. In view of assertion (a) and case b.1, we may assume that: If $(u_{2(n+1)}, u_i) \in A(D)$ for some $i \in \{1, 2, \ldots, 2(n + 1) - 5\}$ then $(u_{2(n+1)}, u_i)$
is not coloured 1.
— $\langle u_2(n+1), u_1 \rangle$ is not coloured 1: It follows from the fact \( \langle u_2(n+1), u_1 \rangle \in A(D) \).
— $\langle u_2(n+1)-5, u_2(n+1) \rangle \in A(D)$: Otherwise it follows from Lemma 2.1 that \( \langle u_2(n+1), u_2(n+1)-5 \rangle \in A(D) \), now \( \langle u_2(n+1)-5, T, u_2(n+1) \rangle \cup \langle u_2(n+1), u_2(n+1)-5 \rangle \) is monochromatic coloured 1 (note that it is a directed cycle of length 6 and it has arcs in $T$), and then \( \langle u_2(n+1), u_2(n+1)-5 \rangle \) is coloured 1, contradicting our assumption.

Let $i_0 = \max\{i \in \{0, 1, 2, \ldots, 2(n+1)-7\} \mid \langle u_2(n+1), u_i \rangle \in A(D)\}$ (notice that $i_0$ is well defined as $\langle u_2(n+1), u_1 \rangle \in A(D)$). Therefore $\langle u_2(n+1), u_{i_0} \rangle \in A(D)$, $\langle u_{i_0+2}, u_2(n+1) \rangle \in A(D)$ and $\langle u_{i_0+1}, u_{i_0} \rangle$ is not coloured 1. Now we have the directed cycle of length 4 $\langle u_2(n+1), u_{i_0}, u_{i_0+1}, u_{i_0+2}, u_2(n+1) \rangle$ which is quasi-monochromatic with $\langle u_2(n+1), u_{i_0} \rangle$ not coloured 1; and $\langle u_{i_0}, u_{i_0+1} \rangle$, $\langle u_{i_0+1}, u_{i_0+2} \rangle$ coloured 1; so $\langle u_{i_0+2}, u_2(n+1) \rangle$ is coloured 1. Thus, $\langle u = u_0, T, u_{i_0+2} \rangle \cup \langle u_{i_0+2}, u_2(n+1) = v \rangle$ is a $uv$-directed path coloured 1 whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

\[ \textbf{Theorem 2.1.} \text{Let $D$ be an $m$-coloured bipartite tournament. Assume that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic and $D$ has no subtournament isomorphic to $T$. Then $C(D)$ is a kernel-perfect digraph.} \]

\[ \textbf{Proof.} \text{We will prove that every directed cycle contained in $C(D)$ has at least one symmetrical arc. Then Theorem 2.1 will follow from Theorem 1.1. We proceed by contradiction, suppose that there exists $C = \langle u_0, u_1, \ldots, u_n, u_0 \rangle$ a directed cycle contained in Asym($C(D)$). Therefore, $n \geq 2$. For each $i \in \{0, 1, \ldots, n\}$ there exists a $u_iu_{i+1}$-monochromatic directed path contained in $D$, and there is no $u_{i+1}u_i$-monochromatic directed path contained in $D$. Thus, it follows from Lemma 2.3 that at least one of the following assertions hold:} \]

(ii) there exists a $u_iu_{i+1}$-directed path of length 2,
(iii) there exists a $u_iu_{i+1}$-monochromatic directed path of length 4. Throughout the proof the indices of the vertices of $C$ are taken mod $n+1$.

For each $i \in \{0, 1, \ldots, n\}$ let
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\[ T_i = \begin{cases} 
(u_i, u_{i+1}) & \text{if } (u_i, u_{i+1}) \in A(D), \\
\text{a } u_{i+1} \text{-directed path of length 2, when } (u_i, u_{i+1}) \notin A(D) \\
\text{and such a path exists,} \\
\text{a } u_{i+1} \text{-monochromatic directed path of length 4, otherwise.} 
\end{cases} \]

Let \( C' = \bigcup_{i=0}^{n} T_i \). Clearly \( C' \) is a closed directed walk; let \( C' = (z_0, z_1, \ldots, z_k, z_0) \). We define the function \( \varphi \colon \{0, 1, \ldots, k\} \to V(C) \) as follows: If \( T_i = (u_i = z_{i_0}, z_{i_0+1}, \ldots, z_{i_0+r} = u_{i+1}) \) with \( r \in \{1, 2, 4\} \), then \( \varphi(j) = z_{i_0} \) for each \( j \in \{i_0, i_0 + 1, \ldots, i_0 + r - 1\} \). We will say that the index \( i \) of the vertex \( z_i \) of \( C' \) is a principal index when \( z_i = \varphi(i) \). We will denote by \( I_p \) the set of principal indices.

I. First observe that for each \( i \in \{0, 1, \ldots, k\} \) we have \( \{i, i+1, i+2, i+3\} \cap I_p \neq \emptyset \). Assume w.l.o.g. that \( 0 \in I_p \) and \( z_0 = u_0 \). In what follows, the indices of the vertices of \( C' \) will be taken modulo \( k+1 \).

Case a. \( k = 3 \).
In this case \( C' \) is a directed cycle of length 4 and hence it is quasi-monochromatic. Since \( n \geq 2 \), then \( u_1 \in \{z_1, z_2\} \) and \( u_n \in \{z_2, z_3\} \). And it is easy to see that there exists a \( u_{i+1}u_i \)-monochromatic directed path in \( D \), for some \( i \in \{0, 1, \ldots, n\} \), a contradiction.

Case b. \( k = 5 \).
In this case \( C' \) is a directed cycle of length 6, and then it is monochromatic, which clearly implies that there exists a \( u_1u_0 \)-monochromatic directed path in \( D \), a contradiction.

Case c. \( k \geq 7 \).
We will prove several assertions.

1(c). For each \( i \in \{0, 1, \ldots, k\} \cap I_p \) we have \( (z_i, z_{i+5}) \in A(D) \). Since \( (i + 5) \equiv 1 \pmod{2} \), it follows that \( (z_i, z_{i+5}) \in A(D) \) or \( (z_{i+5}, z_i) \in A(D) \). Assume, for a contradiction that \( (z_{i+5}, z_i) \in A(D) \). Therefore \( (z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}, z_{i+5}, z_i) \) is monochromatic. Now let \( j \in \{0, 1, \ldots, n\} \) be such that \( u_j = z_i \), then \( u_{j+1} \in \{z_{i+1}, z_{i+2}, z_{i+4}\} \). So there exists a \( u_{j+1}u_j \)-monochromatic directed path, a contradiction.

2(c). For each \( i \in \{0, \ldots, k\} \) such that \( i + 5 \in I_p \) we have \( (z_i, z_{i+5}) \in A(D) \).
Assume, for a contradiction that \((z_{i+5}, z_i) \in A(D)\). Then \((z_i, z_{i+1}, z_i+2, z_{i+3}, z_i+4, z_{i+5}, z_i)\) is monochromatic. Let \(j \in \{0, \ldots, n\}\) be such that \(u_j = z_{i+5}\), thus \(u_{j-1} \in \{z_{i+1}, z_i+3, z_i+4\}\) and there exists a \(u_j u_{j-1}\)-monochromatic directed path, a contradiction.

3(c). For each \(i \in \{5, \ldots, k-2\}\) such that \(i \equiv 1 \pmod{4}\) we have \((z_0, z_i) \in A(D)\). We proceed by contradiction; suppose that for some \(i \in \{5, \ldots, k-2\}\) we have \(i \equiv 1 \pmod{4}\) and \((z_i, z_0) \in A(D)\). Since \(0 \in I_p\), it follows from 1(c) that \((z_0, z_0) \in A(D)\) and then \(i \geq 9\).

Let \(i_0 = \min\{j \in \{5, \ldots, k-6\} \mid j \equiv 1 \pmod{4}\}\) and \((z_j, z_0) \in A(D)\). Notice that \(i_0\) is well defined, as \(i \geq 9\). Thus \((z_0, z_{i_0-4}) \in A(D), (z_0, z_{i_0}) \in A(D)\) and \((z_{i_0}, z_0) \in A(D)\). Now \(C^2 = (z_0, z_{i_0-4}, z_{i_0-3}, z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_0)\) is a directed cycle of length 6 and hence it is monochromatic, w.l.o.g we will assume that it is coloured 1. Now we consider two cases:

3(c).1. \(i_0 \in I_p\).

In this case \(z_0 = u_j\) for some \(j \in \{3, \ldots, n\}\) and from the definition of \(C'\), \(u_{j+1} \in \{z_{i_0+1}, z_{i_0+2}, z_{i_0+4}\}\). In any case, there exists a \(u_{j+1} u_j\)-monochromatic directed path contained in \(C\), a contradiction.

3(c).2. \(i_0 \notin I_p\).

In this case, we have from observation I that \(\{i_0-3, i_0-2, i_0-1\} \cap I_p \neq \emptyset\), let \(\ell \in \{i_0 - 3, i_0 - 2, i_0 - 1\} \cap I_p\) and \(u_j \in V(C)\) such that \(u_j = z_{\ell}\). From 1(c) we have \((z_{\ell}, z_{\ell+5}) \in A(D)\) and \(\ell + 5 \in \{i_0 + 2, i_0 + 3, i_0 + 4\}\) which implies that \(C^3 = (z_{i_0-4}, C', z_0) \cup (z_{\ell}, z_{\ell+5}) \cup (z_{\ell+5}, z_{i_0-4})\) is a closed directed walk of length 6 (as \((z_{i_0-4}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})\) is a closed directed walk of length 10). It follows from Lemma 2.2 that \(C^3\) is a directed cycle and the hypothesis implies that it is monochromatic. Since \((z_{i_0+4}, z_0) \in A(C^2) \cap A(C^3)\), we have that \(C^3\) is coloured 1. Now from the definition of \(C'\) we have \(u_{j+1} \in \{z_{\ell+1}, z_{\ell+2}, z_{\ell+4}\}\) \(\subseteq \{z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\}\).

When \(u_{j+1} \in \{z_{i_0}, z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\}\), we obtain that \((z_0, C^2, z_0) \cup (z_0, C^4, z_0)\) contains a \(u_{j+1} u_j\)-monochromatic directed path, a contradiction.

When \(u_{j+1} \in \{z_{i_0-2}, z_{i_0-1}\}\), we take \(i_1 \in \{i_0 - 2, i_0 - 1\}\) such that \(u_{j+1} = z_{i_1}\). From 1(c) we have \((z_{i_1}, z_{i_1+5}) \in A(D)\) where \(z_{i_1+5} \in \{z_{i_0+3}, z_{i_0+4}\}\), thus \(C^4 = (z_{i_0-4}, C', z_{i_1}) \cup (z_{i_1}, z_{i_1+5}) \cup (z_{i_1+5}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})\) is a directed cycle of length 6 (notice that \((z_{i_0-4}, C^4, z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})\) is a closed directed walk of length 10). From the hypothesis we have that \(C^4\) is monochromatic. Since \((z_{i_0+4}, z_0) \in A(C^4) \cap A(C^3)\) we obtain that \(C^4\) is coloured 1.
Finally, since \( \{u_j, u_{j+1}\} \subseteq V(C^4) \) we have a \( u_{j+1}u_j \)-monochromatic directed path, a contradiction.

4(c). For any \( i \in \{3, \ldots, k-4\} \) such that \( i \equiv k \mod 4 \) we have \((z_i, z_0) \in A(D)\). Since \( i \equiv k \mod 4 \) and \( k \equiv 1 \mod 2 \) we have \( i \equiv 1 \mod 2 \), and from Lemma 2.1 we obtain \((z_0, z_i) \in A(D)\) or \((z_i, z_0) \in A(D)\).

Assume, for a contradiction that \((z_0, z_i) \in A(D)\), for some \( i \in \{3, \ldots, k-4\} \) such that \( i \equiv k \mod 4 \).

Since \( 0 \in I_p \), it follows from 2(c) that \((z_{k-4}, z_0) \in A(D)\), and thus \( i \leq k - 8 \). Let \( i_0 = \max\{i \in \{7, \ldots, k-4\} \mid i \equiv k \mod 4\} \) and \((z_0, z_{i_0-4}) \in A(D)\), \((z_0, z_{i_0}) \in A(D)\) and \((z_{i_0+4}, z_0) \in A(D)\). So \( C^2 = (z_0, z_{i_0-4}, z_{i_0-3}, z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_0) \) is a directed cycle of length 6 and hence it is monochromatic, w.l.o.g. assume that it is coloured 1.

When \( i_0 \notin I_p \), we have \( z_{i_0} = u_j \) for some \( j \in \{2, \ldots, n-2\} \). From the definition of \( C' \) we have \( u_j-1 \in \{z_{i_0-1}, z_{i_0-2}, z_{i_0-4}\} \) and then there exists a \( u_ju_{j-1} \)-monochromatic directed path contained in \( C^2 \), a contradiction.

When \( i_0 \notin I_p \), then from I we have \( \{i_0 - 3, i_0 - 2, i_0 - 1\} \cap I_p \neq \emptyset \). Let \( \ell \in \{i_0 - 3, i_0 - 2, i_0 - 1\} \cap I_p \), so \( u_j = z_\ell \) for some \( u_j \in V(C) \). From 1(c) it follows \((z_\ell, z_{\ell+5}) \in A(D)\) and \( \ell + 5 \in \{i_0 + 2, i_0 + 3, i_0 + 4\} \).

Now \( C^3 = (z_{i_0-4}, C', z_\ell) \cup (z_\ell, z_{\ell+5}) \cup (z_{\ell+5}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4}) \) is a directed cycle of length 6 (as \((z_{i_0-4}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4}) \) is a closed directed walk of length 10), and then it is monochromatic. Since \((z_0, z_{i_0-4}) \in A(C^2) \cap A(C^3) \) we have \( C^3 \) is coloured 1. Observe that \( u_{j+1} \in \{z_{i_0-2}, z_{i_0-1}, z_{i_0}\} \) then there exists a \( u_{j+1}u_j \)-monochromatic directed path contained in \( C^2 \), a contradiction.

If \( u_{j+1} \in \{z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\} \), then we take \( i_1 \in \{i_0 + 1, i_0 + 2, i_0 + 3\} \) such that \( u_{j+1} = z_{i_1} \). From 2(c), \((z_{i_1-5}, z_{i_1}) \in A(D)\), where \( z_{i_1-5} \in \{z_{i_0-4}, z_{i_0-3}, z_{i_0-2}\} \). Now, \( C^4 = (z_{i_0-4}, C', z_{i_1-5}) \cup (z_{i_1-5}, z_{i_1}) \cup (z_{i_1}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4}) \) is a directed cycle of length 6 (as \((z_{i_0-4}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4}) \) is a closed directed walk of length 10), so it is monochromatic and coloured 1 (because \((z_0, z_{i_0-4}) \in A(C^4) \cap A(C^3)\)). We conclude that \((u_{j+1} = z_{i_1}, C^4, z_{i_0-4}) \cup (z_{i_0-4}, C^2, z_\ell = u_j) \) contains a \( u_{j+1}u_j \)-monochromatic directed path, a contradiction.

Now we will analyze the two possible cases:

Case c.1. \( k \equiv 1 \mod 4 \).

Since \( 0 \in I_p \), it follows from 2(c) that \((z_{k-4}, z_0) \in A(D)\). On the other hand,
we have $k - 4 \equiv 1 \pmod{4}$, and from 3(c), $(z_0, z_{k-4}) \in A(D)$, a contradiction (as $D$ is a bipartite tournament).

Case 2. $k \equiv 3 \pmod{4}$.

First, we prove several assertions:

5(c.2). For any $i \in \{3, \ldots, k-4\}$ such that $i \equiv 3 \pmod{4}$ we have $(z_i, z_0) \in A(D)$.

This assertion follows from 4(c) as $i \equiv k \pmod{4}$.

6(c.2). For any $i, j \in \{0, \ldots, k\}$ such that $i \in I_p$ and $j - i \equiv 1 \pmod{4}$, we have $(z_i, z_j) \in A(D)$.

Let $r \in \{0, 1, \ldots, n\}$ be such that $u_r = z_i$, now we rename the vertices of $C$ in such a way that $C$ starts at $u_r$. Joining the corresponding directed paths ($T_i$) between the vertices of $C$, we obtain a closed directed walk $C' = (z_0, z_1, \ldots, z_k, z_0)$ which is the same as $C'$ where the vertices where renamed as follows: for each $t \in \{0, \ldots, k\}$ $z_t = z_{t+i}$, thus $z_0 = z_i$. Let $j \in \{0, \ldots, k\}$ be such that $j - i \equiv 1 \pmod{4}$. It follows from 3(c) that $(z_0, z_{j-i}) \in A(D)$ and that means $(z_i, z_j) \in A(D)$ (as $z_0 = z_i$ and $z_{j-i} = z_j$).

7(c.2). For any $i, j \in \{0, \ldots, k\}$ such that $i \in I_p$ and $j - i \equiv 3 \pmod{4}$, we have $(z_i, z_j) \in A(D)$.

We proceed as in 6(c.2), to obtain $C'$. Taking $j \in \{0, \ldots, k\}$ such that $j - i \equiv 3 \pmod{4}$, we obtain from 5(c.2) that $(z_{j-i}, z_0) \in A(D)$; i.e., $(z_j, z_i) \in A(D)$.

8(c.2). For any $i \in \{0, \ldots, k\}$ we have $(z_i, z_{i-3}) \in A(D)$.

We proceed by contradiction, suppose that for some $i \in \{0, \ldots, k\}$ we have $(z_{i-3}, z_i) \in A(D)$. Since $i - (i - 3) \equiv 3 \pmod{4}$, we have from 7(c.2) that $i - 3 \notin I_p$; and since $(i - 3) - i \equiv 1 \pmod{4}$, we obtain from 6(c.2) that $i \notin I_p$. From 1, $\{i - 3, i - 2, i - 1, i\} \cap I_p \neq \emptyset$. Thus $\{i - 2, i - 1\} \cap I_p \neq \emptyset$.

And here we consider the two possible cases:

Case 8(c.2) a. $i - 2 \in I_p$.

Let $j \in \{0, \ldots, n\}$ be such that $z_{i-2} = u_j$. We have $(z_{i+1}, z_{i-2} = u_j) \in A(D)$ (this follows directly from 7(c.2), observing that $i+1-(i-2) \equiv 3 \pmod{4}$), also $(z_{i-2}, z_{i-5}) \in A(D)$ (from 6(c.2), just observe that $(i - 5) - (i - 2) \equiv 1 \pmod{4}$). Now we have $C^2 = (u_j = z_{i-2}, z_{i-5}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i-2} = u_j)$ is a directed cycle of length 6 and from the hypothesis it is monochromatic, assume w.l.o.g. that it is coloured 1. From the definition of $C'$, $u_{j-1} \in \{z_i, z_{i-1}, z_{i-3}\}$. Since $i - 3 \notin I_p$ we obtain $u_{j-1} \in \{z_{i-6}, z_{i-4}, z_{i-3}\}$.
When $u_{j-1} = z_{i-4}$, we obtain $\{u_{j-1}, u_j\} \subset V(C^2)$. Thus there exists a $u_j u_{j-1}$-monochromatic directed path contained in $C^2$, a contradiction. When $u_{j-1} = z_{i-6}$, we have $(z_{i+1}, z_{i-6} = u_{j-1}) \in A(D)$ (from 7(c.2) as $(i+1)-(i-6) \equiv 3(\text{mod} 4)$). So $C^3 = (u_{j-1} = z_{i-6}, z_{i-5}, z_{i-4}, z_{i-3}, z_{i+1}, z_{i-6} = u_{j-1})$ is a directed cycle of length 6 and hence it is monochromatic; moreover it is coloured 1 (because $(z_{i-3}, z_i) \in A(C^2) \cap A(C^3)$). Therefore, $(u_j = z_{i-2}, C^2, z_{i+1}) \cup (z_{i+1}, z_{i-6} = u_{j-1})$ is a $u_j u_{j-1}$-monochromatic directed path, a contradiction.

Case 8(c.2) b. $i-1 \in I_p$.

Let $j \in \{0, \ldots, n\}$ be such that $z_{i-1} = u_j$. We have $(z_{i+2}, z_{i-1} = u_j) \in A(D)$ (this follows from 7(c.2), as $i + 2 - (i - 1) \equiv 3(\text{mod} 4)$), and $(z_{i-1}, z_{i-4}) \in A(D)$ (this follows from 6(c.2), because $(i-4) - (i-1) \equiv 1(\text{mod} 4)$). Therefore $C^2 = (u_j = z_{i-1}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i+2}, z_{i+3} = u_{j+1})$ is a directed cycle of length 6, hence it is monochromatic say coloured 1. From the definition of $C'$, we have $u_{j+1} \in \{z_i, z_{i+1}, z_{i+3}\}$; moreover $u_{j+1} \in \{z_{i+1}, z_{i+3}\}$ because $i \notin I_p$. If $u_{j+1} = z_{i+1}$, then $\{u_j, u_{j+1}\} \subset V(C^2)$ and thus there exists a $u_{j+1} u_j$-monochromatic directed path, a contradiction. Hence $u_{j+1} = z_{i+3}$. Now observe that $(u_{j+1} = z_{i+3}, z_{i-4}) \in A(D)$ (this follows from 6(c.2) as $i-4-(i+3) \equiv 1(\text{mod} 4)$). Therefore $C^3 = (u_{j+1} = z_{i+3}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i+2}, z_{i+3} = u_{j+1})$ is a directed cycle of length 6 and it is coloured 1 (because $(z_{i-3}, z_i) \in A(C^2) \cap A(C^3)$). We conclude that $(u_{j+1} = z_{i+3}, z_{i-4}) \cup (z_{i-4}, C^2, z_{i-1} = u_j)$ is a $u_{j+1} u_j$-monochromatic directed path, a contradiction.

9(c.2). If for some $i \in \{0, \ldots, k\}$ we have $(z_{i-1}, z_i)$ and $(z_i, z_{i+1})$ have different colours, then $i \in I_p$.

From I we have $\{i-3, i-2, i-1, i\} \cap I_p \neq \emptyset$. Let $r_0 = \min\{r \in \{0,1,2,3\} | i-r \in I_p\}$ and let $j \in \{0,1,\ldots,n\}$ be such that $z_{i-r_0} = u_j$; so we have $u_j \in \{z_{i-3}, z_{i-2}, z_{i-1}, z_i\}$. From the definition of $C'$, $u_j \in \{z_{i-r_0+1}, z_{i-r_0+2}, z_{i-r_0+4}\} \subseteq \{z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}\}$. Now consider $\ell \in \{i-r_0+1, i-r_0+2, i-r_0+4\}$ such that $u_{j+1} = z_{\ell}$. From the definition of $r_0$ and since $\ell \in I_p$, we have $\ell \notin \{i-2, i-1, i\}$, i.e., $u_{j+1} \in \{z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}\}$.

If $T_j$ has length 4, then $T_j$ is monochromatic; and hence $\{(z_{i-1}, z_i), (z_i, z_{i+1})\} \subseteq A(T_j)$, and $z_i = u_j, z_{i+4} = u_{j+1}$. Thus $i \in I_p$.

If $T_j$ has length 1, then $z_i = u_j$, i.e., $i \in I_p$.

If $T_j$ has length 2, then $u_j \in \{z_{i-1}, z_i\}$. When $u_j = z_i$ clearly $i \in I_p$.

When $u_j = z_{i-1}$, we have $u_{j+1} = z_{i+1}$. From 8(c.2) we obtain $(z_{i+2}, z_{i-1}) \in A(D)$ and thus $C^2 = (u_j = z_{i-1}, z_i, z_{i+1} = u_{j+1}, z_{i+2}, z_{i-1} = u_j)$ is
a directed cycle of length 4 (which from the hypothesis is quasi-monochromatic). Since \((z_{i-1}, z_i)\) and \((z_i, z_{i+1})\) have different colours, we conclude that \((u_{j+1}, C^2, u_j)\) is a \(u_{j+1}u_{j-}\)-monochromatic directed path, a contradiction.

10(c.2). There exists a change of colour in \(C'\); i.e., there exists \(i \in \{0, \ldots, k\}\) such that \((z_{i-1}, z_i)\) and \((z_i, z_{i+1})\) have different colours.

Otherwise \(C'\) is monochromatic, and for any \(j \in \{0, \ldots, n\}\), there exists a \(u_{j+1}u_{j-}\)-monochromatic directed path, a contradiction.

We will assume w.l.o.g. that \((z_{i-1}, z_i)\) is coloured 1 and \((z_i, z_{i+1})\) is coloured 2.

11(c.2). \(i \in I_p\).
It follows directly from 9(c.2) and our assumption. Let \(j \in \{0, \ldots, n\}\) be such that \(z_i = u_j\).

12(c.2). \(\{(z_{i+2}, z_{i-1}), (z_{i+1}, z_{i-2}), (z_i, z_{i-3}), (z_{i+3}, z_i)\} \subseteq A(D)\).
This follows directly from 8(c.2).

13(c.2). \((z_{i+1}, z_{i+2})\) and \((z_{i+2}, z_{i-1})\) have the same colour, say \(a\), with \(a \in \{1, 2\}\).

Let \(C' = (z_{i-1}, z_i = u_j, z_{i+1}, z_{i+2}, z_{i-1})\) from 12(c.2), it is a directed cycle of length 4 and then it is quasi-monochromatic. Since \((z_{i-1}, z_i)\) and \((z_i, z_{i+1})\) are coloured 1 and 2 respectively, 13(c.2) follows.

14(c.2). \((z_{i+1}, z_{i-2})\) and \((z_{i-2}, z_{i-1})\) have the same colour, say \(b\), with \(b \in \{1, 2\}\). The proof is similar to that of 13(c.2) by considering the directed cycle of length 4, \(C^3 = (z_{i-2}, z_{i-1}, z_i = u_j, z_{i+1}, z_{i-2})\).

15(c.2). \(\{i - 1, i + 1\} \cap I_p = \emptyset\).
First suppose for a contradiction that \(i - 1 \notin I_p\). From the definition of \(C'\), and since \(z_i = u_j\), we have \(z_{i-1} = u_{j-1}\). From 13(c.2) \((z_{i+1}, z_{i+2})\) and \((z_{i+2}, z_{i-1})\) have the same colour \(a \in \{1, 2\}\). If \(a = 2\), then \((z_i = u_j, z_{i+1}, z_{i+2}, z_{i-1} = u_{j-1})\) is a \(u_ju_{j-1}\)-monochromatic directed path, a contradiction. If \(a = 1\), then from 9(c.2) we have \(i + 1 \in I_p\). So, \(z_{i+1} = u_{j+1}\) and \((u_{j+1} = z_{i+1}, z_{i+2}, z_{i-1}, z_i = u_j)\) is a \(u_{j+1}u_j\)-monochromatic directed path, a contradiction.

Now, suppose for a by contradiction that \(i + 1 \in I_p\). Thus \(z_{i+1} = u_{j+1}\). From 14(c.2) we have \((z_{i+1}, z_{i-2})\) and \((z_{i-2}, z_{i-1})\) have the same colour \(b\), with \(b \in \{1, 2\}\). If \(b = 1\) then \((u_{j+1} = z_{j+1}, z_{i-2}, z_{i-1}, z_i = u_j)\) is a \(u_{j+1}u_j\)-monochromatic directed path, a contradiction. If \(b = 2\) then from 9(c.2) we have \(i - 1 \notin I_p\), but we have proved that this leads to a contradiction.
16(c.2). \((z_{i+1}, z_{i+2})\) is coloured 2.
Otherwise \((z_i, z_{i+1})\) and \((z_{i+1}, z_{i+2})\) have different colours and from 9(c.2) \(i + 1 \in I_p\), contradicting 15(c.2).

17(c.2). \((z_{i-2}, z_{i-1})\) is coloured 1.
Otherwise \((z_{i-2}, z_{i-1})\) and \((z_{i-1}, z_i)\) have different colours, and from 9(c.2) \(i - 1 \in I_p\), contradicting 15(c.2).

18(c.2). \((z_{i+2}, z_{i-1})\) is coloured 2.
This follows directly from 13(c.2) and 16(c.2).

19(c.2). \((z_{i+1}, z_{i-2})\) is coloured 1.
Follows directly from 14(c.2) and 17(c.2). Now we will analyze the two possible cases:

Case c.2.1. \(i + 2 \notin I_p\).
In this case, we have from the definition of \(C'\) that \(i + 4 \in I_p\) and \(z_{i+4} = u_{j+1}\).
And we have the following assertions: 1(c.2.1) to 11(c.2.1).

1(c.2.1). \((z_{i+2}, z_{i+3})\) and \((z_{i+3}, z_{i+4})\) are coloured 2.
Since \(u_{j+1} = z_{i+4}\), then \(T_f = (u_j = z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4} = u_{j+1})\) is monochromatic; moreover it is coloured 2 (as \((z_i, z_{i+1})\) is coloured 2).

2(c.2.1). \((z_{i+4}, z_{i-3})\) \(\in A(D)\).
This follows from 6(c.2) because \(i + 4 \equiv 1 \pmod{4}\).

3(c.2.1). \((z_{i-1}, z_{i+4})\) \(\in A(D)\).
The assertion follows from 7(c.2) as \(i - 1 - (i + 4) \equiv 3 \pmod{4}\).

4(c.2.1). \(\{(z_{i+4}, z_{i+1}), (z_{i+3}, z_i)\} \subseteq A(D)\).
Is a direct consequence of 8(c.2).

5(c.2.1). \((z_{i+4}, z_{i+1})\) is not coloured 1.
Assuming for a contradiction that \((z_{i+4}, z_{i+1})\) is coloured 1, we obtain that \((u_{j+1} = z_{i+4}, z_{i+1}, z_{i-2}, z_{i-1}, z_i = u_j)\) is a \(u_{j+1}u_{j}\)-monochromatic directed path, a contradiction.

6(c.2.1). \((z_{i-1}, z_{i+4})\) is coloured 1.
We have that \((z_{i+1}, z_{i-2}, z_{i-1}, z_{i+4}, z_{i+1})\) is quasi-monochromatic (because it is a directed cycle of length 4). From 19(c.2) \((z_{i+1}, z_{i-2})\) is coloured 1, from 17(c.2), \((z_{i-2}, z_{i-1})\) is coloured 1; and from 5(c.2.1) \((z_{i+4}, z_{i+1})\) is not coloured 1. So, \((z_{i-1}, z_{i+4})\) is coloured 1.

7(c.2.1). \((z_{i+4}, z_{i+1})\) is coloured 2.
We have that: \((z_{i+1}, z_{i+2}, z_{i-1}, z_{i+4}, z_{i+1})\) is quasi-monochromatic (from the hypothesis), \((z_{i+1}, z_{i+2})\) is coloured 2 (16(c.2)), \((z_{i+2}, z_{i-1})\) is coloured 2 (18(c.2)) and \((z_{i-1}, z_{i+4})\) is coloured 1 (6(c.2.1)).

8(c.2.1). \((z_{i-3}, z_{i+2}) \in A(D)\).
Assume, for a contradiction that \((z_{i-3}, z_{i+2}) \notin A(D)\). Then \((z_{i+2}, z_{i-3}) \in A(D)\) and \((z_{i+2}, z_{i-3}, z_{i-2}, z_{i+1}, z_{i+4})\) is a directed cycle of length 6. From the hypothesis we have that it must be monochromatic, but it has two arcs coloured 1 ((\(z_{i-2}, z_{i-1}\)) and \((z_{i-1}, z_i)\)) and two arcs coloured 2 ((\(z_i, z_{i+1}\)) and \((z_{i+1}, z_{i+2})\)), a contradiction.

9(c.2.1). \((z_{i-2}, z_{i+3}) \in A(D)\).
Assuming for a contradiction that \((z_{i-2}, z_{i+3}) \notin A(D)\), we obtain \((z_{i+3}, z_{i-2}) \in A(D)\) and \((z_{i+3}, z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2}, z_{i+3})\) is a directed cycle of length 6. It has two arcs coloured 1 ((\(z_{i-2}, z_{i-1}\)) and \((z_{i-1}, z_i)\)) and two arcs coloured 2 ((\(z_i, z_{i+1}\)) and \((z_{i+1}, z_{i+2})\)), contradicting the hypothesis.

10(c.2.1). \((z_{i+3}, z_i)\) is not coloured 2.
Assume, for a contradiction that \((z_{i+3}, z_i)\) is coloured 2, then \((u_{j+1} = z_{i+4}, z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_i = u_j)\) is a \(u_{j+1}u_j\)-monochromatic directed path, a contradiction.

11(c.2.1). The arcs \((z_{i-2}, z_{i+3})\) and \((z_{i+3}, z_i)\) are coloured 1.
We have \((z_{i+3}, z_i, z_{i+1}, z_{i-2}, z_{i+3})\) a directed cycle of length 4, thus it is quasi-monochromatic. Since \((z_i, z_{i+1})\) is coloured 2 and \((z_{i+1}, z_{i-2})\) is coloured 1 (19(c.2)), then \((z_{i-2}, z_{i+3})\) and \((z_{i+3}, z_i)\) are both coloured 1 or are both coloured 2. And from 10(c.2.1) \((z_{i+3}, z_i)\) is not coloured 2.

12(c.2.1). \((z_{i+4}, z_{i-3})\) and \((z_{i-3}, z_{i-2})\) are both coloured 1 or are both coloured 2.
We have \((z_{i-2}, z_{i+3}, z_{i+4}, z_{i-3}, z_{i-2})\) is quasi-monochromatic; \((z_{i-2}, z_{i+3})\) is coloured 1 (11(c.2.1)) and \((z_{i+3}, z_{i+4})\) is coloured 2 (1(c.2.1)).

If \((z_{i+4}, z_{i-3})\) and \((z_{i-3}, z_{i-2})\) are both coloured 1, then \((u_{j+1} = z_{i+4}, z_{i-3}, z_{i-2}, z_{i-1}, z_i = u_j)\) is a \(u_{j+1}u_j\)-monochromatic directed path (coloured 1), a contradiction. If \((z_{i+4}, z_{i-3})\) and \((z_{i-3}, z_{i-2})\) are both coloured 2, then \((z_{i-1}, z_{i+4}, z_{i-3}, z_{i-2}, z_{i-1})\) is a directed cycle of length 4 with two arcs coloured 1 and two arcs coloured 2, a contradiction to the hypothesis. So case (c.2.1) is not possible.

Case c.2.2. \(i + 2 \in I_p\).
Since \(i + 1 \notin I_p\), then \(z_{i+2} = u_{j+1}\). We have the following assertions:
1(c.2.2). \((z_{i+2}, z_{i-5}) \in A(D)\).
This follows from 6(c.2), as \((i - 5) - (i + 2) \equiv 1(\text{mod } 4)\).

2(c.2.2). \((z_{i-3}, z_{i+2}) \in A(D)\).
Since \((i - 3) - (i + 2) \equiv 3(\text{mod } 4)\), the assertion follows from 7(c.2).

3(c.2.2). \((z_{i-4}, z_{i+1}) \in A(D)\).
Assume, for a contradiction that \((z_{i-4}, z_{i+1}) \notin A(D)\). Then \((z_{i+1}, z_{i-4}) \in A(D)\) and \((z_{i-4}, z_{i-3}, z_{i-2}, z_{i-1}, z_{i}, z_{i+1}, z_{i-4})\) is monochromatic (as it is a directed cycle of length 6), but \((z_{i-1}, z_{i})\) is coloured 1 and \((z_{i}, z_{i+1})\) is coloured 2, a contradiction.

4(c.2.2). \((z_{i-1}, z_{i-4}) \in A(D)\).
It follows from 8(c.2).

5(c.2.2). \((z_{i-5}, z_{i}) \in A(D)\).
Since \((i - 5) - i \equiv 3(\text{mod } 4)\) then the assertion follows from 7(c.2).

6(c.2.2). \((z_{i-2}, z_{i-5}) \in A(D)\).
This follows from 8(c.2).

7(c.2.2). The arcs \((z_{i}, z_{i-3})\) and \((z_{i-3}, z_{i+2})\) are both coloured 2.
We have \((z_{i-1}, z_{i}, z_{i-3}, z_{i+2}, z_{i-1})\) a directed cycle of length 4, thus it is quasi-monochromatic. Since \((z_{i-1}, z_{i})\) is coloured 1 and \((z_{i+2}, z_{i-1})\) is coloured 2 then \((z_{i}, z_{i-3})\) and \((z_{i-3}, z_{i+2})\) are both coloured 1 or are both coloured 2. If they are both coloured 2, then we are done.

Now suppose that \((z_{i}, z_{i-3})\) and \((z_{i-3}, z_{i+2})\) are both coloured 1. Therefore \((z_{i+2}, z_{i-1}, z_{i-4}, z_{i-3}, z_{i+2})\) is quasi-monochromatic. Since \((z_{i+2}, z_{i-1})\) is coloured 2 and \((z_{i-3}, z_{i+2})\) is coloured 1, then \((z_{i-1}, z_{i-4})\) and \((z_{i-4}, z_{i-3})\) are both coloured 1 or are both coloured 2.

We will analyze the two possible cases:

Case 7(c.2.2)a. The arcs \((z_{i-1}, z_{i-4})\) and \((z_{i-4}, z_{i-3})\) are both coloured 2.
In this case we have \((z_{i-3}, z_{i-4}, z_{i-2}, z_{i-1})\) is quasi-monochromatic, \((z_{i-2}, z_{i-1})\) is coloured 1 and \((z_{i-1}, z_{i-4})\) and \((z_{i-4}, z_{i-3})\) are both coloured 2.

So, it follows from 9(c.2) that \(i - 2 \in I_p\). Since \(i - 1 \notin I_p\) (15(c.2)) then \(z_{i-2} = u_{j-1}\). Thus \((u_j = z_i, z_{i+1}, z_{i+2}, z_{i-1}, z_{i-4}, z_{i-3}, z_{i-2} = u_{j-1})\) is a \(u_j u_{j-1}\)-directed path coloured 2, a contradiction. So case 7(c.2.2)a is not possible.
On the other hand we have.

Assume, for a contradiction that \((z_{i-3}, z_{i-2})\) is not coloured 1 (otherwise \((u_j = z_i, z_{i-3}, z_{i-2}, z_{i-1}, z_{i-4})\) is a directed walk coloured 1 which contains \(\{z_{i-2}, z_{i-4}\}\); and from the definition of \(C'\), \(u_{j-1} \in \{z_{i-2}, z_{i-4}\}\) thus there exists a \(u_ju_{j-1}\)-monochromatic directed path; a contradiction). Now from 9(c.2) we have \(\{i - 3, i - 2\} \subseteq I_p\). Since \(i - 1 \notin I_p\) we have \(z_{i-2} = u_{j-1}\) and \(z_{i-3} = u_j\). Therefore \((u_{j-1} = z_{i-2}, z_{i-1}, z_{i-4}, z_{i-3} = u_j)\) is a \(u_{j-1}u_j\)-monochromatic directed path (coloured 1), a contradiction.

We conclude that the arcs \((z_i, z_{i-3})\) and \((z_{i-3}, z_{i+2})\) are both coloured 2.

8(c.2.2). \((z_{i-3}, z_{i-2})\) is coloured 1.

We have \((z_{i-2}, z_{i-1}, z_i, z_{i-3}, z_{i-2})\) which is quasi-monochromatic; \((z_i, z_{i-3})\) coloured 2 and \((z_{i-2}, z_{i-1})\) coloured 1.

9(c.2.2). \((z_{i-2}, z_{i-5})\) and \((z_{i-5}, z_i)\) are both coloured 1.

\((z_{i-2}, z_{i-5})\) and \((z_{i-5}, z_i)\) are both coloured 1 or are both coloured 2: this is because \((z_i, z_{i-3}, z_{i-2}, z_{i-5}, z_i)\) is quasi-monochromatic with \((z_i, z_{i-3})\) coloured 2 and \((z_{i-3}, z_{i-2})\) coloured 1.

Assume, for a contradiction that \((z_{i-2}, z_{i-5})\) and \((z_{i-5}, z_i)\) are both coloured 2.

Denote by \(a\) the colour of the arc \((z_{i+2}, z_{i-5})\). We have \(a \neq 2\) (otherwise \((u_{j+1} = z_{i+2}, z_{i-5}, z_i = u_j)\) is a \(u_{j+1}u_j\)-monochromatic directed path, a contradiction). Now, \((z_{i-5}, z_{i-4})\) and \((z_{i-4}, z_{i-3})\) are both coloured \(b\) with \(b \in \{1, 2\}\) (this is because \((z_{i-5}, z_{i-4}, z_{i-3}, z_{i-2}, z_{i-5})\) is quasi-monochromatic with \((z_{i-3}, z_{i-2})\) coloured 1 and \((z_{i-2}, z_{i-5})\) coloured 2). If \(b = 1\) then \(a = 1\) (notice that \((z_{i+2}, z_{i-5}, z_{i-4}, z_{i-3}, z_{i+2})\) is quasi-monochromatic; with \((z_{i-3}, z_{i+2})\) coloured 2 and \((z_{i-3}, z_{i-4})\) coloured 1; so \(a = 1\)). Thus \((u_{j+1} = z_{i+2}, z_{i-5}, z_{i-4}, z_{i-3}, z_i = u_j)\) is a \(u_{j+1}u_j\)-monochromatic directed path (coloured 1), a contradiction. If \(b = 2\), then \(i - 3 \in I_p\) (from 9(c.2)) and from the definition of \(C'\), \(i - 2 \notin I_p\). Thus \(z_{i-2} = u_{j-1}, z_{i-3} = u_j\) and \((u_{j-1} = z_{i-2}, z_{i-5}, z_{i-4}, z_{i-3}, z_i = u_j)\) is a \(u_{j-1}u_j\)-monochromatic directed path (coloured 2), a contradiction.

10(c.2.2). \((z_{i+2}, z_{i-5})\) is coloured 2.

\((z_i, z_{i+1}, z_{i+2}, z_{i-5}, z_i)\) is quasi-monochromatic with \((z_{i-5}, z_i)\) coloured 1 and \((z_i, z_{i+1})\) and \((z_{i+1}, z_{i+2})\) coloured 2.

11(c.2.2). \((z_{i-4}, z_{i-3})\) is not coloured 2.

Assume, for a contradiction that \((z_{i-4}, z_{i-3})\) coloured 2. Then \(i - 3 \notin I_p\). On the other hand we have \(i - 4 \in I_p\) (because \((z_{i-5}, z_{i-4}, z_{i-3}, z_{i-2}, z_{i-5})\)

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is quasi-monochromatic with \((z_{i-4}, z_{i-3})\) coloured 2 and \(((z_{i-3}, z_{i-2})\) and \((z_{i-2}, z_{i-5})\)) coloured 1; so \((z_{i-5}, z_{i-4})\) is coloured 1 and then (from 9(c.2)) \(i-4 \in I_p\). Now, from the definition of \(C'\), we have \(z_{i-3} = u_r\) and \(z_{i-4} = u_{r-1}\) for some \(r \in \{1, 2, \ldots, n\}\). Thus \((u_r = z_{i-3}, z_{i-2}, z_{i-5}, z_{i-4} = u_{r-1})\) is a \(u_ru_{r-1}\)-monochromatic directed path (coloured 1), a contradiction.

12(c.2.2). \((z_{i-5}, z_{i-4})\) is coloured 2.
\((z_{i-5}, z_{i-4}, z_{i-3}, z_{i+2}, z_{i-5})\) is quasi-monochromatic, with \(((z_{i-3}, z_{i+2})\) and \((z_{i+2}, z_{i-5})\)) coloured 2 and \((z_{i-4}, z_{i-3})\) not coloured 2.

13(c.2.2). \((z_{i-4}, z_{i+1})\) is coloured 1.
\((z_{i+1}, z_{i-2}, z_{i-5}, z_{i-4}, z_{i+1})\) is quasi-monochromatic with \((z_{i-5}, z_{i-4})\) coloured 2 and \(((z_{i+1}, z_{i-2})\) and \((z_{i-2}, z_{i-5})\)) coloured 1.

14(c.2.2). \(D[\{z_i, z_{i+1}, z_{i+2}, z_{i-5}, z_{i-4}, z_{i-2}\}]\) is isomorphic to \(\tilde{T}_6\).
Let \(f : \{z_i, z_{i+1}, z_{i+2}, z_{i-5}, z_{i-4}, z_{i-2}\} \to V(\tilde{T}_6)\) defined as follows: \(f(z_i) = x, f(z_{i+1}) = y, f(z_{i+2}) = v, f(z_{i-5}) = w, f(z_{i-4}) = z, f(z_{i-2}) = u\) is an isomorphism.

Assertion 14(c.2.2) contradicts the hypothesis, so case c(2.2) is not possible; also case c.2 is not possible.

As a direct consequence of Theorem 2.1, we have the following result:

**Theorem 2.2.** Let \(D\) be an \(m\)-coloured bipartite tournament. Assume that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic and \(D\) has no subtournament isomorphic to \(\tilde{T}_6\). Then \(D\) has a kernel by monochromatic paths.

**Remark 2.1.** The hypothesis that every directed cycle of length 6 is monochromatic in Theorem 2.1 is tight.

Let \(D\) be the 3-coloured bipartite tournament defined in [8] as follows: \(V(D) = \{u, v, w, x, y, z\}\), \(A(D) = \{(u, x), (x, v), (v, y), (y, w), (w, z), (z, u), (x, u), (y, u), (z, v)\}\); the arcs \((x, w), (w, z)\) and \((z, u)\) coloured 1; the arcs \((y, u), (u, x)\) and \((x, v)\), coloured 2; and the arcs \((z, v), (v, y)\) and \((y, w)\) coloured 3. \(D\) has a directed cycle of length 6 which is not monochromatic, every directed cycle of length 4 in \(D\) is quasi-monochromatic, \(D\) has no subtournament isomorphic to \(\tilde{T}_6\) and \(\mathcal{C}(D)\) is a complete multidigraph which has no kernel.
Remark 2.2. The hypothesis that every directed cycle of length 6 in a bipartite tournament $D$ is monochromatic, does not imply that every directed cycle of length 4 in $D$ is quasi-monochromatic.

Proof. Let $T = (U, W)$ be the 2-coloured bipartite tournament defined as follows: $U = \{u, v, w, x, y\}$ and $W = \{a, b, c, d, e\}$. In $T$, $C_1 = (u, a, v, b, w, c, u)$ is a directed cycle of length 6 coloured 1, $C_2 = (x, d, y, e, x)$ is a directed cycle of length 4 coloured 2. $T$ has arcs from $U \cap V(C_1)$ to $W \cap V(C_2)$ coloured 1 and finally $T$ contains the arcs $(u, b)$, $(a, w)$, $(c, w)$ coloured 1 (see Figure 2). $C_1$ is the only directed cycle of length 6 contained in $T$, and it is monochromatic. And $C_2$ is a directed cycle of length 4 that is not quasi-monochromatic.

Remark 2.3. For each $m$ there exists an $m$-coloured Hamiltonian bipartite tournament such that: every directed cycle of length 4 is quasi-monochromatic; every directed cycle of length 6 is monochromatic and $D$ has no subtournament isomorphic to $T_6$.

Proof. Let $D = (V_1, V_2)$ be the $m$-coloured bipartite tournament defined as follows:
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\[ V(D) = \bigcup_{i=1}^{6} X_i \] where \( X_i = \{ x_{i,1}, x_{i,2}, \ldots, x_{i,m} \} \),

\[ V_1 = X_1 \cup X_3 \cup X_5, \quad V_2 = X_2 \cup X_4 \cup X_6, \]

\[ A(D) = \bigcup_{i=1}^{5} X_i^\prime \cup X_5^2 \cup X_6^0 \] where \( X_i^\prime = \{ (x_{i,j}, x_{i+1,j}) \mid j \in \{1, \ldots, m\} \} \),

\[ X_5^2 = \{ (x_{\ell,j}, x_{\ell+3,j}) \mid j \in \{1, \ldots, m\} \}, X_6^0 = \{ (x_{6,i}, x_{1,i+1}) \mid i \in \{1, \ldots, m-1\} \} \cup \{ (x_{6,m}, x_{1,1}) \}, \]

where \((x_{1,i}, x_{2,i})\) is coloured \( i \); and any other arc of \( D \) is coloured 1 and in any direction.

References


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