

# Exact integrability of the $su(n)$ Hubbard model

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## Abstract

The bosonic  $su(n)$  Hubbard model was recently introduced. The model was shown to be integrable in one dimension by exhibiting the infinite set of conserved quantities. I derive the  $R$ -matrix and use it to show that the conserved charges commute among themselves. This new matrix is a non-additive solution of the Yang-Baxter equation. Some properties of this matrix are derived.

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# 1 Introduction

The two-dimensional Hubbard model was introduced to describe the effects of correlation for  $d$ -electrons in transition metals [1]. It was then shown to be relevant to the study of high- $T_c$  superconductivity of cuprate compounds.

In one dimension the model is integrable [2, 3, 4]. The integrability framework of the model is the quantum inverse scattering method [5]. However, despite sharing many properties with discrete quantum integrable models, the model has a peculiar integrable structure which defines a class of its own.

In seeking to generalize the Hubbard model in any dimension, it was therefore natural to look for a one-dimensional generalization which is integrable. An  $n$ -state generalized model which contains the usual  $su(2)$  model was recently introduced in [6]. This  $su(n)$  Hubbard model was shown to possess an infinite set of conserved charges and to have an extended  $su(n)$  symmetry. The model is built by coupling two copies of the recently discovered  $su(n)$  XX ‘free-fermions’ model [7]. For  $n = 2$  a fermionic formulation exists, but for  $n > 2$  finding an analogous framework is a tantalizing problem.

In this work I derive the  $R$ -matrix of the model; this provides a direct proof of the commutation of the conserved charges among themselves. Section two gives the definition of the bosonic Hamiltonian and the transfer matrix. The  $R$ -matrix intertwining the monodromy matrices is derived in section three. In section four some properties of this new matrix are given. I conclude with some remarks and outline some outstanding issues.

## 2 The model

Let  $E^{\alpha\beta}$  be the  $n \times n$  matrix with a one at row  $\alpha$  and column  $\beta$  and zeros otherwise. The  $su(n)$  Hubbard Hamiltonian on a ring then reads [6]:

$$\begin{aligned} H_2 &= \sum_i h_{ii+1} + \sum_i h'_{ii+1} + U \sum_i h_i^c \\ &= \sum_i \sum_{\alpha < \beta} \left( x E_i^{\alpha\beta} E_{i+1}^{\beta\alpha} + x^{-1} E_i^{\beta\alpha} E_{i+1}^{\alpha\beta} + (E \rightarrow E') \right) + U \sum_i \left( \rho_i + \frac{n-2}{2} \right) \left( \rho'_i + \frac{n-2}{2} \right) \end{aligned} \quad (1)$$

where  $\rho = \sum_{\alpha < \beta} E^{\alpha\beta} - (n-1)E^{nn}$ , and primed and unprimed quantities correspond to two commuting copies of the  $E$  matrices. The Hamiltonians  $h$  and  $h'$  are  $su(n)$  XX Hamiltonians [7]. The complex free parameter  $x$  is a deformation inherited from the XX model. The Hamiltonian  $H_2$  is defined in one dimension but can be evidently defined on any lattice; integrability is lost however.

For  $n = 2$  and  $x = 1$ , and using Pauli matrices, the Hamiltonian is just the integrable bosonic version of the usual Hubbard Hamiltonian [3]:

$$H_2^{(2)} = \frac{1}{2} \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + (\sigma \rightarrow \sigma') + U \sum_i \sigma_i^z \sigma'_i{}^z \quad (2)$$

The Hamiltonians can be written simply in terms of  $su(n)$  hermitian traceless matrices. For  $|x| = 1$  the Hamiltonians are hermitian.

The transfer matrix is the generator of the infinite set of conserved quantities. Its construction was given in [6]. We recall it here. Consider first the  $R$ -matrix of the  $su(n)$

XX model [7]:

$$\begin{aligned}
R(\lambda) = & a(\lambda) [E^{nn} \otimes E^{nn} + \sum_{\alpha, \beta < n} E^{\beta\alpha} \otimes E^{\alpha\beta}] \\
& + b(\lambda) \sum_{\alpha < n} (xE^{nn} \otimes E^{\alpha\alpha} + x^{-1}E^{\alpha\alpha} \otimes E^{nn}) \\
& + c(\lambda) \sum_{\alpha < n} (E^{n\alpha} \otimes E^{\alpha n} + E^{\alpha n} \otimes E^{n\alpha})
\end{aligned} \tag{3}$$

where  $a(\lambda) = \cos(\lambda)$ ,  $b = \sin(\lambda)$  and  $c(\lambda) = 1$ . The functions  $a$ ,  $b$  and  $c$  satisfy the ‘free-fermion’ condition:  $a^2 + b^2 = c^2$ . For this set of parameters, a Jordan-Wigner transformation turns the  $U = 0$  Hamiltonian density for  $su(2)$  into a fermionic expression for free fermions hopping on the lattice.

Consider also the matrix

$$I_0(h) = \cosh\left(\frac{h}{2}\right) \text{Id} + \sinh\left(\frac{h}{2}\right) C_0 C'_0 = \exp\left(\frac{h}{2} C_0 C'_0\right) \tag{4}$$

where  $C = \sum_{\alpha < n} E^{\alpha\alpha} - E^{nn}$ . We stress that  $C$  turns out to be the fundamental matrix, not the  $su(n)$  generator  $\rho$ . We have  $\rho + \frac{n-2}{2}\text{Id} = \frac{n}{2}C$ , for  $n \geq 2$ . The parameter  $h$  is related to the spectral parameter  $\lambda$  by

$$\sinh(2h) = \frac{n^2 U}{4} \sin(2\lambda) \tag{5}$$

One chooses for  $h(\lambda)$  the principal branch which vanishes for vanishing  $\lambda$  or  $U$ . Then for  $U = 0$  the monodromy matrix becomes a tensor product of two uncoupled XX models. The Lax operator at site  $i$  is given by:

$$L_{0i}(\lambda) = I_0(h) R_{0i}(\lambda) R'_{0i}(\lambda) I_0(h) \tag{6}$$

and the monodromy matrix is a product of Lax operators,  $T(\lambda) = L_{0M}(\lambda) \dots L_{01}(\lambda)$ , where  $M$  is the number of sites on the chain. The transfer matrix is the trace of the monodromy matrix over the auxiliary space 0:  $\tau(\lambda) = \text{Tr}_0 [(L_{0M} \dots L_{01})(\lambda)]$ . One possible set of conserved quantities is given by

$$H_{p+1} = \left( \frac{d^p \ln \tau(\lambda)}{d\lambda^p} \right)_{\lambda=0} \tag{7}$$

The proof that  $H_2$  commutes with  $\tau(\lambda)$  was given in [6]. The derivative of the matrix  $I$  gives the coupling term appearing in (1). Note that the definition involving a logarithm has two benefits. Besides giving the most local operators, it further disentangles the two copies.

### 3 The $R$ -matrix

We derive the  $R$ -matrix intertwining two monodromy matrices at different spectral parameters. To this end we generalize the algebraic method of the Decorated Star Triangle Equation introduced by Shastry [10].

The XX  $R$ -matrix satisfies the regularity property  $\check{R}(0) = \text{Id}$ , the unitarity condition  $\check{R}(\lambda)\check{R}(-\lambda) = \text{Id} \cos^2 \lambda$  and the Yang-Baxter equation

$$\check{R}_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{13}(\mu)R_{23}(\lambda)\check{R}_{12}(\lambda - \mu) \quad (8)$$

where  $R = P\check{R}$  and  $P$  is the permutation operator on the tensor product of two  $n$ -dimensional spaces. It is easy to verify that it also satisfies a decorated Yang-Baxter equation

$$\check{R}_{12}(\lambda + \mu)C_1 R_{13}(\lambda) R_{23}(\mu) = R_{13}(\mu) R_{23}(\lambda) C_2 \check{R}_{12}(\lambda + \mu) \quad (9)$$

We now look for the  $R$ -matrix intertwining two  $L$ -matrices:

$$\check{R}(\lambda_1, \lambda_2) L(\lambda_1) \otimes L(\lambda_2) = L(\lambda_2) \otimes L(\lambda_1) \check{R}(\lambda_1, \lambda_2) \quad (10)$$

The  $su(2)$  case lead us to consider the following Ansatz [10]:

$$\begin{aligned} \check{R}(\lambda_1, \lambda_2) = & I_{12}(h_2)I_{34}(h_1) \left( \alpha \check{R}_{13}(\lambda_1 - \lambda_2)\check{R}_{24}(\lambda_1 - \lambda_2) \right. \\ & \left. + \beta \check{R}_{13}(\lambda_1 + \lambda_2)C_1\check{R}_{24}(\lambda_1 + \lambda_2)C_2 \right) I_{12}(-h_1)I_{34}(-h_2) \end{aligned} \quad (11)$$

The  $R$ -matrix acts on the product of four auxiliary spaces labeled from 1 to 4, and  $\alpha, \beta$  are to be determined. One then requires relation (10) to be satisfied and uses (8) and (9) to derive the following equation:

$$\begin{aligned} & \left( \alpha \check{R}_{13}(\lambda_1 - \lambda_2)\check{R}_{24}(\lambda_1 - \lambda_2) + \beta C_3\check{R}_{13}(\lambda_1 + \lambda_2)C_4\check{R}_{24}(\lambda_1 + \lambda_2) \right) I_{12}(2h_1)I_{34}(2h_2) = \\ & I_{12}(2h_2)I_{34}(2h_1) \left( \alpha \check{R}_{13}(\lambda_1 - \lambda_2)\check{R}_{24}(\lambda_1 - \lambda_2) + \beta \check{R}_{13}(\lambda_1 + \lambda_2)C_1\check{R}_{24}(\lambda_1 + \lambda_2)C_2 \right) \end{aligned}$$

Expanding the exponentials and taking into account all the terms yield only two equations:

$$\frac{\beta}{\alpha} = \frac{b}{B} \tanh(h_1 + h_2) \quad , \quad \frac{\beta}{\alpha} = \frac{a}{A} \tanh(h_1 - h_2) \quad (12)$$

where  $a = \cos(\lambda_1 - \lambda_2)$ ,  $b = \sin(\lambda_1 - \lambda_2)$ ,  $A = \cos(\lambda_1 + \lambda_2)$  and  $B = \sin(\lambda_1 + \lambda_2)$ . The compatibility equation

$$\frac{\tan(\lambda_1 - \lambda_2)}{\tan(\lambda_1 + \lambda_2)} = \frac{\tanh(h_1 - h_2)}{\tanh(h_1 + h_2)} \quad (13)$$

is satisfied provided equation (5) is satisfied for the pairs  $(\lambda_1, h_1)$  and  $(\lambda_2, h_2)$ . One can then pull out  $\alpha = \alpha(\lambda_1, \lambda_2)$  which appears as an arbitrary normalization of the  $R$ -matrix, to obtain:

$$\begin{aligned} \check{R}(\lambda_1, \lambda_2) = & \alpha(\lambda_1, \lambda_2)I_{12}(h_2)I_{34}(h_1) \left( \check{R}_{13}(\lambda_1 - \lambda_2)\check{R}_{24}(\lambda_1 - \lambda_2) + \frac{\sin(\lambda_1 - \lambda_2)}{\sin(\lambda_1 + \lambda_2)} \right. \\ & \left. \times \tanh(h_1 + h_2)\check{R}_{13}(\lambda_1 + \lambda_2)C_1\check{R}_{24}(\lambda_1 + \lambda_2)C_2 \right) I_{12}(-h_1)I_{34}(-h_2) \end{aligned} \quad (14)$$

The monodromy matrix being a tensor product of  $M$  copies of  $L$  matrices, one has

$$\check{R}(\lambda_1, \lambda_2) T(\lambda_1) \otimes T(\lambda_2) = T(\lambda_2) \otimes T(\lambda_1) \check{R}(\lambda_1, \lambda_2) \quad (15)$$

Taking the trace over the auxiliary spaces and using the cyclicity property of the trace one obtains  $[\tau(\lambda_1), \tau(\lambda_2)] = 0$ . We have thus proven that all the conserved charges  $H_p$  mutually commute.

Note that this proof is rigorous and valid *for all values of  $n$* , and for arbitrary values of the complex parameter  $x$ . It only involves the algebraic properties of the operators appearing in the various matrices, not the specific  $n$ -dependent representation. The equations (8) and (9) are the only equations of this type needed for the proof.

## 4 Properties of the $\check{R}$ matrix

I now give some properties of the  $R$ -matrix. At  $U = 0$  the two  $XX$  models decouple and  $h(\lambda, U) = 0$ . Expression (14) indeed decouples as a tensor product of two  $su(n)$   $XX$   $\check{R}$ -matrices.

The matrix also satisfies the regularity property

$$\check{R}(\lambda_1, \lambda_1) = \alpha(\lambda_1, \lambda_1) \text{Id} \quad (16)$$

and the unitarity property:

$$\begin{aligned} \check{R}(\lambda_1, \lambda_2)\check{R}(\lambda_2, \lambda_1) &= \alpha^2(\lambda_1, \lambda_2) \cos^2(\lambda_1 - \lambda_2) \\ &\times \left( \cos^2(\lambda_1 - \lambda_2) - \cos^2(\lambda_1 + \lambda_2) \tanh^2(h_1 - h_2) \right) \text{Id} \end{aligned} \quad (17)$$

The derivation of the last property is straightforward and involves algebraic relation between the building blocks of the  $su(n)$   $XX$   $\check{R}$ -matrix.

One can invoke the associativity of the algebra of  $L$  matrices, which ultimately reduces to the associativity of usual matrix multiplication, to conclude that the intertwiner satisfies a Yang-Baxter relation of its own. The two ways of permuting a product of three  $L$ -matrices imply

$$\begin{aligned} \check{\Pi} L(\lambda_1) \otimes L(\lambda_2) \otimes L(\lambda_3) \check{\Pi}^{-1} &= L(\lambda_1) \otimes L(\lambda_2) \otimes L(\lambda_3) \\ \check{\Pi} &= \left( \check{R}_{12}(\lambda_2, \lambda_3) \check{R}_{23}(\lambda_1, \lambda_3) \check{R}_{12}(\lambda_1, \lambda_2) \right)^{-1} \check{R}_{23}(\lambda_1, \lambda_2) \check{R}_{12}(\lambda_1, \lambda_3) \check{R}_{23}(\lambda_2, \lambda_3) \end{aligned} \quad (18)$$

I am unaware of the existence of an equivalent of the Schur lemma for the algebra of  $L$ -matrices. This would allow to conclude that  $\check{\Pi} \propto \text{Id}$ . Once proportionality is established, the regularity property ensures that the proportionality constant is one. We can argue that  $\check{\Pi} = \text{Id}$  holds because it has been explicitly verified for  $n = 2$  [8], and because the building blocks of the matrix satisfy algebraic relations which are independent of  $n$ . Thus the  $R$ -matrix satisfies the Yang-Baxter equation:

$$\check{R}_{12}(\lambda_2, \lambda_3) \check{R}_{23}(\lambda_1, \lambda_3) \check{R}_{12}(\lambda_1, \lambda_2) = \check{R}_{23}(\lambda_1, \lambda_2) \check{R}_{12}(\lambda_1, \lambda_3) \check{R}_{23}(\lambda_2, \lambda_3) \quad (19)$$

where  $\lambda$  and  $h$  are related through (5).

Using the explicit expression (14), it should be possible and it is instructive to try to check that the above equation is satisfied for any value of  $n$ . The factors  $I$  drop out and half of the terms on both sides of the YBE compensate each other because relations (8) and (9) hold. The eight remaining terms involve highly non-trivial relations. Each term is a product of six  $R$ -matrices, three for every copy, in the ordering dictated by the YBE. The  $C$  factors can be dropped by changing the arguments of the  $R$ -matrices appropriately. However the arguments do not allow the use of (8, 9) because the middle argument is not the sum of the extreme ones. One is forced to expand all the products on a basis and to regroup terms and check that the resulting trigonometric constraints are satisfied. I have not verified whether all these relations hold. Although specific particularities pertaining to the  $XX$  matrix are needed, I stress again that the proof is algebraic. In this respect the proof of [8] for  $n = 2$ , although following a different approach, should generalize in a straightforward way to any value of  $n$ .

## 5 Conclusion

We have shown that all the conserved charges of the  $su(n)$  Hubbard model mutually commute by exhibiting the intertwining matrix. This matrix is the  $su(n)$  generalization of the  $su(2)$  one obtained in [3]. Some properties were then derived. A notable feature of the matrix is its non-additivity property; the  $\lambda$  dependence cannot be reduced to a difference  $(\lambda_1 - \lambda_2)$ .

One can now start diagonalizing the Hamiltonians by the method of the algebraic Bethe Ansatz. Preliminary results suggest an interesting structure for the Bethe eigenstates [9].

One can also consider the extension of the  $su(n)$  model to other algebras. The algebraic underlying structure of the  $su(n)$  XX model should admit generalizations [7, 9].

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**Note:** While this work was being written, Martins exhibited a gauge-transformed version of the foregoing  $\tilde{R}(\lambda_1, \lambda_2)$  matrix. The derivation is also based on a generalization of Shastry's method. However, the proof in [11] was carried out for  $n = 3$ , and 'extensive checks' were made for  $n = 3, 4$ . The expression of  $\tilde{R}$  for all values of  $n$  is left as a (correct) *conjecture*. This reference also used unnecessarily complicated versions of the Yang-Baxter equations.

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