THE NONSTATIONARY IDEAL ON $P_\kappa(\lambda)$  
FOR $\lambda$ SINGULAR

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Abstract

We give a new characterization of the nonstationary ideal on $P_\kappa(\lambda)$ in the case when $\kappa$ is a regular uncountable cardinal and $\lambda$ a singular strong limit cardinal of cofinality at least $\kappa$.

0 Introduction

Let $\kappa$ be a regular uncountable cardinal and $\lambda \geq \kappa$ be a cardinal.

As [10] and [11] of which it is a continuation, this paper investigates ideals on $P_\kappa(\lambda)$ with some degree of normality. For $\delta \leq \lambda$, let $NS^\delta_{\kappa,\lambda}$ denotes the least $\delta$-normal ideal on $P_\kappa(\lambda)$. Thus $NS^\delta_{\kappa,\lambda} = \text{the noncofinal ideal } I_{\kappa,\lambda}$ for any $\delta < \kappa$, and $NS^\lambda_{\kappa,\lambda}$ is the nonstationary ideal $NS_{\kappa,\lambda}$. $NSS_{\kappa,\lambda}$ denotes the least seminormal ideal on $P_\kappa(\lambda)$. It is simple to see that $NSS_{\kappa,\lambda} = NS^\delta_{\kappa,\lambda}$ in case $\text{cf}(\lambda) < \kappa$. If $\lambda$ is regular, then by a result of Abe [1], $NSS_{\kappa,\lambda} = \bigcup_{\delta < \lambda} NS^\delta_{\kappa,\lambda}$.

One problem we address in the paper is whether for $\lambda > \kappa$ $NS_{\kappa,\lambda}$ is the restriction of a smaller ideal, i.e. whether $NS_{\kappa,\lambda} = J \upharpoonright A$ for some ideal $J \subset NS_{\kappa,\lambda}$ and some $A \in NS^*_\kappa\lambda$. The question as stated has a positive answer (see [2]) with $J = \nabla^\lambda I_{\kappa,\lambda}$. By a result of Abe [1] we can also take $J = NSS_{\kappa,\lambda}$ in case $\kappa \leq \text{cf}(\lambda) < \lambda$. We investigate the possibility of taking $J = \bigcup_{\delta < \xi} NS^\delta_{\kappa,\lambda}$ for some $\xi < \lambda$. If $\lambda$ is regular, no such $J$ will work since then, by an argument of [7],

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there is no $A$ such that $NS_{\kappa, \lambda} = NSS_{\kappa, \lambda} | A$.

Let $H_{\kappa, \lambda}$ assert that $\text{cof}(NS_{\kappa, \tau}) \leq \lambda$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$, where $\text{cof}(NS_{\kappa, \lambda})$ denotes the reduced cofinality of $NS_{\kappa, \lambda}^+$. Clearly, $H_{\kappa, \lambda}$ follows from $2^{<\lambda} = \lambda$. But there are other situations in which $H_{\kappa, \lambda}$ holds. For instance, if in $V$, $GCH$ holds and $\lambda$ is a limit cardinal, and $P$ is the forcing notion to add $\lambda^+$ Cohen reals, then in $V^P$, $2^{\aleph_0} > \lambda$ but, by results of [7], for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$, $\text{cof}(NS_{\kappa, \tau}) = \tau^+$ and hence $\text{cof}(NS_{\kappa, \tau}) \leq \lambda$.

It is known ([16], [11]) that if $\text{cf}(\lambda) < \kappa$, then $H_{\kappa, \lambda}$ holds just in case $NS_{\kappa, \lambda} = I_{\kappa, \lambda} | A$ for some $A$. We will prove the following.

**Proposition 0.1.** Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$ and $H_{\kappa, \lambda}$ holds. Then (a) $NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^{\text{cf}(\lambda)} | A$ for some $A$, but (b) there is no $B$ such that $NS_{\kappa, \lambda} = (\bigcup_{\delta < \text{cf}(\lambda)} NS_{\kappa, \lambda}^\delta) | B$.

It is not known whether the converse holds:

**Question.** Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$ and $NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^{\text{cf}(\lambda)} | A$ for some $A$. Does it follow that $H_{\kappa, \lambda}$ holds?

If $\lambda$ is singular and $H_{\kappa, \lambda}$ holds, then by the results above $NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^{\text{cf}(\lambda)} | A$ for some $A$. The following problem is open.

**Question.** Is it consistent that “$\lambda$ is singular but $NS_{\kappa, \lambda} \neq NS_{\kappa, \lambda}^\delta | A$ for every $\delta < \lambda$ and every $A \in NS_{\kappa, \lambda}^*$”?

For any infinite cardinal $\tau < \lambda$, let $u(\tau, \lambda) =$ the least size of any cofinal subset of $(P_\tau(\lambda), \subset)$.

Now suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then by results of [11], there is no $A$ such that $NS_{\kappa, \lambda} = I_{\kappa, \lambda} | A$. And it is shown in [10] that for any $\delta$ such that $\kappa \leq \delta < \text{cf}(\lambda)$ and $u(\delta) \in (\delta^+ \setminus \lambda, \lambda)$, there is no $A$ such that $NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^\delta | A$. Thus assuming Shelah’s Strong Hypothesis (SSH), $NS_{\kappa, \lambda} \neq NS_{\kappa, \lambda}^\delta | A$ for every $\delta < \text{cf}(\lambda)$ and every $A \in NS_{\kappa, \lambda}^*$.

**Question.** Is it consistent relative to some large cardinal that “$\kappa < \text{cf}(\lambda) < \lambda$ and $NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^\delta | A$ for some $\delta < \text{cf}(\lambda)$ and some $A \in NS_{\kappa, \lambda}^*$”?

Another problem we consider is whether $NS_{\kappa, \lambda}^\delta$ is nowhere precipitous, where $\delta \leq \lambda$. As shown by Matsubara and Shioya [14], $I_{\kappa, \lambda}$ is nowhere precipitous, and in fact so is any ideal $J$ on $P_\kappa(\lambda)$ of cofinality $u(\kappa, \lambda)$. Thus for every ideal $J$ on $P_\kappa(\lambda)$,

$$\text{cof}(J) \leq \lambda \Rightarrow \text{cof}(J) = u(\kappa, \lambda) \Rightarrow J \text{ is nowhere precipitous}.$$
We establish the following.

**Proposition 0.2.** Suppose \( H_{\kappa, \lambda} \) holds, and let \( \xi > \kappa \) be such that (a) \( \xi \) is either a successor ordinal, or a limit ordinal of cofinality at least \( \kappa \), and (b) \( \xi \leq \eta \), where \( \eta \) equals \( \lambda + 1 \) if \( \text{cf}(\lambda) < \kappa \), and \( \text{cf}(\lambda) \) otherwise. Then \( \overline{\text{cof}}(\bigcup_{\delta < \xi} NS_{\kappa, \lambda}^\delta) \leq \lambda \).

It follows from Propositions 0.1 and 0.2 that if \( H_{\kappa, \lambda} \) holds, then \( N S_{\kappa, \lambda} | A = NS_{\kappa, \lambda}^\delta | A \) for some \( A \in NS_{\kappa, \lambda}^\ast \), where \( \delta \) equals \( \text{cf}(\lambda) \) if \( \kappa \leq \text{cf}(\lambda) < \lambda \), and \( 0 \) otherwise.

Let us next consider cases when \( \kappa \leq \delta \leq \lambda \) and \( \text{cof}(NS_{\kappa, \lambda}^\delta) > u(\kappa, \lambda) \). Goldring [7] and the second author proved that if \( \lambda \) is regular and \( \mu > \lambda \) is Woodin, then in \( V^{\text{Col}(\lambda, \mu)} \) \( NS_{\kappa, \lambda} \) is precipitous. On the other hand Matsubara and the second author [13] showed (1) that if \( \lambda \) is a strong limit cardinal with \( \kappa \leq \text{cf}(\lambda) < \lambda \), then \( NS_{\kappa, \lambda} \) is nowhere precipitous. We establish the following.

**Proposition 0.3.** Suppose \( \kappa \leq \text{cf}(\lambda) \leq \delta < \lambda \).

(i) If \( |\delta| = \text{cf}(\lambda) \) and \( \tau^{\text{cf}(\lambda)} < \lambda \) for every cardinal \( \tau < \lambda \), then \( NS_{\kappa, \lambda}^\delta \) is nowhere precipitous.

(ii) If \( \text{cf}(\lambda) < |\delta| \) and \( \tau^{\text{cf}(\kappa, |\delta|)} < \lambda \) for every cardinal \( \tau < \lambda \), where \( c(\kappa, |\delta|) \) denotes the least size of any closed unbounded subset of \( P_\kappa(|\delta|) \), then \( NS_{\kappa, \lambda}^\delta \) is nowhere precipitous.

Note that if \( \kappa \leq \text{cf}(\lambda) \leq \delta < \lambda \) and the hypothesis of (i) (respectively, (ii)) of Proposition 0.3. holds, then \( \lambda^{\text{cf}(\lambda)} = \lambda \), so by results of [11],

\[
\text{cof}(NS_{\kappa, \lambda}^\delta) \geq \overline{\text{cof}}(NS_{\kappa, \lambda}^\delta) > \lambda = u(\kappa, \lambda).
\]

The paper is organized as follows. Section 1 collects basic definitions and facts concerning ideals on \( P_\kappa(\lambda) \). It is shown in Section 2 that \( \overline{\text{cof}}(NS_{\kappa, \lambda}^\delta) \) is a non-decreasing function of \( \pi \). In Section 3 we establish that if \( \lambda \) is regular, then \( \overline{\text{cof}}(NS_{\kappa, \lambda}) = \lambda \) just in case \( H_{\kappa, \lambda} \) holds. In Section 4, Proposition 0.2 is proved. In Section 5 we show that it is consistent relative to a large cardinal that “\( \lambda \) is regular and \( \overline{\text{cof}}(NS_{\kappa, \lambda} | A) < \lambda \) for some \( A \)”.

1 At some point the first author claimed to have found an error in the proof but it turned out that the mistake was his.
1 Ideals on $P_\kappa(\lambda)$

In this section we collect basic material concerning ideals on $P_\kappa(\lambda)$.

$NS_\kappa$ denotes the nonstationary ideal on $\kappa$.

For a set $A$ and a cardinal $\rho$, let $P_\rho(A) = \{a \subseteq A : a \prec \rho\}$.

Given four cardinals $\tau, \rho, \chi$, and $\sigma$, we define $\text{cov}(\tau, \rho, \chi, \sigma)$ as follows. If there is $X \subseteq P_\rho(\tau)$ with the property that for any $a \in P_\rho(\tau)$, we may find $Q \subseteq P_\rho(X)$ with $a \subseteq \cup Q$, we let $\text{cov}(\tau, \rho, \chi, \sigma) = \kappa$. Otherwise we let $\text{cov}(\tau, \rho, \chi, \sigma) = \kappa$.

We let $\text{cov}(\tau, \rho, \chi, \sigma) = u(\tau, \chi)$ in case $\rho = \chi$ and $\sigma = 2$.

**LEMMA 1.1.** ([15], pp. 85-86) Let $\tau, \rho, \chi$, and $\sigma$ be four cardinals such that $\tau \geq \rho \geq \chi \geq \sigma \geq 2$. Then the following hold:

(i) If $\tau > \rho$, then $\text{cov}(\tau, \rho, \chi, \sigma) \geq \tau$.

(ii) $\text{cov}(\tau, \rho, \chi, \sigma) = \text{cov}(\tau, \rho, \chi, \omega, \sigma)$.

(iii) $\text{cov}(\tau^+, \rho, \chi, \sigma) = \tau^+ \cdot \text{cov}(\tau, \rho, \chi, \sigma)$

(iv) If $\tau > \rho$ and $\text{cf}(\tau) < \sigma = \text{cf}(\sigma)$, then $\text{cov}(\tau, \rho, \chi, \sigma) = \bigcup_{\rho \leq \tau < \tau} \text{cov}(\tau', \rho, \chi, \sigma)$.

(v) If $\tau$ is a limit cardinal such that $\tau > \rho$ and $\text{cf}(\tau) \geq \chi$, then $\text{cov}(\tau, \rho, \chi, \sigma) = \bigcup_{\rho \leq \tau < \tau} \text{cov}(\tau', \rho, \chi, \sigma)$.

$I_{\kappa, \lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $\{a \in A : b \subseteq a\} = \emptyset$ for some $a \in P_\kappa(\lambda)$.

By an **ideal** on $P_\kappa(\lambda)$, we mean a collection $J$ of subsets of $P_\kappa(\lambda)$ that is closed under subsets (i.e. $P(A) \subseteq J$ for all $A \in J$), $\kappa$-complete (i.e. $\cup X \in J$ for every $X \in P_\kappa(J)$), fine (i.e. $I_{\kappa, \lambda} \subseteq J$) and proper (i.e. $P_\kappa(\lambda) \notin J$).

Given an ideal $J$ on $P_\kappa(\lambda)$, let $J^+ = \{A \subseteq P_\kappa(\lambda) : A \notin J\}$ and $J^* = \{A \subseteq P_\kappa(\lambda) \setminus A \in J\}$. For $A \in J^*$, let $J \mid A = \{B \subseteq P_\kappa(\lambda) : B \cap A \in J\}$. Given a cardinal $\chi > \lambda$ and $f : P_\kappa(\lambda) \to P_\kappa(\chi)$, we let $f(J) = \{X \subseteq P_\kappa(\chi) : f^{-1}(X) \in J\}$.

$\mathcal{M}_J$ denotes the collection of all maximal antichains in the partially ordered set $(J^+, \subseteq)$, i.e. of all $Q \subseteq J^+$ such that (i) $A \cap B \in J$ for any distinct $A, B \in Q$, and (ii) for every $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.

For a cardinal $\rho$, $J$ is $\rho$-saturated if $|Q| < \rho$ for every $Q \in \mathcal{M}_J$.

$\text{cof}(J)$ denotes the least cardinality of any $X \subseteq J$ such that $J = \bigcup_{A \subseteq X} P(A)$.

$\overline{\text{cof}}(J)$ denotes the least size of any $Y \subseteq J$ with the property that for every $A \in J$, there is $y \in P_\kappa(Y)$ with $A \subseteq \cup y$.

$\text{cof}(J)$ denotes the least cardinality of any $A \in J^+$.

$\overline{\text{cof}}(J)$ denotes the least size of any $Y \subseteq J$ with the property that for every $A \in J$, there is $y \in P_\kappa(Y)$ with $A \subseteq \cup y$.

$\text{non}(J)$ denotes the least cardinality of any $A \in J^+$.

Note that $\overline{\text{cof}}(J) \geq \overline{\text{cof}}(J) \geq \overline{\text{non}}(I_{\kappa, \lambda}) = u(\kappa, \lambda)$.
The following is well-known (see e.g. [10] and [11]).

**LEMMA 1.2.**

(i) \( \lambda^{<\kappa} = 2^{<\kappa} \cdot u(\kappa, \lambda) \).

(ii) \( \text{cof}(I_{\kappa, \lambda}) = \lambda \).

(iii) Let \( J \) be an ideal on \( P_\kappa(\lambda) \) such that \( \text{cof}(J) < \lambda \). Then \( \text{cof}(J) = u(\kappa, \lambda) \).

Shelah’s Strong Hypothesis (SSH) asserts that for any two uncountable cardinals \( \chi \) and \( \rho \) with \( \chi \geq \rho = \text{cf}(\rho) \), \( u(\rho, \chi) \) equals \( \chi \) if \( \text{cf}(\chi) \geq \rho \), and \( \chi^+ \) otherwise.

**LEMMA 1.3.** ([8])

(i) Suppose there is a \( \pi \)-saturated ideal on \( P_\nu(\lambda) \), where \( \pi \) and \( \nu \) are two cardinals such that \( \omega < \nu = \text{cf}(\nu) \leq \lambda \) and \( \pi < \nu \cap \kappa + \). Then \( u(\kappa, \lambda) \) equals \( \lambda \) if \( \text{cf}(\lambda) \geq \kappa \), and \( \lambda^+ \) if \( \omega < \text{cf}(\lambda) < \kappa \), and (b) \( \text{cov}(\lambda, \kappa, \kappa, \omega_1) = \lambda \) if \( \text{cf}(\lambda) = \omega \).

Numerous variations on the original notion of ideal normality have been considered over the years. One such variant is the concept of \( \delta \)-normality which has been studied by Abe [1].

Let \( \delta \leq \lambda \). An ideal \( J \) on \( P_\kappa(\lambda) \) is \( \delta \)-normal if given \( A \in J^+ \) and \( f : A \to \delta \) with the property that \( f(a) \in a \) for all \( a \in A \), there exists \( B \in J^+ \cap P(A) \) such that \( f \) is constant on \( B \).

\( N S_\delta \kappa, \lambda \) denotes the smallest \( \delta \)-normal ideal on \( P_\kappa(\lambda) \).

Note that \( \lambda \)-normality is the same as normality, so \( N S_\lambda \kappa, \lambda = N S_\kappa, \lambda \).

\( c(\kappa, \lambda) \) denotes the least size of any closed unbounded subset of \( P_\kappa(\lambda) \).

**LEMMA 1.4.**

(i) ([1]) Let \( \delta \) be an ordinal such that \( \delta + \kappa \leq \lambda \). Then \( NS_{\delta + \kappa}^\delta \kappa, \lambda \neq {}\emptyset \).

(ii) ([10]) Suppose \( \kappa \leq \delta < \lambda \). Then \( NS_{\kappa, \lambda}^\delta \) \( \mid A \) for some \( A \).

(iii) ([10]) Let \( \delta \) and \( \eta \) be two ordinals such that \( | \delta | < | \eta | < \lambda \) and \( \kappa \leq \eta \). Then there is no \( A \) such that \( NS_{\kappa, \lambda}^\eta = NS_{\kappa, \lambda}^\delta \mid A \).

**LEMMA 1.5.**

(i) ([11]) \( \text{cof}(NS_{\kappa, \lambda}^\delta) \geq \lambda \) for every \( \delta \leq \lambda \).

(ii) ([8], [11]) Let \( \delta \leq \lambda \). Then \( \text{cof}(NS_{\kappa, \lambda}^\delta \mid A) = \text{cof}(NS_{\kappa, \lambda}^\delta) \) for every \( A \in NS_{\kappa, \lambda}^\ast \).
The concept of $[\delta]^{<\theta}$-normality generalizes that of $\delta$-normality.

Let $\delta \leq \lambda$, and let $\theta$ be a cardinal with $\theta \leq \kappa$. An ideal $J$ on $P_\kappa(\lambda)$ is $[\delta]^{<\theta}$-normal if given $A \in J^+$ and $f : A \rightarrow P_\delta(\delta)$ with the property that $f(a) \in P_{[a \cap \delta]}(a \cap \delta)$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that $f$ is constant on $B$.

Note that for $\theta = \kappa$, $[\lambda]^{<\theta}$-normality is the same as strong normality.

We set $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and $\kappa$ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.

**LEMMA 1.6.** ([10])

(i) Suppose that $\delta < \kappa$, or $\theta < \kappa$, or $\kappa$ is not a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if $|P_\theta(\rho)| < \kappa$ for every cardinal $\rho < \kappa \cap (\delta + 1)$.

(ii) Suppose that $\delta \geq \kappa$, $\theta = \kappa$ and $\kappa$ is a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if $\kappa$ is a Mahlo cardinal.

(iii) Suppose there exists a $[\kappa]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Then $\kappa^{<\overline{\theta}} = \kappa$, and $(\pi^{<\overline{\theta}})^{<\overline{\theta}} = \pi^{<\overline{\theta}}$ for every cardinal $\pi > \kappa$.

Assuming there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, $NS_{\kappa, \lambda}^{[\delta]^{<\theta}}$ denotes the smallest such ideal.

**LEMMA 1.7.** ([10])

(i) Suppose $\theta < 2$ or $\delta < \kappa$. Then $NS_{\kappa, \lambda}^{[\delta]^{<\theta}} = I_{\kappa, \lambda}$.

(ii) Suppose $2 \leq \theta \leq \omega$. Then $NS_{\kappa, \lambda}^{[\delta]^{<\theta}} = NS_{\kappa, \lambda}^{[\theta]^{<\theta}}$.

(iii) Suppose $|\delta|^{<\overline{\theta}} = |\eta|^{<\overline{\theta}}$, where $\kappa \leq \eta \leq \lambda$ and $\pi$ is a cardinal with $2 \leq \pi \leq \kappa$. Then $NS_{\kappa, \lambda}^{[\delta]^{<\theta}} | A = NS_{\kappa, \lambda}^{[\eta]^{<\theta}} | A$ for some $A \in (NS_{\kappa, \lambda}^{[\gamma]^{<\theta}})^*$, where $\gamma = \delta \cup \eta$ and $\rho = \theta \cup \pi$.

Given an ordinal $\eta$, a cardinal $\pi$ and $f : P_\pi(\eta) \rightarrow P_\kappa(\lambda)$, let $C_{f, \kappa, \lambda}$ be the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \pi \neq \emptyset$ and $f(e) \subseteq a$ for every $e \in P_{[a \cap \pi]}(a \cap \eta)$.

**LEMMA 1.8.** ([10]) Suppose $A \subseteq P_\kappa(\lambda)$, $\kappa \leq \delta \leq \lambda$, and $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$. Then the following are equivalent:

(i) $A \in NS_{\kappa, \lambda}^{[\delta]^{<\theta}}$.

(ii) $A \cap C_{f, \kappa, \lambda} = \emptyset$ for some $f : P_{\pi, \kappa}(\delta) \rightarrow P_\kappa(\lambda)$.
(iii) \( A \cap \{ a \in C^\kappa_\gamma : a \cap \kappa \in \kappa \} = \emptyset \) for some \( g : P_\gamma^\delta (\delta) \to P_\delta (\lambda) \).

**LEMMA 1.9.** ([11]) Let \( \chi \) and \( \theta \) be two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi \leq \lambda \). Then the following hold:

(i) Let \( J \) be a \( [\chi]^{<\theta} \)-normal ideal on \( P_\kappa (\lambda) \). Then either \( \text{cf}(\text{cof}(J)) < \kappa \), or \( \text{cf}(\text{cof}(J)) > \chi^{<\theta} \).

(ii) If \( \chi^{<\theta} < \lambda \), then \( \text{cof}(\text{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) \geq \lambda \).

**LEMMA 1.10.** ([10], [11]) Suppose \( \kappa \leq \delta < \lambda \), and \( \theta \) is a cardinal with \( 2 \leq \theta \leq \kappa \). Then the following hold:

(i) \( \text{cof}(\text{NS}^{[\theta]^{<\kappa}}_{\kappa,\lambda}) = \text{cof}(\text{NS}^{[\delta]^{<\kappa}}_{\kappa,\delta}) \cdot \text{cov}(\lambda, (| \delta |^{<\theta} \cdot | \delta |^{<\theta} + \kappa) \text{ and} \\text{cof}(\text{NS}^{[\theta]^{<\kappa}}_{\kappa,\lambda}) = \text{cof}(\text{NS}^{[\delta]^{<\kappa}}_{\kappa,\delta}) \cdot \text{cov}(\lambda, (| \delta |^{<\theta} \cdot | \delta |^{<\theta} + \kappa ).

(ii) If \( \lambda \) is a limit cardinal and either \( \text{cf}(\lambda) < \kappa \) or \( \text{cf}(\lambda) > \delta |^{<\theta} \), then \( \text{cof}(\text{NS}^{[\theta]^{<\kappa}}_{\kappa,\lambda}) = \bigcup_{\delta < \tau < \lambda} \text{cof}(\text{NS}^{[\delta]^{<\kappa}}_{\kappa,\lambda}).

For a cardinal \( \tau, \mathcal{D}^{\tau}_{\kappa,\lambda} \) denotes the smallest cardinality of any family \( F \) of functions from \( \tau \) to \( P_\kappa (\lambda) \) with the property that for any \( g : \tau \to P_\kappa (\lambda) \), there is \( f \in F \) such that \( g(\alpha) \subseteq f(\alpha) \) for every \( \alpha < \tau \).

**LEMMA 1.11.** ([10])

(i) For any cardinal \( \tau > 0 \), \( \text{cf}(\mathcal{D}^{\tau}_{\kappa,\lambda}) > \tau \).

(ii) Suppose \( 0 < \delta \leq \lambda \), and \( \theta \) is a cardinal with \( 0 < \theta \leq \kappa \). Then \( \text{cof}(\text{NS}^{[\delta]^{<\kappa}}_{\kappa,\lambda} \upharpoonright A) = \mathcal{D}^{[\delta]^{<\kappa}}_{\kappa,\lambda} \) for every \( A \in (\text{NS}^{[\delta]^{<\kappa}}_{\kappa,\lambda})^+ \).

Next let us recall a few facts concerning the notion of precipitousness.

An ideal \( J \) on \( P_\kappa (\lambda) \) is precipitous if whenever \( A \in J^+ \) and \( < Q_n : n < \omega > \) is a sequence of members of \( \mathcal{M}_{J \upharpoonright A} \) such that \( Q_n+1 \subseteq \bigcup_{B \in Q_n} P(B) \) for all \( n < \omega \), there exists \( f \in \prod_{n \in \omega} Q_n \) such that \( f(0) \supseteq f(1) \supseteq \ldots \) and \( \bigcap_{n < \omega} f(n) \neq \emptyset \). \( J \) is nowhere precipitous if for each \( A \in J^+ \), \( J \upharpoonright A \) is not precipitous.

Let \( G(J) \) denote the following two-player game lasting \( \omega \) moves, with player I making the first move : I and II alternately pick members of \( J^+ \), thus building a sequence \( < X_n : n < \omega > \), subject to the condition that \( X_0 \supseteq X_1 \supseteq \ldots \) II wins \( G(J) \) just in case \( \bigcap_{n < \omega} X_n = \emptyset \).

**LEMMA 1.12.** ([5]) An ideal \( J \) on \( P_\kappa (\lambda) \) is nowhere precipitous if and only if the player II has a winning strategy in the game \( G(J) \).
The following is a straightforward generalization of a result of Foreman [4]:

**Proposition 1.13.** Let \( \chi \) and \( \theta \) be two cardinals such that \( \chi \leq \lambda \) and \( \theta \leq \kappa \). Then every \([\chi]^{<\theta}\) -normal, \((\chi^{<\theta})^+\) -saturated ideal on \( P_\kappa(\lambda) \) is precipitous.

**Lemma 1.14.** ([14]) Suppose \( J \) is an ideal on \( P_\kappa(\lambda) \) such that \( \text{cof}(J) = \text{non}(J) \). Then \( J \) is nowhere precipitous.

Thus for an ideal \( J \) on \( P_\kappa(\lambda) \),

\[
\text{cof}(J) \leq \lambda \Rightarrow \text{cof}(J) = u(\kappa, \lambda) \Rightarrow J \text{ is nowhere precipitous}.
\]

Let \( \tau \) be a cardinal with \( \kappa \leq \tau \leq \lambda \). It is simple to see that if GCH holds and either \( \text{cf}(\lambda) < \kappa \) or \( \tau < \text{cf}(\lambda) \), then \( \text{cof}(NS^\tau_{\kappa, \lambda}) = u(\kappa, \lambda) \). Note that if SSH holds and \( \kappa \leq \text{cf}(\lambda) \leq \tau \), then by Lemmas 1.5 (i) and 1.9, \( \text{cof}(NS^\tau_{\kappa, \lambda}) = u(\kappa, \lambda) \).

**Proposition 1.15.** Suppose \( \sigma \) is a strong limit cardinal with \( \text{cf}(\sigma) < \kappa \leq \sigma \leq \lambda \leq 2^\sigma \). Then the following hold:

(i) \( \text{cof}(NS^\tau_{\kappa, \lambda}) = u(\kappa, \lambda) \) for every cardinal \( \tau \) with \( \kappa \leq \tau \leq \sigma \).

(ii) Suppose \( 2^\lambda = 2^\sigma \). Then \( \text{cof}(NS^\tau_{\kappa, \lambda}) = u(\kappa, \lambda) \) for every cardinal \( \tau \) with \( \sigma < \tau \leq \lambda \).

**Proof.**

(i) : Let \( \tau \) be a cardinal with \( \kappa \leq \tau \leq \sigma \). If \( \tau = \lambda \), then

\[
\text{cof}(NS^\tau_{\kappa, \lambda}) \leq 2^\lambda = \lambda^{<\kappa} = u(\kappa, \lambda).
\]

Otherwise by Lemma 1.10

\[
\text{cof}(NS^\tau_{\kappa, \lambda}) = \text{cof}(NS_\kappa(\tau)) \cdot u(\tau^+, \lambda) \leq \lambda^\tau = \sigma^\tau = \sigma^{\text{cf}(\sigma)} \leq \lambda^{<\kappa} = u(\kappa, \lambda).
\]

(ii) : Given a cardinal \( \tau \) with \( \sigma < \tau \leq \lambda \),

\[
\text{cof}(NS^\tau_{\kappa, \lambda}) \leq 2^\lambda = 2^\sigma = \sigma^{\text{cf}(\sigma)} = u(\kappa, \lambda).
\]

\( \square \)

### 2 \( \text{cof}(NS^\chi_{\kappa, \lambda}) \)

By Lemma 1.11 (ii), \( \text{cof}(NS^\chi_{\kappa, \lambda}) = \partial^\chi_{\kappa, \lambda} \) for any cardinal \( \chi \) with \( \kappa \leq \chi \leq \lambda \). We now derive a similar formula for \( \text{cof}(NS^\chi_{\kappa, \lambda}) \).

For a cardinal \( \tau \), \( \mathfrak{D}_{\kappa, \lambda} \) denotes the smallest cardinality of any family \( F \) of functions from \( \tau \) to \( P_\kappa(\lambda) \) with the property that for any \( g : \tau \rightarrow P_\kappa(\lambda) \), there is \( Z \in P_\kappa(F) \) such that \( g(\alpha) \subseteq \bigcup_{f \in Z} f(\alpha) \) for every \( \alpha < \tau \).
LEMMA 2.1. Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Then $\text{cof}(\mathcal{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) \leq \mathfrak{b}_{\kappa,\lambda}^\chi$.

Proof. Select a collection $G$ of functions from $P_{\vec{\tau}_3}(\chi)$ to $P_{\kappa}(\lambda)$ so that $|G| = \mathfrak{b}_{\kappa,\lambda}^\chi$ and for any $k : P_{\vec{\tau}_3}(\chi) \rightarrow P_{\kappa}(\lambda)$, there is $Z_k \in P_{\kappa}(G)$ such that $k(e) \subseteq \bigcup_{g \in Z_k} g(e)$ for all $e \in P_{\vec{\tau}_3}(\chi)$. Then clearly for each $k : P_{\vec{\tau}_3}(\chi) \rightarrow P_{\kappa}(\lambda)$, $\bigcap_{g \in Z_k} C_{\kappa}^{\kappa,\lambda} \subseteq C_{\kappa}^{\kappa,\lambda}$. Hence $\text{cof}(\mathcal{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) \leq |G|$. \hfill $\square$

LEMMA 2.2. Let $\theta$ and $\chi$ be two cardinals such that $\omega \leq \theta = \text{cf}(\theta) < \kappa \leq \chi \leq \lambda$. Then $\mathfrak{b}_{\kappa,\lambda}^{\kappa,\chi} \leq u(\theta, \text{cof}(\mathcal{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}))$.

Proof. Pick a collection $H$ of functions from $P_\theta(\chi) \rightarrow P_3(\lambda)$ so that $|H| = \mathfrak{b}_{\kappa,\lambda}^{\kappa,\chi}$ and for any $A \in (\mathcal{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}})^\kappa$ with $\{b \in \bigcap_{h \in Q} C_{\kappa}^{\kappa,\lambda} : b \cap \kappa \in \kappa\} \subseteq A$. Select $X \subseteq P_{\theta}(H) \setminus \{\emptyset\}$ so that $|X| = u(\theta, |H|)$ and for any $Z \in P_{\theta}(H)$, there is $X \in \mathcal{X}$ with $Z \subseteq X$. For $X \in \mathcal{X}$, define $t_X : P_{\theta}(\chi) \rightarrow P_{\kappa}(\lambda)$ by $t_x(e) = \bigcap t_{X,e}$, where

$$T_{X,e} = \left\{b \in \bigcap_{h \in X} C_{\kappa}^{\kappa,\lambda} : e \cup \theta \subseteq b \text{ and } b \cap \kappa \in \kappa\right\}.$$

Note that $t_X(e) \subseteq T_{X,e}$, and $t_Y(e) \subseteq t_X(e)$ for all $Y \in \mathcal{X} \cap P(X)$. Now fix $f : P_\theta(\chi) \rightarrow P_{\kappa}(\lambda)$. We may find $W \in P_{\kappa}(\mathcal{X})$ such that

$$\left\{b \in \bigcap_{h \in W} C_{\kappa}^{\kappa,\lambda} : b \cap \kappa \in \kappa\right\} \subseteq C_{\kappa}^{\kappa,\lambda},$$

$\theta \leq |W|$ and for any $K \in P_\theta(W)$, there is $Z \in W$ with $U_K \subseteq Z$. For $e \in P_\theta(\chi)$, put $b_e = \bigcup_{X \in W} t_X(e)$. Note that $b_e \cap \kappa \in \kappa$.

Claim. Let $k \in \cup W$. Then $b_e \in C_{\kappa}^{\kappa,\lambda}$.

Proof of Claim. Fix $d \in P_{\theta}(b_e \cap \kappa)$. Pick $\varphi : d \rightarrow W$ so that $\beta \in t_{\varphi(\beta)}(e)$ for every $\beta \in d$. Select $Y \in W$ with $k \in Y$. There must be $Z \in W$ such that $Y \cup \left(\bigcup_{\beta \in d} \varphi(\beta)\right) \subseteq Z$. Then $d \in P_{\theta}(t_Z(e))$ and $t_Z(e) \in C_{\kappa}^{\kappa,\lambda}$, so $k(d) \subseteq t_Z(e) \subseteq b_e$. This completes the proof of the claim.

Thus $b_e \in \bigcap_{h \in U} C_{\kappa}^{\kappa,\lambda}$. Hence $b_e \in C_{\kappa}^{\kappa,\lambda}$, and consequently $f(e) \subseteq b_e$. \hfill $\square$
PROPOSITION 2.3. Let $\chi$ be a cardinal with $\kappa \leq \chi \leq \lambda$. Then $\text{cof}(NS^\chi_{\kappa,\lambda}) = \aleph_\chi^\kappa_{\kappa,\lambda}$.

Proof. By Lemmas 2.1 and 2.2. □

COROLLARY 2.4. Let $\pi$ and $\chi$ be two cardinals such that $\kappa \leq \pi < \chi \leq \lambda$. Then $\text{cof}(NS^\pi_{\kappa,\lambda}) \leq \text{cof}(NS^\chi_{\kappa,\lambda})$.

3 $NSS_{\kappa,\lambda}$

An ideal $J$ on $P_\kappa(\lambda)$ is seminormal if it is $\delta$-normal for every $\delta < \lambda$. $NSS_{\kappa,\lambda}$ denotes the smallest seminormal ideal on $P_\kappa(\lambda)$.

LEMMA 3.1.

(i) (Folklore) Suppose $\text{cf}(\lambda) < \kappa$. Then $NSS_{\kappa,\lambda} = NSS_{\kappa,\lambda}$.

(ii) ([1]) Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then $NSS_{\kappa,\lambda} = NSS_{\kappa,\lambda} | A$ for some $A$.

PROPOSITION 3.2. Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then $\text{cof}(NSS_{\kappa,\lambda}) > \lambda$.

Proof. By Lemmas 1.5 (iv) and 3.1. □

We will see that “$\text{cof}(NSS_{\kappa,\lambda}) > \lambda$” needs not hold in case $\lambda$ is regular. Note that if $\lambda$ is regular, then by Lemma 1.5 (iv), $\text{cof}(NSS_{\kappa,\lambda}) > \lambda$.

LEMMA 3.3. ([1]) Suppose $\lambda$ is regular. Then $NSS_{\kappa,\lambda} = \bigcup_{\delta < \lambda} NSS^\delta_{\kappa,\lambda}$.

Proof. It is immediate that $\bigcup_{\delta < \lambda} NSS^\delta_{\kappa,\lambda} \subseteq NSS_{\kappa,\lambda}$. To show the reverse inclusion, fix $A \in \bigcup_{\delta < \lambda} NSS^\delta_{\kappa,\lambda}$, $\eta$ with $\kappa \leq \eta < \lambda$, and $f : A \rightarrow \eta$ with the property that $f(a) \in a$ for all $a \in A$. For $\xi$ with $\eta \leq \xi < \lambda$, we may find $B_\xi \in (NSS^\xi_{\kappa,\lambda})^+ \cap P(A)$ and $\gamma_\xi < \eta$ such that $f$ takes the constant value $\gamma_\xi$ on $B_\xi$. There must be $\beta < \eta$ and $Z \subseteq \{\xi : \eta \leq \xi < \lambda\}$ such that $|Z| = \lambda$ and $\gamma_\xi = \beta$ for all $\xi \in Z$. Now set $C = \bigcup_{\xi \in Z} B_\xi$. Then clearly $C \in \bigcup_{\delta < \lambda} NSS^\delta_{\kappa,\lambda}$. Moreover $f$ is identically $\beta$ on $C$. □

LEMMA 3.4. ([11]) Suppose $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$, and $J$ is an ideal on $P_\kappa(\lambda)$ such that $J \subseteq NSS_{\kappa,\lambda}^{\lambda < \theta}$ and $\text{cof}(J) \leq \lambda^{< \theta}$. Then $J | A = I_{\kappa,\lambda} | A$ for
some $A \in \left( NS^{[\lambda]^{<\sigma}}_{k,\lambda} \right)^*$. 

In particular, if $J \subseteq NS_{k,\lambda}$ and $\text{col}(J) \leq \lambda$, then $J \mid D = I_{k,\lambda} \mid D$ for some $D \in NS^*_{k,\lambda}$.

**Lemma 3.5.** ([11]) Suppose $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$, and let $\sigma$ be the least cardinal $\tau$ such that $\tau < \theta \geq \lambda$. Then $\text{col}(I_{\kappa,\lambda} \mid A) \geq \sigma$ for every $A \in \left( NS^{[\lambda]^{<\sigma}}_{k,\lambda} \right)^*$.

**Proposition 3.6.** Suppose $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$, and $J$ is an ideal on $P_{\kappa}(\lambda)$ with $J \subseteq NS^{[\lambda]^{<\sigma}}_{k,\lambda}$. Let $\sigma$ be the least cardinal $\tau$ such that $\tau < \theta \geq \lambda$. Then $\text{col}(J) \geq \sigma$.

**Proof.** If $\text{col}(J) > \lambda < \theta$, there is nothing to prove. Otherwise, there is by Lemma 3.4 $A \in \left( NS^{[\lambda]^{<\sigma}}_{k,\lambda} \right)^*$ such that $J \mid A = I_{k,\lambda} \mid A$. Then by Lemma 3.5, $\sigma \leq \text{col}(I_{k,\lambda} \mid A) \leq \text{col}(J)$. \(\Box\)

In particular, $\text{col}(J) \geq \lambda$ for any ideal $J \subseteq NS_{k,\lambda}$.

**Lemma 3.7.** ([8])

(i) Suppose $\lambda$ is a successor cardinal, say $\lambda = \nu^+$. Then $NSS_{k,\lambda} \mid C = I_{k,\lambda} \mid C$ for some $C \in NS^*_{k,\lambda}$ if and only if $\text{col}(NSS_{k,\nu}) \leq \lambda$.

(ii) Suppose $\lambda$ is a regular limit cardinal. Then $NSS_{k,\lambda} \mid C = I_{k,\lambda} \mid C$ for some $C \in NS^*_{k,\lambda}$ if and only if $\text{col}(NSS_{k,\tau}) \leq \lambda = \text{cov}(\lambda, \tau^+, \tau^+, \kappa)$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$.

Recall from the introduction that $H_{k,\lambda}$ is said to hold if $\text{col}(NSS_{k,\tau}) \leq \lambda$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$.

**Proposition 3.8.** Suppose $\lambda$ is a regular cardinal. Then the following are equivalent:

(i) $H_{k,\lambda}$ holds.

(ii) $\text{col}(NSS_{k,\lambda}) = \lambda$.

(iii) $NSS_{k,\lambda} \mid C = I_{k,\lambda} \mid C$ for some $C \in NS^*_{k,\lambda}$.

**Proof.**

(i) $\longrightarrow$ (ii) : By Proposition 3.6, $\text{col}(NSS_{k,\lambda}) \geq \lambda$. For the reverse inequality, we consider two cases. First suppose $\lambda$ is a successor cardinal, say $\kappa = \nu^+$. Then by Lemma 3.3 $NSS_{k,\lambda} = \bigcup_{\nu \leq \delta < \lambda} NS^\delta_{k,\lambda}$. Now for $\nu \leq \delta < \lambda$,

$$\text{col}(NSS^\delta_{k,\lambda}) \leq \text{col}(NSS^\nu_{k,\lambda}) \leq \text{col}(NSS^\nu_{k,\lambda}) \cdot \text{cov}(\lambda, \lambda, \lambda, \kappa) \leq \lambda \cdot \lambda = \lambda$$
by Lemmas 1.4 (ii) and 1.10. Hence \( \text{cof}(\bigcup_{\nu \leq \delta < \lambda} NS_{\kappa,\lambda}^\delta) \leq \lambda \).

Next suppose \( \lambda \) is a limit cardinal. Given a cardinal \( \chi \) with \( \kappa \leq \chi < \lambda \), by Corollary 2.4 \( \text{cof}(NS_{\kappa,\lambda}^\chi) \leq \lambda \) for every cardinal \( \tau \) with \( \chi \leq \tau < \lambda \), so by Lemma 1.10 \( \text{cof}(NS_{\kappa,\lambda}^\chi) \leq \lambda \). It follows that \( \text{cof}(NSS_{\kappa,\lambda}) \leq \lambda \) since by Lemma 3.3

\[ NSS_{\kappa,\lambda} = \bigcup_{\kappa \leq \chi < \lambda} NS_{\kappa,\lambda}^\chi. \]

(ii) \( \rightarrow \) (iii) : By Lemma 3.4.

(iii) \( \rightarrow \) (i) : By Lemmas 1.5 (iii) and 3.7. \( \Box \)

4 Ideals \( J \) on \( P_\kappa(\lambda) \) with \( \text{cof}(J) = \lambda \)

In this section we look for cases when \( \text{cof}(\bigcup_{\delta < \xi} NS_{\kappa,\lambda}^\delta) = \lambda \), where \( \kappa < \xi \leq \lambda + 1 \).

We start with the following observation.

**Lemma 4.1.** Suppose \( K \subseteq NS_{\kappa,\lambda}^\kappa \) is an ideal on \( P_\kappa(\lambda) \) with \( \text{cof}(K) \leq \lambda \), and \( \xi \) is an ordinal such that (a) \( \kappa < \xi \leq \lambda + 1 \), (b) \( \xi \) is either a successor ordinal, or a limit ordinal of cofinality at least \( \kappa \), and (c) \( \bigcup_{\delta < \xi} NS_{\kappa,\lambda}^\delta \subseteq K \). Then

\[ \text{cof}(\bigcup_{\delta < \xi} NS_{\kappa,\lambda}^\delta) = \lambda. \]

**Proof.** By Lemma 3.5 we may find \( A \in NS_{\kappa,\lambda}^\kappa \) such that \( K \restriction A = I_{\kappa,\lambda} \restriction A \). For any cardinal \( \chi \) with \( \kappa \leq \chi < \xi \), \( NS_{\kappa,\lambda}^\chi \restriction A = I_{\kappa,\lambda} \restriction A \), so by Lemma 1.5 (ii) \( \text{cof}(NS_{\kappa,\lambda}^\chi) \leq \lambda \) for every \( \delta \) with \( \kappa \leq \delta < \xi \). It easily follows that \( \text{cof}(\bigcup_{\delta < \xi} NS_{\kappa,\lambda}^\delta) \leq \lambda \). The reverse inequality holds by Proposition 3.6. \( \Box \)

So we are looking for a large \( K \subseteq NS_{\kappa,\lambda}^{\text{cf}(\lambda)} \) with \( \text{cof}(K) \leq \lambda \). Assuming \( \mathcal{H}_{\kappa,\lambda} \) holds, we can take \( K = \bigcup_{\delta < \text{cf}(\lambda)} NS_{\kappa,\lambda}^\delta \) if \( \lambda \) is a singular cardinal of cofinality at least \( \kappa \), and \( K = NSS_{\kappa,\lambda} \) otherwise.

**Lemma 4.2.** ([11]) Let \( \theta \) be a cardinal with \( 2 \leq \theta \leq \kappa \). Suppose \( \text{cf}(\lambda) \leq \theta \leq \kappa \). Then for any cardinal \( \nu \) with \( \kappa \leq \nu < \lambda \),

\[ \text{cof}(NS_{\kappa,\lambda}^{\kappa \leq \nu}) \leq \bigcup_{\nu \leq \tau < \lambda} \text{cof}(NS_{\kappa,\lambda}^{\tau < \nu}). \]

**Proposition 4.3.** Let \( \theta \) be a cardinal with \( 2 \leq \theta \leq \kappa \). Suppose \( \text{cf}(\lambda) \leq \kappa \) and there is a cardinal \( \nu \) with \( \kappa \leq \nu < \lambda \) such that for any cardinal \( \tau \) with \( \nu \leq \tau < \lambda \),

\[ \text{cof}(NS_{\kappa,\lambda}^{\tau < \nu}) \leq \lambda \text{ and } \tau^{\mathcal{H}} < \lambda. \] Then \( \text{cof}(NS_{\kappa,\lambda}^{\kappa \leq \nu}) = \lambda \).
Proof. By Proposition 3.6 and Lemma 4.2.

In particular, if \( \text{cf}(\lambda) < \kappa \) and \( \mathcal{H}_{\kappa,\lambda} \) holds, then \( \overline{\text{col}}(NS_{\kappa,\lambda}) = \lambda \).

Note that if \( \overline{\text{col}}(NS_{\kappa,\lambda}^{<\theta}) = \lambda \), then by Lemma 3.4 \( NS_{\kappa,\lambda}^{<\theta} = I_{\kappa,\lambda} | C \) for some \( C \).

**Lemma 4.4.** ([10]) Let \( A \in I_{\kappa,\lambda}^+ \) be such that \( | \{ a \in A : b \subseteq a \} | = |A| \) for every \( b \in P_\kappa(\lambda) \). Then \( A \) can be decomposed into \( |A| \) pairwise disjoint members of \( I_{\kappa,\lambda}^+ \).

**Proposition 4.5.** Let \( \theta \) be a cardinal with \( 2 \leq \theta \leq \kappa \). Suppose there is \( C \) such that \( NS_{\kappa,\lambda}^{<\theta} = I_{\kappa,\lambda} | C \). Then \( P_\kappa(\lambda) \) can be split into \( \pi \) members of \( (NS_{\kappa,\lambda}^{<\theta})^+ \), where \( \pi \) is the least size of any member of \( (NS_{\kappa,\lambda}^{<\theta})^+ \).

Proof. Pick \( D \in (NS_{\kappa,\lambda}^{<\theta})^+ \). Then by Lemma 4.4, \( C \cap D \) can be decomposed into \( \pi \) pairwise disjoint members of \( (NS_{\kappa,\lambda}^{<\theta})^+ \).

In particular, if \( NS_{\kappa,\lambda} = I_{\kappa,\lambda} | C \) for some \( C \), then \( P_\kappa(\lambda) \) can be split into \( c(\kappa,\lambda) \) disjoint stationary sets.

**Proposition 4.6.** Suppose \( \theta \) and \( \rho \) are two cardinals such that \( \omega \leq \theta = \text{cf}(\theta) < \kappa \leq \rho \leq \lambda \), \( u(\theta,\lambda) = \lambda \), and either \( \text{cf}(\lambda) < \kappa \) or \( \text{cf}(\lambda) > \rho^{<\theta} \). Suppose further that for every cardinal \( \tau \) with \( \rho \leq \tau < \lambda \), \( \text{cof}(NS_{\kappa,\lambda}^{<\tau}) \leq \lambda \). Then \( \text{cof}(NS_{\kappa,\lambda}^{<\rho}) \leq \lambda \).

Proof. It suffices to show that \( \text{cof}(NS_{\kappa,\lambda}^{<\rho}) \leq \lambda \) for any cardinal \( \tau \) with \( \rho \leq \tau < \lambda \) since by Lemmas 1.1 and 1.10 \( \text{cof}(NS_{\kappa,\lambda}^{<\tau}) = \bigcup_{\rho \leq \tau < \lambda} \text{cof}(NS_{\kappa,\lambda}^{<\tau}) \) if \( \lambda \) is a limit cardinal, and \( \text{cof}(NS_{\kappa,\lambda}^{<\rho}) = \lambda \cdot \text{cof}(NS_{\kappa,\lambda}^{<\rho^{<\theta}}) \) if \( \lambda = \nu^+ \). Now for any cardinal \( \tau \) with \( \rho \leq \tau < \lambda \),

\[
\text{cof}(NS_{\kappa,\lambda}^{<\rho}) \leq \text{cf}_{\kappa,\lambda}^\rho \leq \text{cf}_{\kappa,\lambda}^{\rho^{<\theta}} \leq u(\theta,\text{cof}(NS_{\kappa,\lambda}^{<\rho^{<\theta}})) \leq u(\theta,\lambda) = \lambda
\]

by Lemmas 2.1 and 2.2.

**Proposition 4.7.** Suppose that \( \mathcal{H}_{\kappa,\lambda} \) holds, and \( \xi \) is an ordinal such that (a) \( \kappa < \xi \leq \eta \), where \( \eta \) equals \( \lambda + 1 \) if \( \text{cf}(\lambda) < \kappa \), and \( \text{cf}(\lambda) \) otherwise, and (b) \( \xi \) is either a successor ordinal, or a limit ordinal of cofinality at least \( \kappa \). Then \( \overline{\text{col}}\left( \bigcup_{\delta < \xi} NS_{\kappa,\lambda}^\delta \right) = \lambda \).

Proof. By Lemmas 1.4 (ii) and 4.1 and Propositions 3.8, 4.3 and 4.6.
In particular if \( \mathcal{H}_{\kappa,\lambda} \) holds and \( \kappa \leq \text{cf}(\lambda) < \lambda \), then 
\[
\operatorname{cof}\left( \bigcup_{\delta < \text{cf}(\lambda)} NS^\delta_{\kappa,\lambda} \right) = \lambda
\]
and hence by Lemma 1.5 (iv) there is no \( A \) such that \( NS_{\kappa,\lambda} = \left( \bigcup_{\delta < \text{cf}(\lambda)} NS^\delta_{\kappa,\lambda} \right)|A) \).

5 Ideals \( J \) on \( P_\kappa(\lambda) \) with \( \overline{\text{cof}}(J) < \lambda \)

There may exist ideals \( J \) on \( P_\kappa(\lambda) \) such that \( \overline{\text{cof}}(J) < \lambda \). Some examples were presented in [11]. We now give some more.

Given two cardinals \( \pi \leq \kappa \) and \( \chi \geq \lambda \), \( A_{\kappa,\lambda}(\pi, \chi) \) asserts the existence of \( Z \subseteq P_\kappa(\lambda) \) with \( |Z| = \chi \) such that \( |Z \cap P(a)| < \kappa \) for every \( a \in P_\kappa(\lambda) \).

**Lemma 5.1.** ([11]) Let \( \theta \) and \( \chi \) be two cardinals such that (a) \( 2 \leq \theta < \kappa \), \( \lambda < \chi \) and there is a \( [\chi]^<\theta \)-normal ideal on \( P_\kappa(\chi) \), and (b) \( A_{\kappa,\lambda}(\pi, \chi) \) holds for some regular uncountable cardinal \( \pi < \kappa \). Then \( \overline{\text{cof}}(I_{\kappa,\chi} \upharpoonright A) < \lambda \) for some \( A \in (NS_{\kappa,\chi}^{[\chi]^<\theta})^+ \).

**Lemma 5.2.** ([9]) Let \( \tau \) be the largest limit cardinal less than or equal to \( \kappa \). Assume \( \text{cf}(\lambda) < \kappa \) and one of the following conditions is satisfied:

(a) \( \tau = \kappa \).

(b) \( \tau > \text{cf}(\lambda) \) and \( \text{cf}(\lambda) \neq \text{cf}(\tau) \).

(c) \( \tau > \text{cf}(\lambda) = \text{cf}(\tau) \) and \( \min\{\text{pp}(\tau), \tau^+\} < \kappa \).

(d) \( \tau \leq \text{cf}(\lambda) \) and \( \min\{2^{\text{cf}(\lambda)}, (\text{cf}(\lambda))^{+3} \} < \kappa \).

Then \( A_{\kappa,\lambda}(\left(\text{cf}(\lambda)\right)^+, \lambda^+) \) holds.

Suppose for instance that \( \kappa \) is a limit cardinal and \( \text{cf}(\lambda) < \kappa \). Then by Lemmas 5.1 and 5.2, \( \overline{\text{cof}}(I_{\kappa,\lambda^+} \upharpoonright B) \leq \lambda \) for some \( B \in NS_{\kappa,\lambda^+}^+ \).

Note that in case \( \kappa \) is the successor of a cardinal of cofinality \( \text{cf}(\lambda) \), Lemma 5.2 does not apply, as none of the conditions (a) - (d) is satisfied. To handle this case, we introduce the following principle.

Given a cardinal \( \chi \geq \lambda \), \( B_{\kappa,\lambda}(\chi) \) asserts the existence of \( Z \subseteq P_\kappa(\lambda) \) with \( |Z| = \chi \) such that for every \( e \subseteq Z \) with \( |e| = \kappa \), there is a \( \kappa \)-to-one function in \( \prod_{z \in e} z \).

**Lemma 5.3.** ([11]) Let \( \theta \) and \( \chi \) be two cardinals such that (a) \( 2 \leq \theta < \kappa \), \( \lambda < \chi \) and there is a \( [\chi]^<\theta \)-normal ideal on \( P_\kappa(\chi) \), and (b) \( B_{\kappa,\lambda}(\chi) \) holds. Then
\( \text{col}(I_{\kappa,\chi} \mid A) \leq \lambda \) for some \( A \in (NS_{\kappa,\chi}^{[\kappa]^{<\theta}})^+ \).

Note that in case \( \text{cf}(\lambda < \kappa, B_{\kappa,\lambda}(\lambda^+)) \) follows from \( ADS_{\lambda} \), where \( ADS_{\lambda} \) asserts the existence of \( y_\alpha \subseteq \lambda \) for \( \alpha < \lambda^+ \) such that (a) for any \( \alpha < \lambda^+ \), \( \cup y_\alpha = \lambda \) and \( \text{o.t.}(y_\alpha) = \text{cf}(\lambda) \), and (b) given \( \beta < \lambda^+ \), there is \( g : \beta \rightarrow \lambda \) such that

\[
(y_\alpha \setminus g(\alpha)) \cap (y_{\alpha'} \setminus g(\alpha')) = \emptyset
\]

for any \( \alpha, \alpha' \in \beta \) with \( \alpha \neq \alpha' \).

For more on the existence of \( A \in (NS_{\kappa,\chi}^{[\kappa]^{<\theta}})^+ \) such that \( \text{col}(I_{\kappa,\chi} \mid A) < \chi \), see [9] and [11].

**Proposition 5.4.** Suppose \( \theta \) and \( \chi \) are cardinals such that \( 2 \leq \theta \leq \kappa \) and \( \lambda < \chi \), and \( A \in (NS_{\kappa,\chi}^{[\kappa]^{<\theta}})^+ \) is such that \( \text{col}(I_{\kappa,\chi} \mid A) \leq \lambda \). Then there is \( B \in (NS_{\kappa,\chi}^{[\kappa]^{<\theta}})^+ \) and a function \( f \) such that (a) \( f \) is an isomorphism from \( (P_{\kappa}(\lambda), \subseteq) \) onto \( (B, \subseteq) \), and (b) for any \( \delta \leq \lambda \), \( f(NS_{\kappa,\chi}^{[\kappa]^{<\theta}} \mid B) \) (and hence \( \text{col}(NS_{\kappa,\chi}^{[\kappa]^{<\theta}} \mid B) \leq \text{col}(NS_{\kappa,\chi}^{[\kappa]^{<\theta}}) \) and \( \text{col}(NS_{\kappa,\chi}^{[\kappa]^{<\theta}}) \leq \text{col}(NS_{\kappa,\chi}^{[\kappa]^{<\theta}}) \)).

**Proof.** Select \( x_\beta \in P_{\kappa}(\chi) \) for \( \beta < \lambda \) so that for each \( X \in I_{\kappa,\chi} \), there is \( z \in P_{\kappa}(\lambda) \) with \( X \cap \{ y \in A : \bigcup_{\beta \in z} x_\beta \subseteq y \} = \emptyset \). For \( \lambda \leq \alpha < \chi \), pick \( z_\alpha \in P_{\kappa}(\lambda) \) with

\[
\{ y \in A : \bigcup_{\beta \in z_\alpha} x_\beta \subseteq y \} \subseteq \{ t \in P_{\kappa}(\chi) : \alpha \in t \}.
\]

Let \( C \) be the set of all \( x \in P_{\kappa}(\chi) \) such that \( (\bigcup_{\beta \in \alpha} x_\beta) \cup (\bigcup_{\alpha \in z \lambda} z_\alpha) \subseteq x \). Note that \( C \subseteq NS_{\kappa,\chi}^{[\kappa]^{<\theta}} \).

**Claim 1.** Let \( x \in A \cap C \). Then \( \lambda \setminus \alpha = \{ \alpha \in \chi \setminus \lambda : z_\alpha \subseteq \alpha \cap \lambda \} \).

**Proof of Claim 1.** Since \( x \in C \), \( x \setminus \alpha \subseteq \{ \alpha \in \chi \setminus \lambda : z_\alpha \subseteq \alpha \cap \lambda \} \). To show the reverse inclusion, fix \( \alpha \in \chi \setminus \lambda \) with \( z_\alpha \subseteq \alpha \cap \lambda \). Then \( \bigcup_{\beta \in z_\alpha} x_\beta \subseteq x \), and hence \( \alpha \in x \), which completes the proof of Claim 1.

**Claim 2.** Let \( \alpha \in P_{\kappa}(\lambda) \). Then \( | \{ \alpha \in \chi \setminus \lambda : z_\alpha \subseteq \alpha \} | < \kappa \).

**Proof of Claim 2.** Pick \( x \in A \cap C \) with \( a \subseteq x \). Then by Claim 1,

\[
\{ \alpha \in \chi \setminus \lambda : z_\alpha \subseteq a \} \subseteq \{ \alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda \} \subseteq X,
\]

which completes the proof of Claim 2.

Using Claim 2, define \( f : P_{\kappa}(\lambda) \rightarrow P_{\kappa}(\chi) \) by \( f(a) = a \cup \{ \alpha \in \chi \setminus \lambda : z_\alpha \subseteq a \} \). Put \( B = \text{ran}(f) \). By Claim 1, \( x = f(x \cap \lambda) \) for any \( x \in A \cap C \), so \( A \cap C \subseteq B \).
It follows that $B \in \left( NS_{\kappa,\lambda}^{[\delta]} \right)^+$. 

As is easily seen, $f$ is an isomorphism from $(P_\kappa(\lambda), \subset)$ onto $(B, \subset)$, and moreover $f^{-1}(X) \in I_{\kappa,\lambda}$ for any $X \in I_{\kappa,\lambda}$. Now fix $\delta \leq \lambda$. Set $J = NS_{\kappa,\lambda}^{[\delta]}$. It is simple to see that $f(J)$ is an ideal on $P_\kappa(\chi)$ with the property that $B \in (f(J))^+$. 

Claim 3. $f(J)$ is $[\delta]^{<\theta}$-normal.

Proof of Claim 3. Fix $X \in (f(J))^+ \cap P(B)$ and $h : X \rightarrow P_\kappa(\delta)$ such that $h(x) \in P_{[x\in\theta]}(x)$ for every $x \in X$. Define $k : f^{-1}(X) \rightarrow P_\kappa(\delta)$ by $k(a) = h(f(a))$. There must be $A \in J^+ \cap P(f^{-1}(X))$ such that $k$ is constant on $A$. Then clearly $f''A \in (f(J))^+ \cap P(X)$, and moreover $h$ is constant on $f''A$, which completes the proof of the claim.

It immediately follows from Claim 3 that $NS_{\kappa,\lambda}^{[\delta]} \upharpoonright B \subseteq f(J)$.

To establish the reverse inclusion fix $Y \in f(J)$. Since $f^{-1}(Y \cap B) \in J$, we may find $g : P_\kappa(\delta) \rightarrow P_\kappa(\lambda)$ such that $f^{-1}(Y \cap B) \cap C_g^{\kappa,\lambda} = \emptyset$. Then clearly $(Y \cap B) \cap C_g^{\kappa,\lambda} = \emptyset$ and hence $Y \cap B \in NS_{\kappa,\lambda}^{[\delta]}$. 

Let $\kappa = (2^\rho)^+$, where $\rho$ is an infinite cardinal, and suppose $\lambda$ is a strong limit cardinal with $\text{cf}(\lambda) \leq \rho$. Then $A_{\kappa,\lambda}(\rho^+, 2^\lambda)$ holds, since $|P_{\rho^+}(\lambda) \cap P(\lambda)| \leq 2^\rho$ for any $a \in P_{\kappa}(\lambda)$). Hence by Lemmas 4.2 and 5.1 and Proposition 5.4, $\text{cof}(NS_{\kappa,\lambda}^{(\lambda)} \upharpoonright B) \leq \lambda$ for some $B \in NS_{\kappa,\lambda}^{+}$. 

PROPOSITION 5.5. Suppose that $\text{cof}(NS_{\kappa,\lambda}) \leq \lambda^+$, and there is $A \in NS_{\kappa,\lambda}^{+}$ such that $\text{cof}(I_{\kappa,\lambda}^+ \upharpoonright A) \leq \lambda$. Then $\text{cof}(NS_{\kappa,\lambda}^+ \upharpoonright B) \leq \lambda^+$ for some $B \in NS_{\kappa,\lambda}^{+}$.

Proof. By Lemma 3.7 (i), there is $C \in NS_{\kappa,\lambda}^+ \upharpoonright C = I_{\kappa,\lambda}^+ \upharpoonright C$. Then $B = A \cap C$ is as desired. 

For example, suppose $\kappa = \omega_1$ and $\lambda = \omega_2$ for some infinite limit ordinal $\alpha$ of cofinality $\omega$. Then by Lemmas 4.2, 5.1 and 5.2 and Proposition 5.5, $\text{cof}(NS_{\kappa,\lambda}^+ \upharpoonright B) \leq \lambda$ for some $B \in NS_{\kappa,\lambda}^{+}$. 

If $\lambda$ is singular, then by Lemma 3.1 $NS_{\kappa,\lambda} = NSS_{\kappa,\lambda} \upharpoonright B$ for some $B$, so $\text{cof}(NS_{\kappa,\lambda} \upharpoonright A) \leq \lambda$ for some $A \in NS_{\kappa,\lambda}^+$ just in case $\text{cof}(NS_{\kappa,\lambda} \upharpoonright D) \leq \lambda$ for some $D \in NS_{\kappa,\lambda}^+$. 

Suppose $\text{cof}(NS_{\kappa,\lambda} \upharpoonright D) \leq \lambda$ for some $D \in NS_{\kappa,\lambda}^+$. Then setting $\sigma = \text{cof}(NS_{\kappa,\lambda} \upharpoonright D)$,
cof(NS_{\kappa, \lambda}) \leq u(\kappa, \sigma) \leq u(\kappa, \lambda) \leq \text{cof}(NS_{\kappa, \lambda})
by Lemma 1.11 (ii), so \text{cof}(NS_{\kappa, \lambda}) = u(\kappa, \sigma) = u(\kappa, \lambda). Hence by Lemma 1.5 (iv), SSH does not hold.

**PROPOSITION 6.1.** Let \theta and \chi be two cardinals such that \(2 \leq \theta \leq \kappa\) and \(\lambda < \chi\). Suppose that \(\text{cof}(NS_{\kappa, \lambda}) \leq \chi^0\), and there is \(A \in (NS_{\kappa, \lambda}^{\chi^0})^+\) such that \(\text{cof}(I_{\kappa, \lambda} \cap A) \leq \lambda\). Then \(\text{cof}(NS_{\kappa, \lambda}) \mid B \leq \lambda\) for some \(B \in (NS_{\kappa, \lambda}^{\chi^0})^+\).

**Proof.** By Lemma 3.4 there is \(C \in (NS_{\kappa, \lambda}^{\chi^0})^*\) such that \(NS_{\kappa, \lambda} \mid C = I_{\kappa, \lambda} \mid C\). Then \(B = A \cap C\) is as desired. \(\Box\)

Here is an example of a situation where Proposition 5.6 applies. Starting from a \(\mathcal{P}(\nu)\)-hypermeasurable, Cummings [3] constructs a generic extension \(V\) of \(V\) in which for any infinite cardinal \(\rho, 2^\rho = \rho^+\) if \(\rho\) is a successor cardinal, and \(\rho^{++}\) otherwise. In \(V\), let \(\sigma\) be a regular uncountable cardinal, and \(\mu > \sigma\) be a cardinal of cofinality less than \(\sigma\). Suppose that (a) \(\sigma\) is not the successor of a cardinal \(\tau\) with \(\text{cf}(\tau) \leq \text{cf}(\mu)\), and (b) \(\sigma\) is not the successor of the successor of a limit cardinal \(\pi\) with \(\text{cf}(\pi) \leq \text{cf}(\mu)\). Then by Lemmas 5.1 and 5.2 and Proposition 5.6, \(\text{cof}(NS_{\sigma, \mu^+} \mid B) \leq \mu\) for some \(B \in (NS_{\sigma, \mu^+}^{\mu^+})^+\).

**6 Cases when \(NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^{\text{cf}(\lambda)} \mid A\) for some \(A\)**

In this section we establish that if \(\kappa \leq \text{cf}(\lambda) < \lambda\) and \(\mathcal{H}_{\kappa, \lambda}\) holds, then \(NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^{\text{cf}(\lambda)} \mid A\) for some \(A\). Note that if \(\text{cf}(\lambda) < \kappa\) and \(\mathcal{H}_{\kappa, \lambda}\) holds, then by Lemmas 3.5 and 4.1, \(NS_{\kappa, \lambda} = I_{\kappa, \lambda} \mid A\) for some \(A\). Note further that if \(\lambda\) is regular, then trivially \(NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^\lambda \mid P(\lambda)\). By combining the three cases, we obtain that if \(\text{cof}(NS_{\kappa, \tau}) \leq \lambda\) for every cardinal \(\tau\) with \(\kappa \cdot \text{cf}(\lambda) \leq \tau < \lambda\), then \(NS_{\kappa, \lambda} = NS_{\kappa, \lambda}^{\text{cf}(\lambda)} \mid A\) for some \(A\).

**PROPOSITION 6.1.** Let \(\pi, \theta\) and \(\chi\) be three cardinals with \(\kappa \leq \pi < \lambda\) and \(2 \leq \theta \leq \kappa \leq \chi \leq \lambda\). Suppose that (a) \(\lambda\) is singular, (b) \(\theta \leq \text{cf}(\lambda)\) in case \(\chi = \lambda\), and (c) \(\text{cof}(NS_{\kappa, \tau}^{\pi \cdot \chi}) \leq \lambda^0\) for every cardinal \(\tau\) with \(\pi \leq \tau < \lambda\). Then there is \(A \in (NS_{\kappa, \lambda}^{\chi^0})^*\) such that \(NS_{\kappa, \lambda}^{\pi \cdot \chi} \subseteq NS_{\kappa, \lambda}^{\text{cf}(\lambda)} \mid A\).

**Proof.** Set \(\mu = \text{cf}(\lambda)\) and select an increasing sequence of cardinals \(\lambda_0, \lambda_1, \ldots < \lambda\) so that (i) \(\bigcup_{\eta < \mu} \lambda_\eta = \lambda\), (ii) \(\lambda_0 > \pi \cdot \mu\), and (iii) \(\lambda_0 \geq \chi\) in case \(\chi < \lambda\). For \(\eta < \mu\), pick a family \(G_\eta\) of functions from \(P(\nu)\) to \(P(\lambda_\eta)\) so that \(|G_\eta| \leq \text{cof}(NS_{\kappa, \lambda_\eta}^{\chi \cap \lambda_\eta})\) and for every \(H \in (NS_{\kappa, \lambda_\eta}^{\chi \cap \lambda_\eta})^*\), there is \(y \in P(\nu)\) with \(|G_\eta \cap H| \leq |G_\eta| \cdot |H|\) and with \(\text{cof}(P(\nu) \setminus \{0\}) \leq \chi\).
such that \( \{ b \in \bigcap_{g \in y} C^{\kappa,\lambda}_{g} : b \cap \kappa \in \kappa \} \subseteq H \). Let \( \bigcup_{\eta \in \mu} G_{\eta} = \{ g_{\xi} : e \in P_{\eta}^{\mu}(\lambda) \} \).

Let \( A \) be the set of all \( a \in P_{\kappa}(\lambda) \) such that

- \( \overline{\theta} \subseteq a \) in case \( \overline{\theta} < \kappa \);
- \( \omega \subseteq a \);
- \( a \cap \kappa \subseteq \kappa \);
- \( k(a) = a \) for every \( a \in a \), where \( k : \lambda \rightarrow \mu \) is defined by \( k(a) = \) the least \( \eta < \mu \) such that \( a \in \lambda_{\eta} \);
- If \( \chi = \lambda \), then \( \iota(v) \in a \) for every \( v \in P_{|\mathcal{V}(\mathcal{G})|}(a) \), where \( \iota : P_{\mathcal{V}(\mathcal{G})}(\lambda) \rightarrow \mu \) is defined by \( \iota(v) \) as the least \( \eta < \mu \) such that \( v \subseteq \lambda_{\eta} \);
- \( g_{\xi}(u) \subseteq a \) whenever \( e \in P_{|\mathcal{V}(\mathcal{G})|}(a) \) and \( u \in P_{|\mathcal{V}(\mathcal{G})|}(a) \cap \text{dom}(g_{\xi}) \).

It is immediate that \( A \in (NS_{\kappa,\lambda}^{[\chi]_{|\mathcal{V}|}})^{+} \). Let us check that \( A \) is as desired. Thus fix \( f : P_{\mathcal{V}}(\chi) \rightarrow P_{\kappa}(\lambda) \). Given \( \eta < \mu \), define \( p_{\eta} : P_{\mathcal{V}}(\chi \cap \lambda_{\eta}) \rightarrow P_{\kappa}(\lambda_{\eta}) \) by \( p_{\eta}(v) = \{ \zeta \} \), where \( \zeta = \) the least \( \sigma \) such that \( \eta \leq \sigma < \mu \) and \( f(v) \subseteq \lambda_{\sigma} \). Also define \( q_{\eta} : P_{\mathcal{V}}(\chi \cap \lambda_{\eta}) \rightarrow P_{\kappa}(\lambda_{\eta}) \) by \( q_{\eta}(v) = \lambda_{\eta} \cap f(v) \). Select \( x_{\eta}, y_{\eta} \in P_{\kappa}(P_{\mathcal{V}}(\lambda)) \setminus \{ \} \), so that \( \{ g_{\xi} : e \in x_{\eta} \cup y_{\eta} \} \subseteq G_{\eta}, \{ b \in \bigcap_{e \in x_{\eta}} C^{\kappa,\lambda}_{g_{\xi}} : b \cap \kappa \in \kappa \} \subseteq C^{\kappa,\lambda}_{e_{\eta}} \) and \( \{ b \in \bigcap_{e \in y_{\eta}} C^{\kappa,\lambda}_{g_{\xi}} : b \cap \kappa \in \kappa \} \subseteq C^{\kappa,\lambda}_{\eta} \). Finally define \( u : \mu \rightarrow P_{\kappa}(\lambda) \) by \( u(\eta) = \cup(x_{\eta} \cup y_{\eta}) \), and \( t : P_{\kappa}(\mu) \rightarrow P_{\kappa}(\lambda) \) so that for any \( \eta \in \mu \), \( t(\{ \eta \}) \) equals \( u(\eta) \) if \( \overline{\theta} < \kappa \), and \( u(\eta) \cap \{ u(\eta) \} ^{+} \) otherwise.

We claim that \( A \cap C_{t}^{\kappa,\lambda} \subseteq C_{f}^{\kappa,\lambda} \). Thus let \( a \in A \cap C_{t}^{\kappa,\lambda} \) and \( v \in P_{|\mathcal{V}(\mathcal{G})|}(a \cap \chi) \). There must be \( \eta \in a \cap \mu \) such that \( v \subseteq \lambda_{\eta} \). Then \( a \cap \lambda_{\eta} \in C_{\mu_{a}}^{\kappa,\lambda_{\eta}} \) since \( x_{\eta} \subseteq P_{|\mathcal{V}(\mathcal{G})|}(a) \). It follows that \( v \cup f(v) \subseteq \lambda_{\sigma} \) for some \( \sigma \in a \cap \mu \). Now \( a \cap \lambda_{\eta} \in C^{\kappa,\lambda}_{e_{\eta}} \), since \( y_{\eta} \subseteq P_{|\mathcal{V}(\mathcal{G})|}(a) \), so \( f(v) \subseteq a \).

In Proposition 6.1 we assumed that \( \overline{\theta} \leq \text{cf}(\lambda) \) in case \( \chi = \lambda \). Some condition of this kind is necessary. In fact if \( \text{cf}(\lambda) < \kappa \) and \( u(\kappa, \lambda \subseteq \mathcal{V}) = \lambda \subseteq \mathcal{V} \), then for each \( A \in (NS_{\kappa,\lambda}^{[\chi]_{|\mathcal{V}|}})^{+} \), \( NS_{\kappa,\lambda}^{[\chi]_{|\mathcal{V}|}} \neq NS_{\kappa,\lambda}^{\text{c}^{\text{cf}(\lambda)}} | A \) since by Lemma 1.11,

\[
\text{cof}(NS_{\kappa,\lambda}^{[\chi]_{|\mathcal{V}|}}) > \lambda \subseteq \mathcal{V} \geq \lambda \geq \text{cof}(NS_{\kappa,\lambda}^{\text{c}^{\text{cf}(\lambda)}} | A).
\]

**Corollary 6.2.** Suppose that \( (a) \) SSH holds, or \( (b) \) there exists a \( \sigma \)-saturated ideal on \( P_{\nu}(\lambda) \), where \( \sigma \) and \( \nu \) are two cardinals such that \( \omega < \nu = \text{cf}(\nu) < \lambda \) and \( \sigma < \nu \), or \( (c) \) there is a regular uncountable cardinal \( \tau < \lambda \) that is mildly \( \pi \)-indefinable for every cardinal \( \pi \) with \( \pi \leq \pi < \lambda \). Let \( \theta \) and \( \chi \) be two cardinals such that \( 2 \leq \theta \leq \kappa, \kappa \cdot \text{cf}(\lambda) \leq \chi < \lambda \) and \( \text{cof}(NS_{\kappa,\chi}^{\text{c}^{\text{cf}(\lambda)}}) \leq \lambda \subseteq \mathcal{V} \). Then
\[ NS_{\kappa,\lambda}^{\chi,\theta} \mid A = NS_{\kappa,\lambda}^{\chi,\theta} \mid A \text{ for some } A \in \left( NS_{\kappa,\lambda}^{\chi,\theta} \right)^* \]

**Proof.** Use Lemmas 1.3 and 1.10. \[ \square \]

**COROLLARY 6.3.** Suppose \( \text{cf}(\lambda) < \kappa \), and \( \theta \) and \( \chi \) are two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi < \lambda \) and \( \text{cof}(NS_{\kappa,\lambda}^{\chi,\theta}) \leq \lambda^{<\theta} \) for every cardinal \( \tau \) with \( \chi \leq \tau < \lambda \). Then \( \text{cof}(NS_{\kappa,\lambda}^{\chi,\theta}) = \text{u}(\kappa, \lambda) \).

**Proof.** By Proposition 6.1 there is \( A \in \left( NS_{\kappa,\lambda}^{\chi,\theta} \right)^* \) such that \( NS_{\kappa,\lambda}^{\chi,\theta} \mid A = I_{\kappa,\lambda} \mid A \). Then by Lemma 1.11 (ii), \( \text{cof}(NS_{\kappa,\lambda}^{\chi,\theta}) = \text{cof}(I_{\kappa,\lambda} \mid A) = \text{cof}(I_{\kappa,\lambda}) = \text{u}(\kappa, \lambda) \). \[ \square \]

**COROLLARY 6.4.**

(i) Suppose \( \lambda \) is singular and \( \mathcal{H}_{\kappa,\lambda} \) holds. Then \( NS_{\kappa,\lambda} = NS_{\kappa,\lambda}^{\text{cf}(\lambda)} \mid A \) for some \( A \).

(ii) Let \( \chi \) be a cardinal such that \( \kappa \cdot \text{cf}(\lambda) \leq \chi < \lambda \) and \( \text{cof}(NS_{\kappa,\lambda}^{\chi,\theta}) \leq \lambda \) for every cardinal \( \tau \) with \( \chi \leq \tau < \lambda \). Then \( NS_{\kappa,\lambda}^{\chi,\theta} \mid A = NS_{\kappa,\lambda}^{\text{cf}(\lambda)} \mid A \) for some \( A \in NS_{\kappa,\lambda}^* \).

**COROLLARY 6.5.**

(i) Let \( \chi \geq \kappa \) be a cardinal, and \( \alpha < \kappa \) be a limit ordinal such that \( \text{cof}(NS_{\kappa,\chi}) \leq \chi^{+\alpha} \). Then \( NS_{\kappa,\chi}^{\chi+\alpha} \mid A = I_{\kappa,\chi+\alpha} \mid A \) for some \( A \in NS_{\kappa,\chi}^* \).

(ii) Let \( \chi > \kappa \) be a cardinal such that \( \text{cof}(NS_{\kappa,\chi}) < \chi^{+\kappa} \). Then \( NS_{\kappa,\chi+\kappa}^{\chi+\kappa} \mid A = NS_{\kappa,\chi+\kappa}^{\chi} \mid A \) for some \( A \in NS_{\kappa,\chi+\kappa}^* \).

**Proof.** Use Lemmas 1.1 and 1.10. \[ \square \]

Note that we do get a better result by considering the reduced cofinality \( \text{cof} \) instead of the usual one \( \text{cof} \). For example, suppose that \( GCH \) holds in \( V \). By a result of [12], there is a \( < \kappa \)-closed, \( \kappa^{+}\)-cc forcing notion \( P \) such that in \( V^P \), \( \text{cof}(NS_{\kappa,\chi}) = \kappa^{+\omega} \) and \( \text{cof}(NS_{\kappa,\kappa}) = \kappa^{+\omega+1} \). Then in \( V^P \), there is by Corollary 6.5 (i) \( A \in NS_{\kappa,\chi+\kappa}^* \) such that \( NS_{\kappa,\kappa+\omega}^{\chi+\kappa} \mid A = I_{\kappa,\chi+\kappa} \mid A \).

Let us next discuss the condition in Proposition 6.1 that \( \text{cof}(NS_{\kappa,\kappa}^{\chi,\theta}) \leq \lambda^{<\theta} \) for almost all cardinals \( \tau < \lambda \).

**PROPOSITION 6.6.** Let \( \theta \) and \( \chi \) be two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi < \lambda \). Suppose \( \text{cof}(NS_{\kappa,\chi}^{\chi,\theta}) \leq \lambda^{<\theta} \) and \( \chi^{<\theta} \geq \lambda \). Then \( \text{cof}(NS_{\kappa,\lambda}^{\chi,\theta}) = \lambda^{<\theta} \).
\[ \text{cof}(NS_{\kappa,\chi}) \leq \chi. \]

**Proof.** By Lemma 1.6 (iii) \( \chi^\theta = \lambda^\theta \), so by Lemma 3.5 \( NS_{\kappa,\chi} = I_{\kappa,\chi} | A \) for some \( A \). It follows that \( \text{cof}(NS_{\kappa,\chi}) \leq \chi \). Moreover by Lemma 1.10

\[ \text{cof}(NS_{\kappa,\chi}^{(\kappa,\chi)}) = \text{cof}(NS_{\kappa,\chi}^{(\kappa)}) \cdot \text{cov}(\lambda, (\lambda^{\theta}), (\lambda^{\theta}), \kappa) = \text{cof}(NS_{\kappa,\chi}^{(\kappa)}). \]

\[ \square \]

**COROLLARY 6.7.** Let \( \theta \) and \( \chi \) be two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi < \chi = \lambda \). Suppose \( \text{cof}(NS_{\kappa,\chi}^{(\kappa,\chi)}) \leq \lambda < \theta = \lambda \). Then there is \( A \in (NS_{\kappa,\chi}^{(\kappa,\chi)})^* \) such that \( \text{cof}(NS_{\kappa,\lambda} | A) \leq \chi \).

**Proof.** By Lemma 1.7 we may find \( A \in (NS_{\kappa,\chi}^{(\kappa,\chi)})^* \) such that \( NS_{\kappa,\lambda} | A = NS_{\kappa,\chi}^{(\kappa,\chi)} | A \). Then by Proposition 6.6, \( \text{cof}(NS_{\kappa,\lambda} | A) \leq \text{cof}(NS_{\kappa,\chi}^{(\kappa,\chi)}) \leq \chi \). \( \square \)

**Question.** Suppose \( \theta \) and \( \chi \) are two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi \) and \( \text{cof}(NS_{\kappa,\chi}^{(\kappa,\chi)}) \leq \lambda < \theta = \lambda \). Does then \( \chi^\theta = \chi \) hold?

**PROPOSITION 6.8.**

(i) Suppose \( \theta \) and \( \sigma \) are two cardinals such that \( 2 \leq \theta \leq \kappa \leq \sigma < \lambda, \theta \leq \text{cf}(\lambda) \) and \( \text{cof}(NS_{\kappa,\chi}^{(\kappa,\sigma)}) \leq \lambda < \theta \) for every cardinal \( \tau \) with \( \sigma \leq \tau < \lambda \). Then there is a cardinal \( \pi \) with \( \sigma \leq \pi < \lambda \) such that \( \text{cof}(NS_{\kappa,\chi}^{(\kappa,\pi)}) \leq \lambda \) for every cardinal \( \chi \) with \( \pi \leq \chi < \lambda \).

(ii) Let \( \theta \) and \( \pi \) be two cardinals with \( 2 \leq \theta \leq \kappa \leq \pi < \lambda \). Suppose \( \kappa \leq \text{cf}(\lambda) < \lambda \), and \( \text{cof}(NS_{\kappa,\chi}^{(\kappa,\pi)}) \leq \lambda \) for every cardinal \( \chi \) with \( \pi \leq \chi < \lambda \). Then \( \text{cof}(NS_{\kappa,\lambda}^{(\kappa,\pi)}) \leq \pi \cdot \text{cf}(\lambda) < \rho < \lambda \).

**Proof.**

(i) : If \( \nu^\pi < \lambda \) for every cardinal \( \nu < \lambda \) and every cardinal \( \rho < \theta \), then \( \lambda^{\theta} = \lambda \), and \( \pi = \sigma \) is as desired. Now suppose there are two cardinals \( \nu < \lambda \) and \( \rho < \theta \) such that \( \nu^\rho \geq \lambda \). Set \( \pi = \nu \cdot \sigma \). Let \( \chi \) be a cardinal with \( \pi \leq \chi < \lambda \). Then \( \chi^\theta = \lambda^{\theta} \), so by Proposition 6.6 \( \text{cof}(NS_{\kappa,\chi}^{(\kappa,\pi)}) \leq \chi \).

(ii) : By Lemma 1.9. \( \square \)

In particular, if \( \kappa \leq \text{cf}(\lambda) < \lambda \), then \( H_{\kappa,\lambda} \) holds just in case \( \text{cof}(NS_{\kappa,\tau}) < \lambda \) for every cardinal \( \tau \) with \( \kappa \leq \tau < \lambda \).
Suppose \( \lambda \) is a limit cardinal and \( \chi \) is a cardinal with \( \kappa \leq \chi \leq \lambda \). If either \( \text{cf}(\lambda) < \kappa \) or \( \text{cf}(\lambda) > \chi \), then by Lemmas 1.10 and 4.1
\[
\text{cof}(NS^\kappa_{\kappa,\lambda}) \leq \bigcup_{\pi \leq \tau < \lambda} \text{cof}(NS^\kappa_{\kappa,\tau}),
\]
where \( \pi \) equals \( \kappa \) if \( \chi = \lambda \), and \( \chi \) otherwise. We will now deal with the case when \( \kappa \leq \text{cf}(\lambda) \leq \chi \). The proof of the following is a modification of that of Proposition 6.1.

**Proposition 6.9** Let \( \chi \) be a cardinal such that \( \kappa \cdot \text{cf}(\lambda) \leq \chi \leq \lambda \). Set \( \pi = \kappa \) if \( \chi = \lambda \), and \( \pi = \chi \) otherwise. Then \( \text{cof}(NS^\kappa_{\kappa,\lambda}) \leq \text{cof}(NS^\kappa_{\kappa,\rho}) \) and \( \text{cof}(NS^\kappa_{\kappa,\lambda}) \leq \text{cof}(NS^\kappa_{\kappa,\rho}) \) where \( \rho = \bigcup_{\pi \leq \tau < \lambda} \text{cof}(NS^\kappa_{\kappa,\tau}) \).

**Proof.** We can assume that \( \text{cf}(\lambda) < \chi \) since otherwise the result is trivial. We show that \( \text{cof}(NS^\kappa_{\kappa,\lambda}) \leq \text{cof}(NS^\kappa_{\kappa,\rho}) \) and leave the proof of the other assertion to the reader. Put \( \mu = \text{cf}(\lambda) \) and pick an increasing sequence \( \langle \lambda_n : \eta < \mu \rangle \) of cardinals cofinal in \( \lambda \) so that \( \lambda_0 > \kappa \cdot \mu \), and \( \lambda_0 \geq \chi \) in case \( \chi < \lambda \). For \( \eta < \mu \), select a family \( G_\eta \) of functions from \( P_3(\chi \cap \lambda_\eta) \) to \( P_3(\lambda_\eta) \) so that \( |G_\eta| \leq \text{cof}(NS^\kappa_{\kappa,\lambda_\eta}) \) and for any \( H \in (NS^\kappa_{\kappa,\lambda_\eta})^* \), there is \( y \in P_3(G_\eta) \setminus \{\emptyset\} \) with \( \{b \in \bigcap_{\eta < \mu} C^\kappa_{\eta,\lambda_n} : b \cap \kappa \in \kappa \} \subseteq H \). Let \( \bigcup_{\eta < \mu} G_\eta = \{g_\xi : \xi < \rho\} \).

By Proposition 2.3 we may find a collection \( T \) of functions from \( \mu \) to \( P_\kappa(\rho) \) such that \( |T| = \text{cof}(NS^\mu_{\kappa,\rho}) \) and for any \( u : \mu \to P_\kappa(\rho) \), there is \( z \in P_\kappa(T) \) with the property that \( u(\eta) \subseteq \bigcup_{t \in z} t(\eta) \) for every \( \eta \in \mu \). For \( t \in T \), let \( D_t \) be the set of all \( a \in P_\kappa(\lambda) \) such that for any \( \eta \in a \cap \mu \) and any \( \xi \in t(\eta) \), \( a \cap \lambda_\xi \in C^\kappa_{g_\xi,\lambda_\xi} \). Note that \( D_t \in (NS^\kappa_{\kappa,\lambda})^* \).

Now fix \( f : P_3(\chi) \to P_3(\lambda) \). Given \( \eta < \mu \), define \( p_\eta : P_3(\chi \cap \lambda_\eta) \to P_3(\lambda_\eta) \) by \( p_\eta(v) = \{\xi\} \), where \( \xi = \) the least \( \sigma \) such that \( \eta \leq \sigma < \mu \) and \( f(v) \leq \lambda_\sigma \), and \( q_\eta : P_3(\chi \cap \lambda_\eta) \to P_3(\lambda_\eta) \) by \( q_\eta(v) = \lambda_\eta \cap f(v) \). Select \( x_\eta, y_\eta \in P_\kappa(\rho) \setminus \{\emptyset\} \) so that \( \{g_\xi : \xi \in x_\eta \cup y_\eta\} \subseteq G_\eta \), \( \{b \in \bigcap_{\xi \in x_\eta} C^\kappa_{g_\xi,\lambda_\xi} : b \cap \kappa \in \kappa \} \subseteq C^\kappa_{p_\eta,\lambda_\eta} \) and \( \{b \in \bigcap_{\xi \in y_\eta} C^\kappa_{g_\xi,\lambda_\xi} : b \cap \kappa \in \kappa \} \subseteq C^\kappa_{q_\eta,\lambda_\xi} \).

We may find \( z \in P_\kappa(T) \) such that \( x_\eta \cup y_\eta \subseteq \bigcup_{t \in z} t(\eta) \) for every \( \eta \in \mu \).
Let us show that $A \cap (\bigcap_{\xi \in z} D_\xi) \subseteq C^{\kappa,\lambda}_f$. Thus let $a \in A \cap (\bigcap_{\xi \in z} D_\xi)$ and $\nu \in P_\lambda(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $\nu \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in \bigcap_{\xi \in z} C^{\kappa,\lambda_\eta}_\xi$, so $\nu \cap f(\nu) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now $a \cap \lambda_\sigma \in \bigcap_{\xi \in y} C^{\kappa,\lambda_\sigma}_\xi$, and therefore $f(\nu) \subseteq a$. □

7 Nowhere precipitousness of $NS^{\nu}_{\kappa,\lambda}$

Throughout this section it is assumed that $\kappa \leq \text{cf}(\lambda) < \lambda$. Let $\nu$ be a cardinal with $\text{cf}(\lambda) \leq \nu < \lambda$. We will show that under certain conditions, $NS^{\nu}_{\kappa,\lambda}$ is nowhere precipitous. Our proof will follow that of Theorem 2.1 in [13], except that we do not appeal to pcf theory.

Set $\mu = \text{cf}(\lambda)$. We assume that $c(\kappa, \nu) < \lambda$ in case $\nu > \mu$. Let $\rho < \lambda$ be a regular cardinal such that $\rho > \mu$ if $\nu = \mu$, and $\rho > c(\kappa, \nu)$ otherwise.

Select a continuous, increasing sequence $< \lambda_\beta : \beta < \mu >$ of cardinals so that $\bigcup \lambda_\beta = \lambda$ and $\lambda_0 > \rho$. Let $E$ be the set of all limit ordinals $\alpha < \mu$ with $\text{cf}(\alpha) < \kappa$. We define $D$ as follows. If $\nu = \mu$, we set $D = E$. Otherwise we pick $D$ in $NS^*_\lambda$, so that $|D| = c(\kappa, \nu)$. For $d \in D$, put $\alpha(d) = \cup(d \cap \mu)$. Note that $\alpha(d) \in E$. Moreover $\alpha(d) = d$ in case $\nu = \mu$.

Let $B$ be the set of all $a \in P_\lambda(\lambda)$ such that (i) $0 \in a$, (ii) $\gamma + 1 \in a$ for every $\gamma \in a \cap \nu$, (iii) $a \cap \kappa \subseteq \{ \beta \in \mu : \lambda_\beta \in a \}$, and (iv) $a \cap \nu \in D$ in case $\nu > \mu$. Then clearly, $B \in (NS^*_\kappa,\lambda)^*$. For $d \in D$, define $W_d$ by letting $W_d = \{ a \in B : \cup(a \cap \mu) = d \}$ if $\nu = \mu$, and $W_d = \{ a \in B : a \cap \nu = d \}$ otherwise. Note that $\cup(a \cap \lambda_\alpha(d)) = \lambda_\alpha(d)$ for every $a \in W_d$.

**LEMMA 7.1.** Suppose there is $T \subseteq P_\kappa(\lambda)$ such that (a) $|T \cap P(a)| < \rho$ for any $a \in P_\kappa(\lambda)$, and (b) $u(\rho, \tau) \leq |T|$ for every cardinal $\tau$ with $\rho \leq \tau < \lambda$. Then for every $R \in (NS^*_\kappa,\lambda)^+$

$$\{ d \in D : \{ a \cap \lambda_\alpha(d) : a \in R \cap W_d \} \geq u(\rho, \lambda_\alpha(d)) \}$$

lies in $NS^*_\mu$ if $\nu = \mu$, and in $NS^*_\kappa,\nu$ otherwise.

**Proof.** For $\eta \in \mu$, select $Z_\eta \in J^*_\rho,\lambda_\eta$ with $|Z_\eta| \leq |T|$. Then clearly there is $Q \subseteq T$ with $|\bigcup_{\eta < \mu} Z_\eta| = |Q|$. Pick a bijection $i : \bigcup_{\eta < \mu} Z_\eta \to Q$ and let $j$ denote the inverse of $i$. For $\alpha \in E$, define $k_\alpha : P_\kappa(\lambda_\alpha) \to P_\rho(\lambda_\alpha)$ by $k_\alpha(b) = \bigcup_{\epsilon \in Q \cap P(b)} (\lambda_\alpha \cap j(\epsilon))$. 

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Claim. Let $S \in (NS^\nu_{\kappa, \lambda})^+$, then there is $d \in D$ such that

$$k_{\alpha(d)}''((a \cap \lambda_{\alpha(d)} : a \in S \cap W_d)) \in I_{\rho, \lambda_{\alpha(d)}}^+.$$ 

Proof of the claim. Assume otherwise. For $d \in D$, select $y_d \in P_\rho(\lambda_{\alpha(d)})$ so that $y_d \setminus k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) \neq \emptyset$ for every $a \in S \cap W_d$. Set $y = \bigcup_{d \in D} y_d$. Note that $y \in P_\rho(\lambda)$. For $\eta \in \mu$, pick $z_\eta \in Z_\eta$ so that $y \cap \lambda_\eta \subseteq z_\eta$. Now let $H$ be the set of all $a \in P_\rho(\lambda)$ such that $i(z_\eta) \in \bigcup_{\zeta \in \alpha \cap \mu} P(a \cap \lambda_\zeta)$ for every $\eta \in \alpha \cap \mu$. Since $H \in (NS^\nu_{\kappa, \lambda})^*$, we can find $a \in S \cap B \cap H$. Set $d = \cup (a \cap \mu)$ if $\nu = \mu$, and $d = a \cap \nu$ otherwise. Then $a \in W_d$ and

$$y_d \subseteq y \cap \lambda_{\alpha(d)} = \bigcup_{\eta \in \alpha \cap \mu} (y \cap \lambda_\eta) \subseteq \bigcup_{\eta \in \alpha \cap \mu} z_\eta = \bigcup_{\eta \in \alpha \cap \mu} j(i(z_\eta)) \subseteq k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}).$$

This contradiction completes the proof of the claim.

It is now easy to show that the conclusion of the lemma holds: Fix $R \in (NS^\nu_{\kappa, \lambda})^+$, and $A$ such that $A \in NS^\mu_\kappa$ if $\nu = \mu$, and $A \in NS^\mu_{\kappa, \nu}$ otherwise. Set $Y = \bigcup_{d \in D \cap A} W_d$. Since $Y \in (NS^\nu_{\kappa, \lambda})^*$, there must be some $d \in D$ such that

$$k_{\alpha(d)}''((a \cap \lambda_{\alpha(d)} : a \in (R \cap Y) \cap W_d)) \in I_{\rho, \lambda_{\alpha(d)}}^+.$$ 

Then clearly, $d \in A$ and $[(a \cap \lambda_{\alpha(d)} : a \in R \cap W_d)] \geq u(\rho, \lambda_{\alpha(d)}).$ \hfill $\square$

Concerning condition (a) in Lemma 7.1 let us observe the following:

**Lemma 7.2.** Let $T \subseteq P_\kappa(\lambda)$ be such that $| T \cap P(a) | \leq u(\kappa, \lambda)$ for any $a \in P_\kappa(\lambda)$. Then $| T | \leq u(\kappa, \lambda)$.

**Proof.** Suppose otherwise. Select $A \in I_{\kappa, \lambda}^+$ with $| A | = u(\kappa, \lambda)$. Define $g : T \to A$ so that $t \leq g(t)$ for all $t \in T$. Then $g$ must be constant on some $D \subseteq T$ with $| D | = | A |^\dagger$. Contradiction. \hfill $\square$

Consider for instance the following situation: In $V$, GCH holds, $\sigma$ is a strong cardinal with $\rho < \sigma < \lambda$, and $\pi$ a cardinal greater than $\lambda$. Then by a result of Gitik and Magidor [6], there is a cardinal preserving, $\sigma^+\text{-cc}$ forcing notion $P$ such that in $V^P$, (a) no new bounded subsets of $\sigma$ are added, (b) $\sigma$ changes its cofinality to $\omega$, and (c) $2^\sigma \geq \pi$. Now working in $V^P$, let $T = P_{\omega_1}(\sigma)$. Then clearly $| T \cap P(\sigma) | \leq 2^{[\sigma]} \leq \kappa$ for any $a \in P_\kappa(\lambda)$. Moreover for any two uncountable cardinals $\chi$ and $\tau$ with $\text{cf}(\chi) = \chi < \sigma \leq \tau \leq \pi$,

$$u(\chi, \tau) = 2^{<\chi} \cdot u(\chi, \tau) = \tau^{<\chi} = \sigma^{<\chi} = \sigma^{\text{cf}(\chi)} = | T |.$$ 

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Hence by Lemma 7.1, for any \( R \in (NS^{\kappa}_{\kappa, \lambda})^+ \),
\[
\{ d \in D : \{ a \cap \lambda^{(d)} : a \in R \cap W_d \} \geq u(\rho, \lambda^{(d)}) \}
\]
lies in \( NS^\mu_\kappa \) if \( \nu = \mu \), and in \( NS^{\kappa \cdot \nu}_{\kappa, \nu} \) otherwise. Note that for any cardinal \( \chi \) with \( \kappa \leq \chi \leq \sigma \), \( \text{cof}(NS^\chi_{\kappa, \lambda}) = u(\kappa, \lambda) \) since \( \text{cof}(NS^\chi_{\kappa, \lambda}) \leq (\lambda^{< \kappa})^\chi = (2^\sigma)^\chi = 2^{\sigma \cdot \chi} \), and moreover, by Lemma 1.9 and Proposition 3.6, \( \text{cof}(NS^\chi_{\kappa, \lambda}) > \lambda \) in case \( \mu \leq \chi \).

**Proposition 7.3.** Suppose there is \( T \subseteq P_\kappa(\lambda) \) and a cardinal \( \pi \) with \( \rho \leq \pi < \lambda \) such that (a) \( |T \cap P(\alpha)| < \rho \) for any \( \alpha \in P_\kappa(\lambda) \), and (b) \( \pi^\mu < u(\rho, \tau) \leq |T| \) for every cardinal \( \tau \) with \( \pi < \tau < \lambda \). Then \( NS^{\kappa \cdot \nu}_{\kappa, \lambda} \) is nowhere precipitous.

**Proof.** By Lemma 1.12 it suffices to show that II has a winning strategy in the game \( G(NS^{\kappa \cdot \nu}_{\kappa, \lambda}) \). We can assume without loss of generality that \( \lambda_0 > \pi \). For \( g : P_\kappa(\nu) \rightarrow P_\kappa(\lambda) \) and \( \alpha < \mu \), define \( \alpha : P_\kappa(\nu) \rightarrow P_\kappa(\lambda) \) by \( \alpha(\varepsilon) = \lambda_0 \cap g(\varepsilon) \).

**Claim 1.** Let \( g : P_\kappa(\nu) \rightarrow P_\kappa(\lambda) \). Then
\[
\{ d \in D : \{ a \cap \lambda^{(d)} : a \in W_d \cap \lambda^{(d)} \in C_{g^{(\alpha)}}^{\kappa, \lambda}(d) \} \subseteq C^{\kappa, \lambda}_g \}
\]
lies in \( (NS^\mu_\kappa, E_\kappa) \) if \( \nu = \mu \), and in \( NS^{\kappa \cdot \nu}_{\kappa, \nu} \) otherwise.

**Proof of Claim 1.** We prove the claim in the case when \( \nu > \mu \), and leave the proof in the case when \( \nu = \mu \) to the reader. Define \( h : P_\kappa(\nu) \rightarrow \mu \) by \( h(\varepsilon) = \lambda_0 \cap g(\varepsilon) \). Let \( Q \) be the set of all \( d \in D \) such that \( h(\varepsilon) \in \varepsilon \) for every \( e \in P_\kappa(d) \). Then clearly \( Q \in NS^{\kappa \cdot \nu}_{\kappa, \nu} \). Now fix \( d \in Q \) and \( a \in W_d \) such that \( a \cap \lambda^{(d)} \in C_{g^{(\alpha)}}^{\kappa, \lambda}(d) \). Let \( e \in P_\kappa(a \cap \nu) \). Then \( h(\varepsilon) \in d \), so \( g(\varepsilon) \subseteq \lambda^{(d)} \). It follows that \( g(\varepsilon) \subseteq a \), since \( \lambda^{(d)} \cap g(\varepsilon) \subseteq a \). Thus \( a \in C^{\kappa, \lambda}_g \). This completes the proof of Claim 1.

**Claim 2.** Let \( X \in (NS^{\kappa \cdot \nu}_{\kappa, \lambda})^+ \) and \( Y \subseteq B \). Suppose that
\[
\{ a \in Y \cap W_d : a \cap \lambda^{(d)} \in C_{\kappa}^{\kappa, \lambda}(d) \} \neq \emptyset
\]
whenever \( d \in D \) and \( k : P_\kappa(\nu) \rightarrow P_\kappa(\lambda^{(d)}) \) are such that
\[
|\{ a \cap \lambda^{(d)} : a \in A \cap W_d \} | \geq \lambda^{(d)}.
\]
Then \( Y \in (NS^{\kappa \cdot \nu}_{\kappa, \lambda})^+ \).

**Proof of Claim 2.** Fix \( g : P_\kappa(\nu) \rightarrow P_\kappa(\lambda) \). By Lemma 7.1 and Claim 1, there must be \( d \in D \) such that
\[
|\{ a \cap \lambda^{(d)} : a \in (X \cap C_{\kappa}) \cap W_d \} | \geq \lambda^{(d)}
\]
and
\[
\{ a \in W_d : a \cap \lambda^{(d)} \in C_{\kappa}^{\kappa, \lambda}(d) \} \subseteq C^{\kappa, \lambda}_g.
\]
Then
\[
\{ a \cap \lambda^{(d)} : a \in (X \cap C_{\kappa}) \cap W_d \} \neq \emptyset
\]
and
\[
\{ a \cap \lambda^{(d)} : a \in C_{\kappa}^{\kappa, \lambda}(d) \} \neq \emptyset
\]
and
\[
\{ a \cap \lambda^{(d)} : a \in (X \cap C_{\kappa}) \cap W_d \} \neq \emptyset
\]
and
\[
\{ a \cap \lambda^{(d)} : a \in C_{\kappa}^{\kappa, \lambda}(d) \} \neq \emptyset
\]
and
\[
\{ a \cap \lambda^{(d)} : a \in (X \cap C_{\kappa}) \cap W_d \} \neq \emptyset
\]
and
\[
\{ a \cap \lambda^{(d)} : a \in C_{\kappa}^{\kappa, \lambda}(d) \} \neq \emptyset
\]
and
\[
\{ a \cap \lambda^{(d)} : a \in (X \cap C_{\kappa}) \cap W_d \} \neq \emptyset
\]
For $d \in D$, consider the following two-person game $G_d$ consisting of $\omega$ moves, with player I making the first move: I and II alternately pick subsets of $W_d$, thus building a sequence $< X_n : n < \omega >$ subject to the following two conditions:

1. $X_0 \supseteq X_1 \supseteq \ldots$, and
2. $\{a \in X_{2n+1} : a \cap \lambda(d) \in C_{k}^{\kappa,\lambda(d)} \} \neq \emptyset$ for every $k : P_3(\nu) \rightarrow P_3(\lambda(d))$ such that

$$\{|{a \cap \lambda(d) : a \in X_{2n} \text{ and } a \cap \lambda(d) \in C_{k}^{\kappa,\lambda(d)}}| = \lambda^\nu$$

II wins the game if and only if $\bigcap_{n<\omega} X_n = \emptyset$.

**Claim 3.** Let $d \in D$. Then II has a winning strategy $\tau_d$ in the game $G_d$.

**Proof of Claim 3.** Let $X_0, X_1, X_2, \ldots$ be the successive moves of player I. For $n \in \omega$, let $K_n$ be the set of all $k : P_3(\nu) \rightarrow P_3(\lambda(d))$ such that

$$\{|{a \cap \lambda(d) : a \in X_{n} \text{ and } a \cap \lambda(d) \in C_{k}^{\kappa,\lambda(d)}}| = \lambda^\nu$$

Case 1 : $|K_n| = \lambda^\nu$ for every $n < \omega$.

Given $n < \omega$, set $K_n = \{k_{n,\xi} : \xi < \lambda^\nu \}$ and let

$$\tau_d(X_0, \ldots, X_n) = \{y_{n,\xi} : \xi < \lambda^\nu \},$$

where $y_{n,\xi} \in X_n$ and

$$y_{n,\xi} \cap \lambda(d) \in C_{k_{n,\xi}}^{\kappa,\lambda(d)} \setminus \{y_{q,\zeta} : q < n \text{ and } \zeta \leq \xi\}.$$

Case 2 : There is $n \in \omega$ such that $|K_n| < \lambda^\nu$.

Let $m$ be the least such $n$. Pick $A \subseteq X_m$ with $|A| < \lambda^\nu$ so that

$$\{a \in A : a \cap \lambda(d) \in C_{k}^{\kappa,\lambda(d)} \} \neq \emptyset$$

for every $k \in K_m$. Then set $\tau_d(X_0, \ldots, X_m) = A$ and $\tau_d(X_0, \ldots, X_m, X_{m+1}) = \emptyset$.

The proof of Claim 3 is now complete.

Finally, consider the strategy $\tau$ for player II in $G(NS_{k,\lambda}^\nu)$ defined by

$$\tau(X_0, \ldots, X_n) = \bigcup_{d \in D} \tau_d(X_0 \cap W_d, X_1 \cap W_d, \ldots, X_n \cap W_d).$$

Using Claims 2 and 3, it is easy to check that the strategy $\tau$ is a winning one. \[\square\]

**COROLLARY 7.4.**
(i) Suppose $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Then $NS_{\kappa,\lambda}^\mu$ is nowhere precipitous.

(ii) Suppose $\nu > \mu$, and $\tau^{c(\kappa,\nu)} < \lambda$ for every cardinal $\tau < \lambda$. Then $NS_{\kappa,\lambda}^\nu$ is nowhere precipitous.

(iii) Suppose $\mathcal{H}_{\kappa,\lambda}$ holds, and $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Then $NS_{\kappa,\lambda}^\mu$ is nowhere precipitous.

**Proof.** By Proposition 7.3 and Corollary 6.4 (i). □

**Question.** Suppose $\mathcal{H}_{\kappa,\lambda}$ holds and $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Does it then follow that $\lambda$ is a strong limit cardinal?

We conclude the section with the following remark. Suppose there exist $T$ and $\pi$ as in the statement of Proposition 7.3. Then either $\text{cof}(NS_{\kappa,\lambda}^\nu) = u(\kappa, \lambda)$, or $\lambda^{<\mu} = \lambda$. To establish this, note that $u(\kappa, \lambda) \leq \lambda^{<\kappa} \leq \lambda^{<\mu} \leq |T|$, so by Proposition 7.2 $|T| = u(\kappa, \lambda) = \lambda^{<\nu}$. It is now simple to see that $|T| = \lambda$ if $\tau^\nu < \lambda$ for every cardinal $\tau < \lambda$, and $|T| = \lambda^\nu$ otherwise.

**References**


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