LPM-97/03
hep-th/9712256

ON STRUCTURE CONSTANTS AND FUSION RULES IN THE $SL(2,\mathbb{C})/SU(2)$ WZNW MODEL

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Abstract. A closed formula for the structure constants in the SL(2,C)/SU(2) WZNW model is derived by a method previously used in Liouville theory. With the help of a reflection amplitude that follows from the structure constants one obtains a proposal for the fusion rules from canonical quantization. Taken together these pieces of information allow an unambigous definition of any genus zero n-point function.

1. Introduction

Little is known about conformal field theories that have continuous families of primary fields if they are not related to free field theories in a simple way. As the existence of continuous spectra of representations is connected with noncompactness of zero-mode configuration space, one may call such theories non-compact CFT's. But such theories promise to have a multitude of interesting applications and connections to other branches: Let me mention various quantum gravity models, connections with massive integrable theories via perturbed CFT, connections with integrable models such as (non-compact versions of) Gaudin, Calogero-Moser, Hitchin etc..

Non-compact conformal field theories are expected to be in many respects qualitatively different from the well studied rational or compact CFT's: The representations of the current algebra that the primary field correspond to will in general have no highest or lowest weight vector (so also no singular vectors). The fusion rules are therefore not determined algebraically but rather analytically. One expects that the operator product expansions of primary fields will involve integrals over continuous sets of operators. Non-vanishing of three point functions does not imply that any one of the three operators actually appears in the OPE of the other two.

The aim of the present work is to obtain some exact results for one of the simplest examples of a noncompact CFT, the WZNW model corresponding to the coset $H_3^+ = SL(2,\mathbb{C})/SU(2)$. It will become clear that the H_3^+ WZNW model itself is not a good physical theory: It is not unitary. There are two main reasons for studying it nevertheless:

One important physical motivation for studying the H_3^+ WZNW model comes from its relation to the euclidean black hole CFT, which is expected to be unitary (cf. discussion in [Ga]). The point is that the $\mathfrak{sl}_2$-current algebra symmetries are explicit in the H_3^+ WZNW model and therefore help to construct primary fields and correlation functions. Many of the results obtained here can be carried over to the euclidean black hole CFT. There is also an intimate relation to Liouville theory, which will be discussed further.

The second reason is that it seems to be (besides Liouville) one of the simplest noncompact CFT's to study. It is therefore a natural starting point for the developement of methods for the investigation of noncompact CFT's, just as the study of representation theory of $SL(2,\mathbb{C})$

WORK SUPPORTED BY THE EUROPEAN UNION UNDER CONTRACT FMRX-CT96-0012

and $SL(2,\mathbb{R})$ by Gelfand, Naimark and Bargmann was the starting point for the representation theory of noncomapct groups in general.

The present paper is part of a series of three papers on that subject. The first one [T1] studies the semiclassical or mini-superspace limit in detail. The theory essentially reduces to harmonic analysis on the symmetric space H_3^+, so a rather complete and rigorous treatment is possible in this limit. This nicely illustrates how the new qualitative features of the H_3^+ WZNW model compared to compact CFT's can be understood from the point of view of harmonic analysis.

The second one [T2] is devoted to certain mathematical aspects of the relevant current algebra representations and the construction of conformal blocks. It treats the cases of current algebra representations induced from $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$ and $SU(2)$ zero mode representations in a uniform manner and therefore lays some rigorous ground not only for the present work but also for a forthcoming treatment of the $SL(2,\mathbb{R})$ WZNW model.

The contents of the present paper may be summarized as follows: The second section is a scetch of the canonical quantization of the model. The current algebra symmetry is established and the stage is set for the later discussion of fusion rules. The spectrum of representations is discussed, based on the result of [Ga] on the partition function.

Since the relevant representations are neither highest nor lowest weight representations one needs a generalization of the bootstrap approach [BPZ] that will be introduced in the following third section. It may be considered as "affinization" of $SL(2,\mathbb{C})$ representation theory.

The following fourth section describes a derivation of the structure constants resp. three point functions. The method was previously used for Liouville theory in [TL]. It is based on the consideration of four point functions with one degenerate field which satsify additional differential equations. Assuming crossing symmetry for these four point functions leads to functional relations for the structure constants of three generic primary fields which have a unique solution for irrational central charge.

Finally, the fifth section is devoted to a discussion of the issue of fusion rules. The results of [Ga] suggests that any normalizable state can be expanded in terms of states from irreducible representations $\mathcal{P}_j$ of the current algebra, where $j = -1/2 + i\rho$, $\rho \in \mathbb{R}$ and the $\mathcal{P}_j$ are induced from principal series representations of $SL(2,\mathbb{C})$. The canonical normalization of states involves integration over zero modes. A simple condition for normalizability is found by considering the zero mode asymptotics of the state created by action of primary field Φ^{j_2} on a primary state Ψ^{j_1}. This defines a certain range of values for j_2, j_1 for which the fusion rules are simply given by expansion over all of the spectrum. The fusion rules for more general values of j_1, j_2 can be obtained by analytic continuation.

Acknowledgments

I would like to thank A.B. and Al.B. Zamolodchikov for valuable discussions. They independently found some crucial results such as the structure constants.

2. Canonical Quantization, Symmetries

The coset $H_3^+ \equiv SL(2,\mathbb{C})/SU(2)$ is the set of all hermitian two-by-two matrices with determinant one. A convenient global coordinate system for $H_3^+$is provided by the parametrization

$$h = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} \tag{1}$$

The starting point of the present discussion will be the following action

$$S = \frac{1}{\pi}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \big(\partial_+\phi\partial_-\phi + \partial_+ u\partial_-\bar{u}e^{-2\phi}\big), \tag{2}$$

where τ and σ denote the time and (periodic) space variables respectively. This action may be obtained [Ga] from a $SL(2,\mathbb{C})$-WZNW model by gauging the $SU(2)$ subgroup in the Lagrangian formalism. Anyway, action (2) defines a sigma-model with target H_3^+ that will be shown to possess conformal invariance via canonical quantization.

2.1. **Canonical quantization.** Introducing the canonical momenta as

$$\Pi_\phi = 2\pi\dot{\phi} \qquad \Pi_u = \pi(\dot{\bar{u}} - \bar{u}')e^{-2\phi} \qquad \Pi_{\bar{u}} = \pi(\dot{u} + u')e^{-2\phi} \tag{3}$$

leads to the following expression for the Hamiltonian

$$H_{cl} = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\Big(\tfrac{1}{2}(\Pi_\phi^2 + (\phi')^2) + 2\Pi_u\Pi_{\bar{u}}e^{2\phi} + \Pi_{\bar{u}}\bar{u}' - \Pi_u u'\Big) \tag{4}$$

The Hamiltonian system thereby defined is quantized by postulating canonical commutation relations at time $\tau = 0$:

$$\begin{aligned} &[u(\sigma), \Pi_u(\sigma')] = 2\pi i b^2\delta(\sigma - \sigma') \\ &[\bar{u}(\sigma), \Pi_{\bar{u}}(\sigma')] = 2\pi i b^2\delta(\sigma - \sigma') \end{aligned} \qquad [\phi(\sigma), \Pi_\phi(\sigma')] = 2\pi i b^2\delta(\sigma - \sigma'), \tag{5}$$

all other commutators vanishing. Planck's constant $\hbar$ was for later convenience written as $\hbar = b^2$. In the description of the quantum theory the rescaled fields $\varphi = b^{-1}\phi$, $\Pi_\varphi = b^{-1}\Pi_\Phi$, $v = b^{-1}u$, $\Pi_v = b^{-1}\Pi_u$ and $\bar{v} = b^{-1}\bar{u}$, $\Pi_{\bar{v}} = b^{-1}\Pi_{\bar{u}}$ will be used.

Introduce modes for φ by

$$\begin{aligned} \varphi(\sigma) &= q + i\sum_{n\neq 0}\frac{1}{n}\big(a_n e^{-in\sigma} + \bar{a}_n e^{in\sigma}\big) \\ \Pi_\varphi(\sigma) &= 2P + \sum_{n\neq 0}\big(a_n e^{-in\sigma} + \bar{a}_n e^{in\sigma}\big) \end{aligned} \tag{6}$$

and modes for v, $\bar{v}$ by

$$\begin{aligned} v(\sigma) &= v + i\sum_{n\neq 0}\frac{1}{n}v_n e^{in\sigma} & \bar{v}(\sigma) &= \bar{v} - i\sum_{n\neq 0}\frac{1}{n}\bar{v}_n e^{-in\sigma} \\ \Pi_v(\sigma) &= \Pi_{v,0} + \sum_{n\neq 0}\Pi_{v,n}e^{in\sigma} & \Pi_{\bar{v}}(\sigma) &= \Pi_{\bar{v},0} + \sum_{n\neq 0}\Pi_{\bar{v},n}e^{-in\sigma} \end{aligned} \tag{7}$$

The only nonvanishing commutators are

$$\begin{aligned} [P, q] &= -\tfrac{i}{2} \\ [\Pi_{v,0}, v] &= -i \\ [\Pi_{\bar{v},0}, \bar{v}] &= -i \end{aligned} \qquad \begin{aligned} [a_n, a_m] &= \tfrac{n}{2}\delta_{n,-m} \\ [\bar{a}_n, \bar{a}_m] &= \tfrac{n}{2}\delta_{n,-m} \end{aligned} \qquad \begin{aligned} [v_n, \Pi_{v,m}] &= n\delta_{n,-m} \\ [\bar{v}_n, \Pi_{\bar{v},m}] &= -n\delta_{n,-m} \end{aligned} \tag{8}$$

Let $\mathcal{F}$ be the Fock space generated from the Fock vacuum Ω defined by

$$\begin{aligned} a_n\Omega &= 0 & v_n\Omega &= 0 & \bar{v}_n\Omega &= 0 \\ \bar{a}_n\Omega &= 0 & \Pi_{v,n}\Omega &= 0 & \Pi_{\bar{v},n}\Omega &= 0 \end{aligned} \qquad \text{for } n > 0 \tag{9}$$

The quantum Hamiltonian, normal ordered corresponding to the choice of Fock vacuum then reads

$$
\begin{aligned}
H =& H_F + H_I \\
H_F =& 2(P^2 + ibP) + \sum_{k=1}^{\infty}\big(2a_{-k}a_k + 2\bar{a}_{-k}\bar{a}_k + \Pi_{v,-k}v_k + v_{-k}\Pi_{v,k} - \Pi_{\bar{v},-k}\bar{v}_k - \bar{v}_{-k}\Pi_{\bar{v},k}\big) \\
H_I =& 2\int_0^{2\pi}\frac{d\sigma}{2\pi}\Pi_v(\sigma)\Pi_{\bar{v}}(\sigma) :e^{2b\varphi}:
\end{aligned} \tag{10}
$$

The operator H will initially be considered to act in the space $\mathcal{S}$ of all

$$
\Psi = \sum_I \mathcal{A}_I\Omega\;\Psi^I(q,v,\bar{v}), \tag{11}
$$

where the $\mathcal{A}_I$ are monomials in the oscillators labelled by suitable multi-indices I, and the functions $\Psi^I(h) \in \mathcal{C}_c^{\infty}(H_3^+)$ are assumed to be nonzero only for finitely many I.

One has to note however the following problem: There does not seem to be a simple scalar product that makes the Hamiltonian symmetric. Instead one has an indefinite, but nondegenerate hermitian form $(.,.)$ w.r.t. which the Hamiltonian is symmetric: First define a hermitian form $(.,.)_{\mathcal{F}}$ on $\mathcal{F}$ by the following hermiticity relations for the oscillators:

$$
a_n^{\dagger} = \bar{a}_{-n} \qquad \Pi_{v,n}^{\dagger} = \Pi_{\bar{v},-n} \qquad v_n^{\dagger} = -\bar{v}_{-n} \tag{12}
$$

The hermitian form thereby defined is diagonalized by forming the linear combinations

$$
\begin{aligned}
e_n^{\pm} =& \tfrac{i}{2}\big(v_n + \bar{v}_n \mp (\Pi_{v,n} - \Pi_{\bar{v},n})\big) \\
f_n^{\pm} =& \tfrac{1}{2}\big(v_n - \bar{v}_n \pm (\Pi_{v,n} + \Pi_{\bar{v},n})\big)
\end{aligned}
\qquad b_n^+ = a_n + \bar{a}_n, \quad b_n^- = i(a_n - \bar{a}_n), \tag{13}
$$

which diagonalize algebra and hermiticity relations:

$$
\begin{aligned}
[e_n^{\pm}, e_m^{\pm}] &= \pm\, n\delta_{n+m} & (e_n^{\pm})^{\dagger} =& e_{-n}^{\pm} \\
[f_n^{\pm}, f_m^{\pm}] &= \pm\, n\delta_{n+m} & (f_n^{\pm})^{\dagger} =& f_{-n}^{\pm} \\
[b_n^{\pm}, b_m^{\pm}] &= \pm\, n\delta_{n+m} & (b_n^{\pm})^{\dagger} =& b_{-n}^{\pm}.
\end{aligned} \tag{14}
$$

It is easy to see that only the oscillators with superscript $(-)$ generate elements F of negative norm $(F,F)_{\mathcal{F}}$. The form $(.,.)_{\mathcal{F}}$ is then diagonal on monomials $\mathcal{B}_I$ in the oscillators $e_{-n}^{\pm}$, $f_{-n}^{\pm}$, $b_{-n}^{\pm}$, labelled by a suitable multi-index I:

$$
(\mathcal{B}_I\Omega, \mathcal{B}_J\Omega)_{\mathcal{F}} = \delta_{I,J}\mathcal{N}_I. \tag{15}
$$

The hermitian form $(.,.)$ is then defined on $\mathcal{S}$ by

$$
(\Psi_2, \Psi_1) = \sum_I \mathcal{N}_I\big(\Psi_2^I, \Psi_1^I\big)_{H_3^+}, \tag{16}
$$

where $(.,.)_{H_3^+}$ is the $SL(2,\mathbb{C})$-invariant scalar product on H_3^+,

$$
(\Psi_2^I, \Psi_1^I)_{H_3^+} = \int_{\mathbb{R}} dq e^{-2bq} \int_{\mathbb{C}} d^2v\;\big(\Psi_2^I(q,v,\bar{v})\big)^*\Psi_1^I(q,v,\bar{v}) \tag{17}
$$

Even though (Ψ,Ψ) is clearly indefinite, there is a canonical norm $\|.\|$ associated to it:

$$
\|\Psi\|^2 = \sum_I |\mathcal{N}_I|\,\|\Psi^I(h)\|^2_{H_3^+}. \tag{18}
$$

Note that $\|\Psi\|^2 \neq (\Psi,\Psi)$ due to taking the absolute value of $(\mathcal{A}_I\Omega, \mathcal{A}_I\Omega)_{\mathcal{F}}$.

One may then finally define the space of states $\mathcal{V}$ as the completion of $\mathcal{S}$ w.r.t. $\|.\|$

2.2. **Current algebra.** In terms of the canonical variables one may construct the currents

$$\begin{aligned} J^-(\sigma) =& i\Pi_{\bar{v}}(\sigma) \\ J^0(\sigma) =& + i : \bar{v}(\sigma)\Pi_{\bar{v}}(\sigma) : + i\tfrac{b^{-1}}{2}\big(\Pi_\varphi(\sigma) + \varphi'(\sigma)\big) \\ J^+(\sigma) =& - i\Big(k\bar{v}'(\sigma) + : (\bar{v}(\sigma))^2\Pi_{\bar{v}}(\sigma) : + b^{-1}\bar{v}(\sigma)\big(\Pi_\varphi(\sigma) + \varphi'(\sigma)\big) + \Pi_v(\sigma) : e^{2b\varphi(\sigma)} :\Big), \end{aligned} \tag{19}$$

as well as currents $\bar{J}^a(\sigma)$ obtained by hermitian conjugation. The two crucial properties satisfied by these definitions are:

(1) *The currents have the following commutation relations with H:*

$$[H, J^a(\sigma)] = -i\partial_\sigma J^a(\sigma) \qquad [H, \bar{J}^a(\sigma)] = i\partial_\sigma \bar{J}^a(\sigma). \tag{20}$$

From this it follows that the equations of motion of these observables are simply

$$\partial_- J^a(\sigma,\tau) = 0 \qquad \partial_+ \bar{J}^a(\sigma,\tau) = 0 \tag{21}$$

(2) *These currents generate a $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ current algebra with central charge k related to b by $b^{-2} = -(k+2)$: The modes J^a_n of $J^a(\sigma)$, $a = -,0,+$ defined by $J^a(\sigma) = \sum_n e^{-in\sigma} J^a_n$ satisfy*

$$\begin{aligned} [J^0_n, J^0_m] &= \tfrac{k}{2} n\delta_{n+m,0} \\ [J^0_n, J^\pm_m] &= \pm J^\pm_{n+m} \end{aligned} \qquad [J^+_n, J^-_m] = 2J^0_{n+m} + kn\delta_{n+m,0}, \tag{22}$$

the modes $\bar{J}^a_n$ of $\bar{J}^a(\sigma)$ commute with the J^a_n and satisfy the same algebra.

The verification of these assertions may be simplified by observing that the currents J^a_F obtained from the expressions (19) by dropping the terms containing $e^{2b\varphi}$ are similiar to the standard free field constuctions [Wa][BF][FF][BO] of $\mathfrak{sl}_2$ current algebras. These free field currents J^a_F may be deformed as

$$\begin{aligned} J^+_\mu(\sigma) =& J^+_F(\sigma) + \mu\Pi_v(\sigma) : e^{2b\varphi(\sigma)} : \\ \bar{J}^+_\mu(\sigma) =& \bar{J}^+_F(\sigma) + \mu\Pi_{\bar{v}}(\sigma) : e^{2b\varphi(\sigma)} :, \end{aligned} \tag{23}$$

without changing the algebra. The check that the modes of $J^a_\mu(\sigma)$ commute with those of $\bar{J}^a_\mu(\sigma)$ essentially boils down to the fact that the deformation of $J^-_F(\sigma)$ is the screening charge for the algebra generated by $\bar{J}^-_F(\sigma)$ and vice versa.

One should note however an important difference between the free field representation that appears here and the usual free field representation of $\mathfrak{sl}_2$ current algebra: The definition of the Fock vacuum Ω does not involve the condition $\Pi_{v,0}\Omega = 0$, $\Pi_{\bar{v},0}\Omega = 0$, which is usually imposed to get highest weight representations of the current algebra. Not imposing these conditions is due to the fact that neither v nor $\Pi_{v,0}$ take values in compact sets so will get continuous spectra upon quantization. One will therefore have to deal with current algebra representations that are neither of highest nor lowest weight type.

Associated with the current algebras one has two commuting Virasoro algebras by the Sugawara construction: In terms of the canonical variables the standard Sugawara energy-momentum tensors express as

$$T_\mu(\sigma) = + : \Pi_{\bar{v}}\bar{v}' : + \tfrac{1}{4} : \big(\Pi_\varphi + \varphi'\big)^2 : - \tfrac{b}{2} e^{i\sigma}\partial_\sigma e^{-i\sigma}\big(\Pi_\varphi + \varphi'\big) + \mu\Pi_{\bar{v}}\Pi_v : e^{2b\varphi} : \tag{24}$$

and the corresponding expression for $\bar{T}_\mu$. The operator

$$H_\mu = \int_0^{2\pi} \frac{d\sigma}{2\pi}\big(T_\mu(\sigma) + \bar{T}_\mu(\sigma)\big)$$

satisfies

$$[H_\mu, J^a_\mu(\sigma)] = -i\partial_\sigma J^a_\mu(\sigma) \qquad [H_\mu, \bar{J}^a_\mu(\sigma)] = i\partial_\sigma \bar{J}^a_\mu(\sigma) \tag{25}$$

for any value of μ. By taking the value $\mu = 1$ one recovers the canonical Hamiltonian of the H_3^+-WZNW model.

2.3. **The Spectrum.** The decompostion of the space of states into *irreducible* representations of the current algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ was found in [Ga] by explicitely evaluating the path integral on the torus. The result was then shown to agree with the definition of the partition function via

$$\mathcal{Z}(q,u) = |q|^{-\frac{c_k}{12}} \operatorname{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi i (uJ_0^0 - \bar{u}\bar{J}_0^0)} \tag{26}$$

if the trace is performed over the space $\mathcal{H}$ spanned by $L^2(H_3^+)$ and the descendants obtained by repeated action of the J^a_n, $\bar{J}^a_n$ with $n < 0$. If $\mathfrak{n}_-$ is the Lie subalgebra spanned by the J^a_n, $\bar{J}^a_n$ with $n < 0$ and $\mathrm{Sym}(\mathfrak{n}_-)$ the corresponding symmetric algebra one may write the definition of $\mathcal{H}$ formally as

$$\mathcal{H} = \mathrm{Sym}(\mathfrak{n}_-) \otimes L^2(H_3^+) \tag{27}$$

In order to introduce an action of the current algebra in $\mathcal{H}$ note that any representation V of a finite dimensional Lie algebra $\mathfrak{g}$ (generators J^a) can be extended to a representation $\mathcal{P}(V, \mathfrak{g}, k)$ of the affine algebra $\hat{\mathfrak{g}}$ (generators J^a_n) corresponding to $\mathfrak{g}$ by demanding $J^a_n V = 0$, $n > 0$ and by extending V by the linear span of expressions of the form

$$J^{a_1}_{-n_1} \dots J^{a_k}_{-n_k} v \quad \text{for any} \quad v \in V.$$

These kind of current algebra representations have been named prolongation modules in [LZ]. To write the definition of $\mathcal{P}(V, \mathfrak{g}, k)$ more precisely, introduce the Lie subalgebra $\mathfrak{b}_+$ spanned by the J^a_n, $\bar{J}^a_n$ with $n \geq 0$. One then has

$$\mathcal{P}(V, \mathfrak{g}, k) = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} V \tag{28}$$

In the present case one may choose V to be a suitable dense subspace $\mathcal{S}(H_3^+)$ of $L^2(H_3^+)$, which is invariant under the action of the zero mode subalgebra $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ by the differential operators

$$\begin{aligned} J^+ &= - e^{2\varphi} \frac{\partial}{\partial v} - \bar{v}^2 \frac{\partial}{\partial \bar{v}} - \bar{v} \frac{\partial}{\partial \varphi} & \quad J^0 &= \bar{v} \frac{\partial}{\partial \bar{v}} + \frac{1}{2} \frac{\partial}{\partial \varphi} & \quad J^- &= \frac{\partial}{\partial \bar{v}} \\ \bar{J}^+ &= - e^{2\varphi} \frac{\partial}{\partial \bar{v}} - v^2 \frac{\partial}{\partial v} - v \frac{\partial}{\partial \varphi} & \quad \bar{J}^0 &= v \frac{\partial}{\partial v} + \frac{1}{2} \frac{\partial}{\partial \varphi} & \quad \bar{J}^- &= \frac{\partial}{\partial v}. \end{aligned} \tag{29}$$

The corresponding prolongation module

$$\hat{\mathcal{S}}(H_3^+) \equiv \mathcal{P}\big(\mathcal{S}(H_3^+), \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, k\big) \tag{30}$$

is then a dense subspace of $\mathcal{H}$ on which the current algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2 \oplus \widehat{\mathfrak{sl}}_2$ is represented.

The decomposition of $L_2(H_3^+)$ into irreducible representations [GGV] (see also discussion in [T1]) reads

$$\mathcal{H} \equiv L^2(H_3^+, dh) = \int_{\rho > 0}^{\oplus} d\rho \rho^2 \ \mathcal{H}_{-\frac{1}{2} + i\rho}, \tag{31}$$

where $\mathcal{H}_j$ is a representation of the princial series of $SL(2, \mathbb{C})$. This decomposition then induces a corresponding decomposition of $\hat{\mathcal{S}}(H_3^+)$ into current-algebra representations

$$\mathcal{P}_j \equiv \mathcal{P}\big(\mathcal{H}_j, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, k\big), \tag{32}$$

which are irreducible by the results in [T2]. It is important to note that although one has chiral factorization on the level of the algebra, one has no factorization of the representation $\mathcal{P}_j$ in the form $\mathcal{R}_j \otimes \bar{\mathcal{R}}_j$, where $\mathcal{R}_j$ (resp. $\bar{\mathcal{R}}_j$) are irreducible representations of the current algebras generated by J^a_n (resp. $\bar{J}^a_n$).

By the Sugawara construction one then also gets an action of two commuting Virasoro algebras (generators L_n, $\bar{L}_n$) on $\hat{\mathcal{S}}(H_3^+)$ resp. $\mathcal{P}_j$. States $\Psi \in \mathcal{S}_3^+$ satisfy

$$L_0\Psi = -t^{-1}Q\Psi, \quad \bar{L}_0\Psi = -t^{-1}Q\Psi, \quad \text{and } L_n\Psi = 0, \quad \bar{L}_n\Psi = 0 \quad \text{for each } n > 0, \tag{33}$$

where Q is the Laplacian on H_3^+, $t = -(k+2)$. They will also be called lowest level or *primary* states. Primary states $\Psi_j \in \mathcal{P}_j$ satisfy $L_0\Psi = h_j\Psi$ with $h_j = t^{-1}j(j+1)$.

Let me finish this section by noting that the representations $\mathcal{P}_j$ are equivalent to a representation $\mathcal{F}_j$ of the current algebra $\hat{\mathfrak{g}}$ in the space $\mathcal{F} \otimes \mathcal{S}(\mathbb{C})$ by means of the modes I^a_n of

$$\begin{aligned} I^-(\sigma) =& i\Pi_{\bar{v}}(\sigma) \\ I^0(\sigma) =& + i : \bar{v}(\sigma)\Pi_{\bar{v}}(\sigma) : + i\tfrac{b^{-1}}{2}\big(\Pi_\varphi(\sigma) + \varphi'(\sigma)\big) \\ I^+(\sigma) =& - i\Big(k\bar{v}'(\sigma) + : (\bar{v}(\sigma))^2\Pi_{\bar{v}}(\sigma) : + b^{-1}\bar{v}(\sigma)\big(\Pi_\varphi(\sigma) + \varphi'(\sigma)\big)\Big), \end{aligned} \tag{34}$$

as well as their "antiholomorphic" counterparts $\bar{I}^a_n$ defined analogously. The generators v_n, $\bar{v}_n$, $\Pi_{v,n}$, $\Pi_{\bar{v},n}$, a_n, $\bar{a}_n$ as well as the definition of the Fock-space $\mathcal{F}$ are as in section 2.1. The zero mode generator P acts by mutliplication with ibj.

3. The bootstrap

The construction and calculation of correlation functions is not easy to achieve directly in the framework of canonical quantization since no explicit construction of primary states or fields is known.

Instead it has in the case of RCFT turned out to be extremely useful to exploit as much as possible the Ward identities from the current algebra symmetries by following a strategy similar to that introduced in [BPZ]. The aim of the present section will be to generalize the usual formalism to the case where one is no longer dealing with lowest- or highest weight representations.

3.1. **Primary and secondary fields.**

Primary fields $\Phi[f|z)$ can be associated to each vector in the zero mode representation, here sufficiently differentiable functions f on H_3^+. The vector f to which $\Phi[f|z)$ corresponds is recovered by the usual prescription for operator-state correpondence:

$$\lim_{z\to 0}\Phi[f|z) = f \tag{35}$$

They transform under the current algebra in a particularly simple way:

$$[J^a_n, \Phi[f|z)] = z^n\Phi[J^a_0 f|z) \qquad [\bar{J}^a_n, \Phi[f|\sigma)] = \bar{z}^n\Phi[\bar{J}^a_0 f|z), \tag{36}$$

where the action of J^a_0, $\bar{J}^a_0$ on f is by (29).

A convenient plane-wave normalizable basis for $L^2(H_3^+)$ was in [GGV], see also [T1], shown to be given by the functions

$$\Psi(j;x|h) = \frac{2j+1}{\pi}\left((1,x)\cdot h\cdot\begin{pmatrix}1\\ \bar{x}\end{pmatrix}\right)^{2j} \tag{37}$$

The corresponding primary fields $\Phi[\Psi(j;x|.)|z)$ will be denoted $\Phi^j(x|z)$. The transformation law (36) may then be reformulated as the following OPE

$$J^a(z)\Phi^j(x|w) = \frac{1}{z-w}\mathcal{D}^a_j\Phi^j(x|w), \qquad \bar{J}^a(\bar{z})\Phi^j(x|w) = \frac{1}{\bar{z}-\bar{w}}\bar{\mathcal{D}}^a_j\Phi^j(x|w) \tag{38}$$

where the differential operators $\mathcal{D}^a_j$ representing $\mathfrak{sl}_2$ are

$$\mathcal{D}^+_j = -\bar{x}^2\partial_{\bar{x}} + 2j\bar{x} \qquad \mathcal{D}^0_j = -\bar{x}\partial_{\bar{x}} - j \qquad \mathcal{D}^-_j = \partial_{\bar{x}}, \tag{39}$$

the $\bar{\mathcal{D}}^a_j$ their complex conjugates.

The representations with spin j and $-j-1$ are equivalent for $j \notin \mathbb{Z}$. This implies that the operators $\Phi^j(x|z)$ and $\Phi^{-j-1}(x|z)$ must be related by a relation of the form

$$\Phi^{-j-1}(x|z) = -R(j)\frac{2j+1}{\pi}\int_{\mathbb{C}} d^2x' \; |x-x'|^{-4j-4}\Phi^j(x'|z), \tag{40}$$

thereby defining a "reflection" amplitude $R(j)$ not restricted by current algebra symmetry.

Descendant (secondary) fields will be defined for each monomial $\mathcal{J}_{I_L}\bar{\mathcal{J}}_{I_R}$ where $\mathcal{J}_I \equiv J^{a_1}_{-n_1}\dots J^{a_k}_{-n_k}$ for multi-index $I = (n_1,\dots,n_i,\dots,n_k;a_1,\dots,a_i,\dots,a_k)$, $n_i < 0$, and correspondingly for $\bar{\mathcal{J}}_I$:

$$\begin{aligned}\big[\mathcal{J}_{I_L}\bar{\mathcal{J}}_{I_R}\Phi\big](x|z) \equiv \prod_i \frac{1}{(n_i-1)!} :&\big((\partial_z)^{n_1-1}J^{a_1}(z)\big)\dots\big((\partial_z)^{n_k-1}J^{a_k}(z)\big)\\ &\big((\partial_{\bar{z}})^{\bar{n}_1-1}J^{\bar{a}_1}(\bar{z})\big)\dots\big((\partial_{\bar{z}})^{\bar{n}_{\bar{k}}-1}J^{\bar{a}_{\bar{k}}}(\bar{z})\big)\Phi^j(x|z):\end{aligned} \tag{41}$$

3.2. **Correlation functions.** The assumption of invariance of correlation functions under the symmetries generated by J^a_0, $\bar{J}^a_0$, L_n, $\bar{L}_n$, $n=-1,0,1$ determines two and three-point functions up to certain functions of the j_i:

$$\begin{aligned}<\Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1)> =& N(j_1)|z_1-z_2|^{4h_1}\delta^{(2)}(x_1-x_2)\delta(j_1,-j_2-1)\\ &+B(j_1)|z_1-z_2|^{4h_1}|x_1-x_2|^{4j_1}\delta(j_1,j_2)\\ <\Phi^{j_3}(x_3|z_3)\Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1)>=&\\ &|x_1-x_2|^{2(j_1+j_2-j_3)}|x_1-x_3|^{2(j_1+j_3-j_2)}|x_2-x_3|^{2(j_2+j_3-j_1)}\times\\ &|z_1-z_2|^{2(h_3-h_1-h_2)}|z_1-z_3|^{2(h_2-h_1-h_3)}|z_2-z_3|^{2(h_1-h_2-h_3)}C(j_1,j_2,j_3),\end{aligned} \tag{42}$$

where $h_i = h(j_i)$, $i=1,2,3$. The two terms in the two point function again arise due to the equivalence of representations with spin j and $-j-1$. I will assume the operators to be normalized by

$$N(j) \equiv 1 \qquad \text{such that} \qquad R(j) = B(j) \tag{43}$$

Furthermore I will assume the $C(j_1,j_2,j_3)$ to be symmetric in its variables as is necessary for the primary fields to be mutually local.

Correlation functions of descendant fields may as usually be reduced to those of primary fields by using (41) and the OPE (38).

In order to use the current algebra symmetries to get information on n-point functions for $n > 3$ one postulates operator product expansions of the form

$$\begin{aligned}\Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) = &\int_{\mathcal{S}} d\mu(j)\ |z_2 - z_1|^{2(h_j - h_2 - h_1)} \\ &\times \sum_{n,\bar{n}=0}^{\infty} (z_2 - z_1)^n (z_2 - z_1)^{\bar{n}} \mathcal{O}^j_{n\bar{n}}\left[\begin{smallmatrix} j_2 & j_1 \\ x_2 & x_1\end{smallmatrix}\right](z_1).\end{aligned} \tag{44}$$

The measure $d\mu(j)$ was introduced to simplify notation as one will in general have to sum over discrete as well as continuous sets of j.

Requiring that both sides of (44) transform the same way under the current algebra allows to express the operators $\mathcal{O}^j_{n\bar{n}}$ as linear combinations of $\Phi^j(x|z_1)$ and its descendants [T2]:

$$\mathcal{O}^j_{n\bar{n}}\left[\begin{smallmatrix} j_2 & j_1 \\ x_2 & x_1\end{smallmatrix}\right](z_1) = \sum_{I_L \in \mathbf{I}_n} \sum_{I_R \in \mathbf{I}_{\bar{n}}} \int_{\mathbb{C}} d^2x\, \mathcal{C}_{I_L I_R}\left(\begin{smallmatrix} j & j_2 & j_1 \\ x & x_2 & x_1\end{smallmatrix}\right) \left[\mathcal{J}_{I_L} \bar{\mathcal{J}}_{I_R} \Phi\right](x|z_1), \tag{45}$$

where the set $\mathbf{I}_n$ contains all multi-indices $I = (n_1, \dots, n_k; a_1, \dots, a_k)$ such that $n = \sum_i n_i$ and the coefficients $\mathcal{C}_{I_L I_R}$ are uniquely defined in terms of $\mathcal{C}_{0,0}$, which reads

$$\begin{aligned}\mathcal{C}_{0,0}\left(\begin{smallmatrix} j & j_2 & j_1 \\ x & x_2 & x_1\end{smallmatrix}\right) =& D(j; j_2, j_1) C\left(\begin{smallmatrix} -j-1 & j_2 & j_1 \\ x & x_2 & x_1\end{smallmatrix}\right) \\ \equiv& D(j; j_2, j_1)|x_1 - x_2|^{2(j_1+j_2+j+1)}|x_1 - x_3|^{2(j_1-j_2-j-1)}|x_2 - x_3|^{2(j_2-j_1-j-1)}\end{aligned} \tag{46}$$

The only remaining freedom is given by the structure constants $D(j; j_2, j_1)$. In order to define them uniquely, one has to assume that only one of the two linearly dependent operators $\Phi^j(x|z_1)$ and $\Phi^{-j-1}(x|z_1)$ appears in (44). One might i.e. take the integration region $\mathcal{S}$ in (44) as subset of $\left\{j \in \mathbb{C};\ \arg(2j+1) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$. By using (44) in a three point function one then finds that

$$D(j; j_2, j_1) = C(-j-1, j_2, j_1). \tag{47}$$

Four point functions may then be expanded in terms of three point functions by i.e. using the OPE (44) of operators $\Phi^{j_2}(x_2|z_2)$ and $\Phi^{j_1}(x_1|z_1)$. One arrives at a representation of the four point function in the form

$$\begin{aligned}&< \Phi^{j_4}(x_4|z_4) \dots \Phi^{j_1}(x_1|z_1) >= \\ &= \int_{\mathcal{S}_s} d\mu(j_{21})\quad C(j_4, j_3, j_{21}) D(j_{21}; j_2, j_1)\quad \left|\mathcal{F}_{s,j_{21}}\left[\begin{smallmatrix} j_4 & j_3 & j_2 & j_1 \\ x_4 & x_3 & x_2 & x_1\end{smallmatrix}\right](z_4, \dots z_1)\right|^2\end{aligned} \tag{48}$$

This representation splits the information involved in the definition of the four point function into a piece determined directly by the current algebra symmetries (the conformal blocks $\mathcal{F}_{s,j_{21}}$, the subscript s refers to the "s-channel") and two pieces of information that one should expect to be determined in terms of the conformal blocks only rather indirectly: The structure constants $C(j_3, j_2, j_1)$ and the set $\mathcal{S}_s$ of intermediate representations. The latter is of course equivalent to knowledge of the fusion rules, i.e. the set of representations appearing in operator product expansions.

3.3. **Crossing symmetry.** An alternative representation of the four point function is obtained by using the OPE (44) of operators $\Phi^{j_3}(x_3|z_3)$ and $\Phi^{j_2}(x_2|z_2)$ to get an expansion in terms of "t-channel" conformal blocks:

$$\begin{aligned}&< \Phi^{j_4}(x_4|z_4) \dots \Phi^{j_1}(x_1|z_1) >= \\ &= \int_{\mathcal{S}_t} d\mu(j_{32})\quad C(j_4, j_1, j_{32}) D(j_{32}; j_3, j_2)\quad \left|\mathcal{F}_{t,j_{32}}\left[\begin{smallmatrix} j_4 & j_3 & j_2 & j_1 \\ x_4 & x_3 & x_2 & x_1\end{smallmatrix}\right](z_4, \dots z_1)\right|^2\end{aligned} \tag{49}$$

A fundamental physical requirement, which is equivalent to mutual locality of primary fields, is that the expansions (48) and (49) into s-channel and t-channel conformal blocks produce the same correlation functions (crossing symmetry). One may hope to infer from the equality of the two decompositions (48), (49) the existence of fusion relations by an argument similar to that given for RCFT in [MS]:

$$\mathcal{F}_{s,j_{21}}\begin{bmatrix} j_4\, j_3\, j_2\, j_1 \\ x_4 x_3 x_2 x_1 \end{bmatrix}(z_4,\dots,z_1) = \int d\mu(j_{32})\;\; F_{j_{21}j_{32}}\begin{bmatrix} j_3 j_2 \\ j_4 j_1 \end{bmatrix} \mathcal{F}_{t,j_{32}}\begin{bmatrix} j_4\, j_2\, j_3\, j_1 \\ x_4 x_2 x_3 x_1 \end{bmatrix}(z_4,\dots,z_1) \tag{50}$$

Indeed, in order to make up an argument of the type given in [MS] one only needs *existence* of an extension of the set of conformal blocks $\mathcal{F}_{t,j_{32}}$ to a basis, with respect to which the $\mathcal{F}_{s,j_{21}}$ can be expanded. Let me note that existence of fusion relations in the mini-superspace limit was shown in [T1].

Given fusion relations (50) the requirement of crossing symmetry translates itself into a system of equations for the structure constants:

$$\begin{aligned} \int_{\mathcal{S}_s} d\mu(j_{21})\; F_{j_{21}j_{32}}\begin{bmatrix} j_3 j_2 \\ j_4 j_1 \end{bmatrix} \bar{F}_{j_{21}j'_{32}}\begin{bmatrix} j_3 j_2 \\ j_4 j_1 \end{bmatrix} \; C(j_4, j_3, j_{21}) D(j_{21}; j_2, j_1) \\ = \delta(j_{32}, j'_{32})\; C(j_4, j_{32}, j_1) C(j_{32}; j_3, j_2). \end{aligned} \tag{51}$$

Viewing the fusion transformations as being given by the conformal blocks, therefore indirectly from the current algebra symmetries, one should read (51) as possible starting point for the determination of the structure constants.

4. Structure constants

The aim of the present section will be to derive an explicit expression for the structure constants $C(j_3, j_2, j_1)$ that appear in the expansion (48) for the four point function of four arbitrary primary fields.

4.1. **Degenerate fields.** The representations $\mathcal{P}_j$ are irreducible for generic j. They become degenerate if and only if j equals any of the $j_{r,s}$, where

$$2j_{r,s} + 1 = r - st \text{ where } \begin{cases} \text{either} & r \geq 1, s \geq 0 \\ \text{or} & r < -1, s < 0 \end{cases} \tag{52}$$

In a formal sense the degeneracy may be seen to arise due to the existence of null vectors. However, these are not found among the normalizable vectors of the representations $\mathcal{P}_j$ but are distributional objects instead, cf. [T2].

Correspondingly there exist fields that satisfy additional differential equations [T2]. These fields will be called degenerate fields in the following and denoted $\Phi_{r,s}$. I will need only the following two simple examples of degenerate fields, corresponding to $j = j_{2,1} = 1/2$ and $j_{1,2} = -t/2$ respectively.

The degenerate field $\Phi_{2,1}(x|z)$ satisfies

$$\partial_x^2 \Phi_{2,1}(x|z) = 0 \qquad \partial_{\bar{x}}^2 \Phi_{2,1}(x|z) = 0. \tag{53}$$

It transforms in the finite dimensional spin 1/2 representation of $SL(2,\mathbb{C})$, and is therefore identified with the quantum analogue of the fundamental field h.

In the other case $\Phi_{1,2}(x|z)$ the differential equation expressing degeneracy reads

$$:\Big(J^+(x|z)\partial_x^2 - 2(1+t)J^0(x|z)\partial_x - t(1+t)J^-(x|z)\Big)\Phi_{1,2}(x|z): \;=\; 0, \tag{54}$$

where $J^a(x|z) = e^{xJ_0^-} J^a(z) e^{-xJ_0^-}$.

4.2. **The method, assumptions.** The method to be used will consist of considering four point functions in which one of the operators is a degenerate field $\Phi_{r,s}$, the others generic. The main assumption will be that these four point functions are crossing symmetric. More precisely the assumption is that the conditions (51) for crossing symmetry with four generic fields are compatible with those considered here.

The considered four point functions will satisfy degeneracy equations in addition to the KZ equations. The set of conformal blocks that solves both equations will be finite. Moreover one has fusion and braiding relations relating different bases for the conformal blocks corresponding to the different possible factorization patterns.

The requirement of crossing symmetry thereby takes the form

$$\sum_{s\in\mathbf{F}_{r,s}} C(j_4,j_3,j_1+s)D(j_1+s;j_2,j_1)\; F_{st}\Big[\begin{smallmatrix} j_3 j_2\\ j_4 j_1\end{smallmatrix}\Big]\; F_{st'}\Big[\begin{smallmatrix} j_3 j_2\\ j_4 j_1\end{smallmatrix}\Big] = \delta_{t,t'}\; C(j_4,j_1,j_3+t)D(j_3+t;j_3,j_2). \tag{55}$$

For both cases $\Phi_{2,1}$ and $\Phi_{1,2}$ it will be possible to calculate conformal blocks and fusion matrices explicitely. Given matrices F_{st}, the equations (55) are finite difference equations for the unknown $C(j_3,j_2,j_1)$. It will be shown that a solution to both the equations from $j_2 = 1/2$ and $j_2 = -t/2$ exists and is unique when t is irrational.

4.3. **Differential equations for the conformal blocks.** It was shown in [T2] that the conformal blocks of any collection of primary fields satisfy a generalization of the KZ equation previously introduced in [FZ]. In terms of the cross-ratios x, z the KZ equation takes the form

$$\begin{aligned} tz(z-1)\partial_z\mathcal{F} =& \mathcal{D}_x^{(2)}\mathcal{F}, \qquad \text{where}\\ \mathcal{D}_x^{(2)} =& x(x-1)(x-z)\partial_x^2\\ &-((\Delta-1)(x^2-2zx+z)+2j_1x(z-1)+2j_2x(x-1)+2j_3z(x-1))\partial_x\\ &+2j_2\Delta(x-z)+2j_1j_2(z-1)+2j_2j_3z \end{aligned} \tag{56}$$

Conformal blocks for correlation functions involving degenerate fields satisfy additional differential equations:

In the case of the (2,1) degenerate field the decoupling equation reads simply $\partial_x^2\mathcal{F} = 0$. The two linearly independent solutions corresponding to s-channel conformal blocks with $j_{21} = j_1 \pm 1/2$ are denoted $\mathcal{F}^{2,1}_{s,\pm}$. For the t-channel one has $j_{32} = j_3 \pm 1/2$ with notation $\mathcal{F}^{2,1}_{t,\pm}$ respectively. Their explicit expressions as well as the fusion matrices defined by

$$\mathcal{F}^{2,1}_{s,\sigma}(x,z) = \sum_{\tau=\pm} F^{2,1}_{\sigma\tau}\mathcal{F}^{2,1}_{t,\tau}(x,z) \tag{57}$$

are given in the appendix.

The conformal blocks of a four point function involving the (1,2) degenerate field satsisfy a third order differential equation $\mathcal{D}_x^{(3)}\mathcal{F} = 0$, where

$$\begin{aligned} \mathcal{D}_x^{(3)} =& x(x-1)(x-z)\partial_x^3\\ &-((\Delta-2)(x^2-2zx+z)+2j_1x(z-1)-2(1+t)x(x-1)+2j_3z(x-1))\partial_x^2\\ &-(2(1+t)(j_1(z-1)+j_3z-(\Delta-1)(z-x))-t(1+t)(x+z+1))\partial_x\\ &-t(1+t)\Delta, \end{aligned} \tag{58}$$

and $\Delta = j_1 + j_2 + j_3 - j_4$. There are now three independent solutions in the s-channel case corresponding to $j_{21} = j_1 \mp t/2$ as well as $j_{21} = -j_1 - 1 + t/2$. They will be denoted $\mathcal{F}^{1,2}_{s\pm}$ and $\mathcal{F}^{1,2}_{s\times}$ respectively. The fusion matrices $F^{1,2}_{\sigma\tau}$ are defined similarly as in (57), with obvious modifications. Again the explicit expressions for conformal blocks and fusion matrices are to be found in the appendix.

4.4. **Liouville case.** It will turn out that parts of the analysis are closely related to the corresponding analysis for Liouville theory [TL]. This will not only allow to simplify some calculations but also shed some interesting light on the relationship of these theories.

The relation between the central charges is given by

$$t = -b^{-2} \quad \text{when the Liouville central charge is} \quad c_L = 1 + 6Q^2, \quad Q = b + b^{-1}$$

whereas the relations between Liouville-momentum α resp. conformal dimension h^L_α and the WZNW-spin j are

$$\alpha \equiv -bj; \qquad h^L_\alpha \equiv \alpha(Q - \alpha)$$

The degenerate fields $\Phi^L_{2,1}$ and $\Phi^L_{1,2}$ have $\alpha_{2,1} = -b/2$ and $\alpha_{1,2} = -b^{-1}/2$ respectively. The decoupling equation for the four-point function in case (2,1) reads

$$\left(b^{-2}\partial_z^2 + \frac{2z-1}{z(1-z)}\partial_z + \frac{h^L_{\alpha_3}}{(1-z)^2} + \frac{h^L_{\alpha_1}}{z^2} + \frac{\kappa_L}{z(1-z)} \right) \mathcal{F}(z) = 0, \tag{59}$$

where $\kappa_L = h^L_{\alpha_1} + h^L_{\alpha_2} + h^L_{\alpha_3} - h^L_{\alpha_4}$. For the case (1,2) one obtaines the corresponding equation by $b \to b^{-1}$. The notation for solutions, fusion matrices, structure constants etc. differs from that introduced previously just by adding a superscript L, i.e. $\mathcal{F} \to \mathcal{F}^L$, $F \to F^L$.

4.5. **Crossing symmetric combinations of conformal blocks.** It will be useful to reconsider the Liouville case along the lines of [TL] first since it can be used to facilitate the analysis of the other cases.

4.5.1. *Liouville case revisited.* The off-diagonal ($\epsilon \neq \epsilon'$) part of (55) reads

$$\frac{E^{L2,1}_{s+}}{E^{L2,1}_{s-}} = -\frac{F^{L2,1}_{-+}F^{L2,1}_{--}}{F^{L2,1}_{+-}F^{L2,1}_{++}}, \qquad \text{where} \quad E^{L2,1}_{s\sigma} \equiv C^L\left(\alpha_4, \alpha_3, \alpha_1 - \sigma\tfrac{b}{2}\right) C^L\left(\alpha_1 - \sigma\tfrac{b}{2}; -\tfrac{b}{2}, \alpha_1\right) \tag{60}$$

where the right hand side is explicitly given by ($\gamma(y) = \frac{\Gamma(y)}{\Gamma(1-y)}$)

$$\frac{F^{L2,1}_{-+}F^{L2,1}_{--}}{F^{L2,1}_{+-}F^{L2,1}_{++}} = -\frac{\gamma\left(b(\alpha_1 + \alpha_3 - \alpha_4 - \frac{b}{2})\right)\gamma\left(b(\alpha_1 + \alpha_4 - \alpha_3 - \frac{b}{2})\right)\gamma\left(b(\alpha_1 + \alpha_3 + \alpha_4 - \frac{3b}{2}) - 1\right)}{\gamma\left(b(2\alpha_1 - b)\right)\gamma\left(b(2\alpha_1 - b) - 1\right)\gamma\left(b(\alpha_3 + \alpha_4 - \alpha_1 - \frac{b}{2})\right)}.$$

In order to determine $C^{L2,1}_\sigma(\alpha_1) \equiv C^L\left(\alpha_1 - \sigma\frac{b}{2}; -\frac{b}{2}, \alpha_1\right)$ one may consider the special case $\alpha_1 = \alpha_4 = \alpha$, $\alpha_3 = -b/2$. Using that $C^L(\alpha;\alpha_3,\alpha_{21}) = C^L(Q-\alpha,\alpha_3,\alpha_{21})$ one now gets an equation that involves $C^{L2,1}_\epsilon(\alpha_1)$ only. It is clear that the crossing symmetry relations (55) can not determine the normalization of operators. The resulting freedom is fixed by imposing the normalization condition $C^{L2,1}_+(\alpha) = 1$. Given that normalization, equations (60) determine $C^{L2,1}_-(\alpha)$ to be

$$C^{L2,1}_-(\alpha) = \nu_L(b)\frac{\gamma\left(b(2\alpha - b - b^{-1})\right)}{\gamma(2b\alpha)},$$

where $\mu_L(b)$ represents the only leftover freedom, which corresponds to the cosmological constant. Inserting this into (60) yields the functional relation for $C^L(\alpha_3,\alpha_2,\alpha_1)$ derived in [TL]:

$$\frac{C^L(\alpha_3,\alpha_2,\alpha_1+b)}{C^L(\alpha_3,\alpha_2,\alpha_1)}=\frac{\big(\mu_L(b)\big)^{-1}\gamma(b(2\alpha_1+b))\gamma(2b\alpha_1)\gamma(b(\alpha_3+\alpha_4-\alpha_1-b))}{\gamma(b(\alpha_1+\alpha_3-\alpha_4))\gamma(b(\alpha_1+\alpha_4-\alpha_3))\gamma(b(\alpha_1+\alpha_3+\alpha_4-b)-1)}.$$

A second functional equation is of course obtained by $b\to b^{-1}$ and $\mu_L(b)\to\tilde{\mu}_L(b^{-1})$ These two functional equations are solved by an expression of the form

$$C^L(\alpha_1,\alpha_2,\alpha_3)=\frac{(\mu_L(b))^{-b^{-1}(\alpha_1+\alpha_2+\alpha_3)}\;\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-b-b^{-1})\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_1+\alpha_3-\alpha_2)\Upsilon(\alpha_2+\alpha_3-\alpha_1)},$$

if ν_L and $\tilde{\nu}_L$ are related by $(\tilde{\nu}_L(b^{-1}))^b=(\nu_L(b))^{b^{-1}}$ and $\Upsilon(x)$ satisfies the functional relations $\Upsilon(x+b)=\gamma(bx)b^{1-2bx}\Upsilon(x)$ and $b\to b^{-1}$. Such a function was introduced in [DO] and [ZZ]. In the latter reference the function $\Upsilon(x)$ was defined by

$$\log\Upsilon(x)\equiv\int_0^\infty\frac{du}{u}\left[\left(\frac{Q}{2}-x\right)^2e^{-u}-\frac{\sinh^2\left(\frac{Q}{2}-x\right)\frac{u}{2}}{\sinh\frac{bu}{2}\sinh\frac{b^{-1}u}{2}}\right].$$

Taking into account the requirement of symmetry of $C^L(\alpha_1,\alpha_2,\alpha_3)$ it was shown in [TL] that for irrational values of b the solution is unique up to a possibly b-dependent factor.

Let me summarize for future reference the properties of $\Upsilon(x)$ that will needed:

1. Symmetries $\Upsilon(Q-x)=\Upsilon(x)$, $\Upsilon_b(x)=\Upsilon_{b^{-1}}(x)$.
2. Functional relations $\Upsilon(x+b)=\gamma(bx)b^{1-2bx}\Upsilon(x)$ and $b\to b^{-1}$.
3. Poles at $x=x_{m,n}=-mb^{-1}-nb$ and $Q-x_{m,n}$ for $m,n\in\mathbb{Z}^{\geq 0}$.

4.5.2. *Case (2,1).* The analysis is completely analogous to the Liouville case, with few changes:

$$\frac{F^{2,1}_{-+}F^{2,1}_{--}}{F^{2,1}_{+-}F^{2,1}_{++}}=-\frac{\gamma(b(\alpha_1+\alpha_3-\alpha_4-b/2))\gamma(b(\alpha_1-\alpha_3+\alpha_4-b/2))\gamma(1-b(\alpha_3+\alpha_4-\alpha_1-b/2))}{\gamma^2(b(2\alpha_1-b))\gamma(1-b(\alpha_1+\alpha_3+\alpha_4-3b/2))}.$$

Normalization condition $C^{2,1}_+(\alpha)=1$ and consideration of the special case $\alpha_3=-b/2$, $\alpha_1=\alpha_4$ now leads to

$$C^{2,1}_-(\alpha_1)=\nu(b)\frac{\gamma(b(2\alpha_1-b))}{\gamma(2b\alpha_1)}.$$

The resulting functional relation differs only very slightly from the Liouville case:

$$\frac{C(\alpha_4,\alpha_3,\alpha_1+b)}{C(\alpha_4,\alpha_3,\alpha_1)}=\frac{(\nu(b))^{-1}\gamma(b(2\alpha_1))\gamma(b(2\alpha_1+b))\gamma(b(\alpha_3+\alpha_4-\alpha_1-b))}{\gamma(b(\alpha_1+\alpha_3-\alpha_4))\gamma(b(\alpha_1+\alpha_4-\alpha_3))\gamma(b(\alpha_3+\alpha_4+\alpha_1-b))}$$

4.5.3. *Case (1,2).* The following remarkable fact facilitates the analysis considerably: In the appendix it is shown that there exists a linear combination

(61) $$\mathcal{G}^{1,2}_{s-}(x,z)=a_{s-}\mathcal{F}^{1,2}_{s-}(x,z)+a_{s\times}\mathcal{F}^{1,2}_{s\times}(x,z)$$

such that $\mathcal{G}^{1,2}_{s+}\equiv\mathcal{F}^{1,2}_{s+}$ and $\mathcal{G}^{1,2}_{s-}(x,z)$ have the same monodromies as the (1,2) Liouville conformal blocks $\mathcal{F}^{L1,2}_{s+}$ and $\mathcal{F}^{L1,2}_{s-}$. Moreover, a second linear combination

(62) $$\mathcal{G}^{1,2}_{s\times}(x,z)=b_{s-}\mathcal{F}^{1,2}_{s-}(x,z)+b_{s\times}\mathcal{F}^{1,2}_{s\times}(x,z)$$

has one dimensional monodromy representation.

In terms of the $\mathcal{G}$'s it is therefore easy to write the most general crossing invariant and single valued combination of conformal blocks: To simplify notation I will drop the superscript (1,2) and subscript s in the following.

$$\Psi \equiv <\Phi^{j_4}\dots\Phi^{j_1}> = \sum_{\sigma=+,-} E^L_\sigma \mathcal{G}_\sigma(x,z)\mathcal{G}_\sigma(\bar{x},\bar{z}) + E^D\mathcal{G}_s(x,z)\mathcal{G}_s(\bar{x},\bar{z}),$$

where E^D is arbitrary. Written in the $\mathcal{F}$-basis of conformal blocks there appear non-diagonal terms:

$$\begin{aligned}\Psi =& E^L_+\mathcal{F}_+\bar{\mathcal{F}}_+ + \big(E^L_- a_-^2 + E^D b_-^2\big)\mathcal{F}_-\bar{\mathcal{F}}_- + \big(E^L_- a_\times^2 + E^D b_\times^2\big)\mathcal{F}_\times\bar{\mathcal{F}}_\times \\ &+ (E^L_- a_- a_\times + E^D b_- b_\times)\Big(\mathcal{F}_-\bar{\mathcal{F}}_\times + \mathcal{F}_\times\bar{\mathcal{F}}_-\Big),\end{aligned} \tag{63}$$

where the abbreviations $\mathcal{F}_\sigma \equiv \mathcal{F}_\sigma(x,z)$, $\bar{\mathcal{F}}_\sigma \equiv \mathcal{F}_\sigma(\bar{x},\bar{z})$ have been used.

However, it follows from the construction of conformal blocks given in [T2] that only diagonal combinations of holomorphic and antiholomorphic conformal blocks can appear in correlation functions of the H_3^+ WZNW model. Choose therefore E^D such that the off-diagonal terms in (63) vanish. With this choice one finds that

$$E_- = E^L_-\frac{a_-}{b_\times}(a_-b_\times - b_-a_\times) = E^L_-\frac{\gamma\big(b^{-1}(\alpha_1+\alpha_3+\alpha_4-\frac{3b^{-1}}{2})-1\big)\gamma\left(2b^{-1}\alpha_1-1\right)}{\gamma\big(b^{-1}(\alpha_1+\alpha_3+\alpha_4-\frac{b^{-1}}{2})-1\big)\gamma\left(b^{-1}(2\alpha_1-b^{-1})-1\right)}$$

from which it follows that

$$\begin{aligned}\frac{E_+}{E_-} =& -\frac{\gamma\big(b^{-1}(\alpha_1+\alpha_3-\alpha_4-\frac{b^{-1}}{2})\big)\gamma\big(b^{-1}(\alpha_1-\alpha_3+\alpha_4-\frac{b^{-1}}{2})\big)}{\gamma\left(b^{-1}(2\alpha_1-b^{-1})\right)\gamma\left(2b^{-1}\alpha_1-1\right)\gamma\big(b^{-1}(\alpha_3+\alpha_4-\alpha_1-\frac{b^{-1}}{2})\big)} \\ &\times\gamma\big(b^{-1}(\alpha_1+\alpha_3+\alpha_4-\tfrac{b^{-1}}{2}-b)\big)\end{aligned}$$

Analogous to the Liouville case one finds

$$C^{1,2}_-(\alpha) = \tilde{\nu}(b^{-1})\frac{\gamma\big(b^{-1}(2\alpha-b)\big)}{\gamma(2b^{-1}\alpha)}$$

leading to the functional equation

$$\frac{C(\alpha_4,\alpha_3,\alpha_1+b^{-1})}{C(\alpha_4,\alpha_3,\alpha_1)} = \frac{(\tilde{\nu}(b^{-1}))^{-1}\gamma(b^{-1}(2\alpha_1))\gamma(b^{-1}(2\alpha_1+b^{-1}))\gamma(b^{-1}(\alpha_3+\alpha_4-\alpha_1-b^{-1}))}{\gamma(b^{-1}(\alpha_1+\alpha_3-\alpha_4))\gamma(b^{-1}(\alpha_1+\alpha_4-\alpha_3))\gamma(b^{-1}(\alpha_3+\alpha_4+\alpha_1-b))}$$

Comparing the functional equations found in the (2,1) and (1,2) cases one observes that one is obtained from the other by $b\to b^{-1}$. They can therefore be solved as in the Liouville case:

$$C(\alpha_1,\alpha_2,\alpha_3) = \frac{C_0(b)(\nu(b))^{-b^{-1}(\alpha_1+\alpha_2+\alpha_3)}\,\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-b)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_1+\alpha_3-\alpha_2)\Upsilon(\alpha_2+\alpha_3-\alpha_1)} \tag{64}$$

where $(\tilde{\nu}(b^{-1}))^b = (\nu(b))^{b^{-1}}$.

4.6. **Reflection amplitude, two point function.** The amplitude $B(j) = R(j)$ can now be explicitely calculated from the three point function and the relation

$$C\begin{pmatrix} j & j_2 & j_1 \\ x' & x_2 & x_1\end{pmatrix} = -\pi\frac{\gamma(j_1-j_2-j)\gamma(j_2-j_1-j)}{\gamma(-2j-1)}\int_{\mathbb{C}} d^2x'\;|x-x'|^{-4j-4}C\begin{pmatrix} j & j_2 & j_1 \\ x' & x_2 & x_1\end{pmatrix} \tag{65}$$

One finds

$$R(j) = (\nu(b))^{-2j-1}\frac{\Gamma(1-t^{-1}(2j+1))}{\Gamma(1+t^{-1}(2j+1))}. \tag{66}$$

5. Fusion rules

Fusion rules are the rules determining the set of irreducible representations contributing in the decomposition of the vector obtained by acting with a primary field $\Phi^{j_2}(x|z)$ on a primary state Ψ^{j_1}.

In most current algebra representations relevant in the present context there are no nullvectors (cf. [T2]). The determination of fusion rules is therefore not an algebraic issue: In [T2] it is shown that chiral vertex operators between three representations with spins j_3, j_2, j_1 exist for a certain range of *complex* values for the j_i around the axis $j_i = -1/2 + i\rho_i$ that furthermore allow meromorphic continuation to generic j_i.

Instead it will be argued that the issue of fusion rules is intimately linked to the issue of spectral decomposition: If $\Phi^{j_2}[v|\sigma)\Psi^{j_1}$ is a normalizable vector then it should be possible to expand it in terms of contributions from the representations $\mathcal{P}_j$, $j = -1/2 + i\rho$ constituting the spectrum. In this case one will generically expect all representations appearing in the spectrum to contribute, as is the case in the mini-superspace limit [T1].

From this point of view the problem is mainly to find criteria for the normalizability of the state $\Phi^{j_2}[v|\sigma)\Psi^{j_1}$. The present section will present a heuristic argument based on the representation of primary fields and -states in canonical quantization that will lead to a precise conjecture on the fusion rules.

5.1. **Spectral decomposition of $\mathcal{V}$.** Motivated by the above mentioned result of Gawedzki and Kupiainen [Ga], I will assume that the space $\mathcal{V}$ can be decomposed into irreducible representations $\mathcal{P}_j$:

$$\mathcal{V} = \int_{\rho>0}^{\oplus} d\rho\rho^2 \; \mathcal{P}_{-\frac{1}{2}+i\rho}. \tag{67}$$

More explicitely I will assume that for a suitable subspace $\Phi \subset \mathcal{V}$ of "test-functions" and its hermitian dual $\Phi^\dagger$ one has a set of maps

$$\alpha_j : \Phi \to \mathcal{P}_j \qquad \text{and} \qquad \beta_j : \mathcal{P}_j \to \Phi^\dagger \tag{68}$$

that intertwine the $\hat{\mathfrak{g}}$ actions on Φ and $\mathcal{P}_j$ resp. $\mathcal{P}_j$ and $\Phi^\dagger$, and allow to write the decomposition of elements of Φ in the form

$$\Psi = -\frac{i}{4} \int_{\frac{1}{2}+i\mathbb{R}^+} dj \;\; \beta_j\big(\alpha_j(\Psi)\big). \tag{69}$$

The intertwining property for β_j in particular implies that eigenvectors of $L_0 + \bar{L}_0$ are mapped to (generalized) eigenvectors of H, so that expansion (69) can be recast as an expansion into (generalized) eigenvectors of the Hamiltonian.

Consider therefore the eigenvalue condition $H\Psi_E = E\Psi_E$ for $\Psi \in \mathcal{F} \otimes \mathcal{C}^\infty(H_3^+)$. It can be viewed as a system of differential equations for the coefficients $\Psi_E^I(q,v,\bar{v})$ in the expansion

$$\Psi_E = \sum_I \mathcal{A}_I \; \Omega \; \Psi_E^I(q,v,\bar{v}). \tag{70}$$

The eigenvalue condition simplifies for $q \to -\infty$, so one expects Ψ_e to be asymptotic to

$$\Psi_E \sim \Psi_{F,E} = e^{-2bjq} F_N^+(v,\bar{v}) + e^{-2b(-j-1)q} F_N^-(v,\bar{v})\big), \tag{71}$$

where the $F_N^\pm(v,\bar{v}) \in \mathcal{F}_j$ have level N, $E = t^{-1}j(j+1) + N$. Moreover, a qualitative analysis of the behavior for $q \to \infty$ suggests that there are two linearly independent solutions with asymptotic behavior of the following type: One horribly diverging (like $\sim \exp(e^{bq})$), the other rapidly converging (like $\sim \exp(-e^{bq})$). Suppressing the diverging solution means that there must be a fixed relation $F_N^- = \mathcal{R}(j)F_N^+$ between $F_N^\pm$, thereby defining an operator $\mathcal{R}(j)$. This is nothing but the statement that the interaction term in the H_3^+-Hamiltonian acts perfectly reflecting as the "Liouville-wall" does [S][P].

These considerations lead to the conjecture that for any given value of j and vectors $F(v,\bar{v}) \in \mathcal{F}_j$ there exists a unique solution $\Psi_j[F]$ of $H\Psi_E = E\Psi_E$, $E = t^{-1}j(j+1)$ in $\mathcal{F}\otimes\mathcal{C}^\infty(H_3^+)$ that has asymptotics (71) with $F^+ \equiv F$, $F_N^- = \mathcal{R}(j)F_N^+$ and which converges rapidly to zero for $q\to\infty$. One has thereby found a representation of the map β_j:

$$\beta_j : \mathcal{F}_j \to \mathcal{F}\otimes\mathcal{C}^\infty(H_3^+), \qquad \beta_j(F) = \Psi_j[F]. \tag{72}$$

It is analogous to the construction of a harmonic function from its boundary values. The intertwining property of the map Ψ_j is nothing but the statement that the generators J_n^a and $\bar{J}_n^a$ go into the free field realizations I_n^a, $\bar{I}_n^a$ for $q\to-\infty$. From this it also follows that the operator $\mathcal{R}(j)$ must be the intertwining operator that establishes the equivalence between the Fock modules $\mathcal{F}_j$ and $\mathcal{F}_{-j-1}$. It is completely determined by the $\hat{\mathfrak{g}}$-intertwining property up to an overall factor $r(j)$ [T2]. To unambigously define $r(j)$, consider the action of $\mathcal{R}(j)$ on the level zero subspace of $\mathcal{F}_j$: It must be proportional to the $SL(2,\mathbb{C})$-intertwining operator $\mathcal{I}_j$, so $r(j)$ will be defined by

$$F_j^-(v,\bar{v}) = r(j)\mathcal{I}_j\big[F^+\big](v,\bar{v}) \equiv r(j)\frac{2j+1}{\pi}\int_{\mathbb{C}} d^2v'\; |v-v'|^{4j}\; F_j^+(v',\bar{v}'). \tag{73}$$

An observation that will be needed in the next section is that the intertwining operator $\mathcal{I}_j$ is diagonalized by the Fourier transform

$$\tilde{F}(\mu,\bar{\mu}) = \frac{1}{2\pi}\int d^2v\; e^{\bar{\mu}\bar{x}-\mu x}F(v,\bar{v}). \tag{74}$$

This follows from the fact that

$$\mathcal{I}_j\big[e^{\mu x-\bar{\mu}\bar{x}}\big] = (\mu\bar{\mu})^{2j+1}\frac{\Gamma(+2j+1)}{\Gamma(-2j-1)}e^{\mu x-\bar{\mu}\bar{x}}. \tag{75}$$

Let $\Psi^j_{\mu\bar{\mu}}$ denote the primary (generalized) eigenstate of H that has $q\to-\infty$ asymptotics

$$\begin{aligned}\Psi^j_{\mu\bar{\mu}} \sim e^{\mu v-\bar{\mu}\bar{v}}\Big(: e^{-2bj\varphi(\sigma)} : + S(j)(\mu\bar{\mu})^{+2j+1} : e^{-2b(-j-1)\varphi(\sigma)} :\Big),\\ S(j) = r(j)\frac{\Gamma(+2j+1)}{\Gamma(-2j-1)}\end{aligned} \tag{76}$$

The state $\Psi^j_{\mu\bar{\mu}}$ thereby defined satisfies a simple reflection property:

$$\Psi^j_{\mu\bar{\mu}} = S(j)(\mu\bar{\mu})^{2j+1}\Psi^{-j-1}_{\mu\bar{\mu}}. \tag{77}$$

It will be found in the next section that indeed the reflection amplitude $r(j)$ considered here is equal to the reflection amplitude $R(j)$ previously calculated in (66).

5.2. **Primary fields.** The argument proposed here to find the fusion rules will require some qualitative information on how primary fields are represented in $\mathcal{V}$. It will be convenient to consider the Fourier transform of the operators $\Phi^j(x|\sigma)$:

$$\Phi^j_{\mu\bar{\mu}}(z) = \frac{1}{2\pi}(\mu\bar{\mu})^{2j+1}\int d^2x\; e^{\bar{\mu}\bar{x}-\mu x}\;\Phi^j(x|\sigma) \tag{78}$$

In terms of the $\Phi^j_{\mu\bar\mu}(z)$ the primary field transformation law takes the following form:

$$
(79)\qquad \begin{aligned} [J_n^+,\Phi^j_{\mu\bar\mu}(z)] &= z^n\left(\bar\mu\frac{\partial^2}{\partial\bar\mu^2}-2j\frac{\partial}{\partial\bar\mu}\right)\Phi^j_{\mu\bar\mu}(z) \\ [J_n^-,\Phi^j_{\mu\bar\mu}(z)] &= -\bar\mu\Phi^j_{\mu\bar\mu}(z) \end{aligned} \qquad [J_n^0,\Phi^j_{\mu\bar\mu}(z)] = z^n\left(\bar\mu\frac{\partial}{\partial\bar\mu}-j\right)\Phi^j_{\mu\bar\mu}(z).
$$

It is a difficult task to find such operators in terms of the elementary operators used in canonical quantization. What can be found is again the $q\to-\infty$ asymptotics of such an operator: In this asymptotics one may replace J_n^a, $\bar J_n^a$ by I_n^a, $\bar I_n^a$. The conditions obtained from (79) by that replacement are solved by

$$
(80)\qquad V^j_{\mu\bar\mu}(\sigma)\sim e^{\mu v(\sigma)-\bar\mu\bar v(\sigma)}\Big(A^j_{\mu\bar\mu}:e^{-2bj\varphi(\sigma)}:+B^j_{\mu\bar\mu}(\mu\bar\mu)^{2j+1}:e^{-2b(-j-1)\varphi(\sigma)}:\Big),
$$

with a priori undetermined coefficients $A^j_{\mu\bar\mu}$, $B^j_{\mu\bar\mu}$. These are fixed by requiring that the operator $V^j_{\mu\bar\mu}(z)$ corresponding to $V^j_{\mu\bar\mu}(\sigma)$ by (euclidean) time-evolution creates the state $\Psi^j_{\mu\bar\mu}$ by the usual state-operator correspondence. For this one needs to consider the state Ψ_0 (the $SL(2,\mathbb{C})$-invariant "vacuum") that is defined by $q\to-\infty$ asymptotics $\Psi_0\sim$ const.. Since this state transforms in the trivial representation of $\mathfrak{g}=\mathfrak{sl}\oplus\mathfrak{sl}$, the action of $V^j_{\mu\bar\mu}(\sigma)$ can only produce states in the representation $\mathcal{P}_j$. Considering the $q\to-\infty$ asymptotics of $V^j_{\mu\bar\mu}(\sigma)\Psi_0$ yields $A^j_{\mu\bar\mu}=1$, $B^j_{\mu\bar\mu}=S(j)(\mu\bar\mu)^{2j+1}$. Comparing the reflection relation that $V^j_{\mu\bar\mu}(\sigma)$ satisfies with that of the state $\Psi^j_{\mu\bar\mu}$ finally allows to determine $r(j)$ resp. $S(j)$ as

$$
(81)\qquad S(j)=(\nu(b))^{2j+1}\frac{\Gamma(+2j+1)\Gamma(1+t^{-1}(2j+1))}{\Gamma(-2j-1)\Gamma(1-t^{-1}(2j+1))}.
$$

It is no accident that the amplitude $S(j)$ appearing in (76) is up to inessential factors equal to the Liouville reflection amplitude discussed in [ZZ].

5.3. **Normalizability of fused state.** In view of the previous discussion it suffices to find out whether the vector $\Phi^{j_2}_{\mu_2\bar\mu_2}(\sigma)\Psi^{j_1}_{\mu_1\bar\mu_1}$ is in $\mathcal{V}$ and can therefore be expanded according to (69). The aim of the present subsection will be to find necessary and (conjecturally) sufficient criteria for normalizability of the state $\Psi_{21}\equiv\Phi^{j_2}_{\mu_2\bar\mu_2}(\sigma)\Psi^{j_1}_{\mu_1\bar\mu_1}$ by considering once more the $q\to-\infty$ asymptotics of its representation in $\mathcal{F}\otimes\mathcal{C}^\infty(H_3^+)$. It is given by

$$
(82)\qquad \Psi_{21}\sim e^{(\mu_2+\mu_1)v-(\bar\mu_2+\bar\mu_1)\bar v}\sum_{s,s'=+,-}F_{s,s'}\exp\Big(-2b\big(s\big(j_1+\tfrac12\big)+s'\big(j_2+\tfrac12\big)-1\big)Q\Big)
$$

with $F_{s,s'}\in\mathcal{F}$. By using knowledge of the reflection amplitude $S(j)$ one finds as necessary condition for normalizability

$$
(83)\qquad |\Re(j_1+j_2+1)|<\tfrac12 \qquad\qquad |\Re(j_1-j_2)|<\tfrac12.
$$

Note that the case that the j_i correspond to representations from the spectrum, $j_i=-\frac12+i\rho$ is well contained in that range, but no case where $j_i=j_{r,s}$ is contained in it.

One is thereby lead to the conjecture that for j_1, j_2 satisfying (83) the operator product expansion (44) involves integration over all j with $j=-1/2+i\rho$, $\rho\in\mathbb{R}$.

In order to extend this conjecture on the fusion rules to general j_2, j_1 one should note that the coefficients appearing in the operator product expansion (44) can be meromorphically continued to general complex j_2, j_1. In the process of analytic continuation it can happen that poles hit the contour of integration $j=-1/2+i\rho_i$, $\rho\in\mathbb{R}$. By deforming the contour one can always rewrite the integral in terms of an integral over the original contour plus a finite sum of residue contributions. This procedure allows to unambigously define the fusion rules of two generic operators.

5.4. **Consistency of fusion rules with structure constants.** The fusion rules for the degenerate fields are severely restricted by the additional differential equations they satisfy. It is a nontrivial check on the previously obtained results on structure constants and fusion rules that by analytic continuing the OPE from the range (83) to i.e. $j_1 = j^{\pm}_{r,s}$ one indeed exactly recovers the restricted fusion rules of degenerate fields.

The basic mechanism is as follows: If one sets i.e. j_1 in the expression for the structure constants (64) to any of the values $j_{r,s}$ then the structure constants will generically vanish. This need not to be true for the residue terms picked up in the analytic continuation from range (83) to $j_1 = j^{\pm}_{r,s}$. The integration in (48) therefore reduces to a sum over a finite number of terms only.

Consider i.e. analytic continuation of j_1 to $j_{r,s}$ while keeping j_2 generic. Consider i.e. the poles of (64) from $\alpha_1+\alpha_2+\alpha_3-b = -mb-nb^{-1}$ (Recall $\alpha_i = -bj_i$). The corresponding residue is proportional to

$$\prod_{u=0}^{3}\left(\prod_{i=0}^{m-1}\frac{1}{\gamma(b(2\alpha_u+ib+nb^{-1}))}\prod_{j=0}^{n}\frac{1}{\gamma(b^{-1}(2\alpha_u+jb^{-1}))}\right)$$

It is easy to check that one needs $m \leq r$, $n \leq s-1$ to get non-vanishing of these residue terms. Similarly one finds non-vanishing residues

$$\text{from the poles at}\left\{\begin{array}{l}\alpha_1+\alpha_2-\alpha_3 = -mb-nb^{-1}\\ \alpha_2+\alpha_3-\alpha_1 = (m+1)b+(n+1)b^{-1}\\ \alpha_1+\alpha_3-\alpha_2 = -mb-nb^{-1}\end{array}\right\}\text{only for}\left\{\begin{array}{l}m\leq r,\quad n\leq s\\ m\leq r,\ \ n\leq s-1\\ m\leq r,\ \ n\leq s\end{array}\right\}$$

The range of values for j_3 for which the residue terms may be non-vanishing coincides with that given by the fusion rules [T2][AY] for fusion of degenerate with generic fields.

Up to this point the choice of contour for the integration over j (the fusion rules) did not enter the discussion at all. The question that does crucially depend on the choice of contour is whether these residues are really picked up in the process of analytic continuation and contour deformation sketched above.

To analyze this question it is convenient to start by considering the case that the analytic continuation has been performed such that $|\alpha_1-\alpha_2| < \frac{b}{2}$. Under that condition one picks up only poles of $\Upsilon^{-1}(\alpha_1+\alpha_2-\frac{b}{2}-i\sigma)\Upsilon^{-1}(\alpha_1+\alpha_2-\frac{b}{2}+i\sigma)$. One should imagine $\alpha_1+\alpha_2$ to move slightly off the real axis in order to avoid that the poles of these two Υ-functions occur simultaneously.

The condition that in analytically continuing α_1 to $-r\frac{b}{2}-s\frac{b^{-1}}{2}$ one has hit the pole with $\Re(\alpha_1+\alpha_2-\frac{b}{2}) = -mb-nb^{-1}$ is

$$\alpha_1+\alpha_2-\frac{b}{2} = -rb-sb^{-1}-\frac{1-\epsilon}{2} < -mb-nb^{-1} \tag{84}$$

if $\alpha_2 = \alpha_1 + b\frac{\epsilon}{2}$ for some $\epsilon \in (-1,1)$. Integers m, n such that $0 \leq m \leq r$ and $0 \leq n \leq s$ will satisfy (84) for *any* value of $b > 0$. For any such m, n one gets residue terms from the poles at both $\alpha_1+\alpha_2+\alpha_3-b = -mb-nb^{-1}$ and $\alpha_1+\alpha_2-\alpha_3-b = -mb-nb^{-1}$. In the special case $|\alpha_1-\alpha_2| < \frac{b}{2}$ considered one will therefore indeed pick up as many residue terms as are allowed by the fusion rules. If one then considers more general cases for $|\alpha_1-\alpha_2|$ it is easy to convince oneself that for any pole of $\Upsilon^{-1}(\alpha_1+\alpha_2-\frac{b}{2}i\sigma)\Upsilon^{-1}(\alpha_1+\alpha_2-\frac{b}{2}+i\sigma)$ that is missed one gets an additional pole from $\Upsilon^{-1}(\alpha_1-\alpha_2+\frac{b}{2}-i\sigma)\Upsilon^{-1}(\alpha_2-\alpha_1+\frac{b}{2}-i\sigma)$.

It follows that analytic continuation of the operator product expansion (44) indeed yields the correct fusion rules for the operator product expansion of a degenerate and a generic field when the proposed fusion rules and structure constants are used.

To see that also the coefficients are correctly reproduced it suffices to note that recursion relations for the structure constants of degenerate fields can be derived exactly as the functional relations for the structure constants of generic fields were derived. These recursion relations will coincide with those satisfied by the residues of $C(j_3,j_2,j_1)$. Moreover, they allow to express the structure constants of any degenerate field $\Phi_{r,s}$ in terms of the $C_{\pm}^{1,2}(\alpha)$, $C_{\pm}^{2,1}(\alpha)$ found above. Structure constants of degenerate field and residues of structure constants of generic fields must therefore coincide.

6. Conclusions

Sufficient information has been obtained to define any n-point correlation in the H_3^+-WZNW model on the sphere:

As discussed in more detail in [T2], one may characterize the conformal blocks as solutions of a KZ-type system of equations, which here generically has continuous sets of solutions. There exist unique power series solutions that can be identified with conformal blocks corresponding to a fixed intermediate representation.

An explicit expression for the structure constants $C(j_3,j_2,j_1)$ was derived from the assumption that the degenerate fields are part of the algebra of mutually local primary fields.

Finally a discussion of the issue of fusion rules from the point of view of canonical quantization was given. Together with the pole structure of the $C(j_3,j_2,j_1)$ this led to a precise conjecture for the fusion rules for general primary fields.

Clearly the task remains to prove crossing symmetry of the so defined four-point functions resp. mutually locality of generic primary fields. However, the assumptions that were made to derive structure constants and fusion rules turned out to be consistent with each other in a remarkable way: *The results of the previous subsection imply that the proposed fusion rules are just right to make the conjecture of crossing symmetry of (48) for generic fields compatible with the crossing symmetry in the case of one degenerate field, which was the basis for the present derivation of the structure constants.*

To put the theory on firmer mathematical ground one may try to establish and characterize duality transformations for the conformal blocks. The possibility of doing this by establishing relations to (noncompact) quantum groups is currently under investigation. This may ultimately lead to a rigorous proof of the results on structure constants and fusion rules discussed here.

From the point of view of physical applications let me note that one obtains almost immediately an important subset of the structure constants for the euclidean black hole CFT from the H_3^+structure constants, namely those for which the winding number is conserved. One interesting consequence is already visible from the results presented here: The exact reflection amplitude differs by certain quantum corrections from the corresponding quantity proposed in [DVV]. There it was obtained from a quantum mechanics which was proposed to describe tachyons in a black hole background. Quantum corrections of the reflection amplitude indicate that the geometry probed by the tachyons is not the geometry described by the classical sigma model metric. It would be extremely interesting to see whether the resulting effective geometry can be efficiently described by some noncommutative geometry as proposed in [FG]

7. Appendix

7.1. Solutions to the Liouville decoupling equations. In order to write down the solutions, introduce the parameters

$$\begin{aligned} u &= b(\alpha_1+\alpha_3+\alpha_4-3b/2)-1 & \bar{u} &= b^{-1}(\alpha_1+\alpha_3+\alpha_4-3b^{-1}/2)-1 \\ v &= b(\alpha_1+\alpha_3-\alpha_4-b/2) & \bar{v} &= b^{-1}(\alpha_1+\alpha_3-\alpha_4-b^{-1}/2) \\ w &= b(2\alpha_1-b) & \bar{w} &= b^{-1}(2\alpha_1-b^{-1}) \end{aligned}$$

Then the solutions to the decoupling equation (59) that correspond to s-channel conformal blocks are given by

$$\begin{aligned} \mathcal{F}_{s+}^{L2,1} &= z^{b\alpha_1}(1-z)^{b\alpha_3}F(u,v,w;z) \\ \mathcal{F}_{s-}^{L2,1} &= z^{1-b(\alpha_1-b)}(1-z)^{b\alpha_3}F(u-w+1,v-w+1,2-w;z), \end{aligned}$$

whereas the solutions for the t-channel are

$$\begin{aligned} \mathcal{F}_{t+}^{L2,1} &= z^{b\alpha_1}(1-z)^{b\alpha_3}F(u,v,u+v-w+1;1-z) \\ \mathcal{F}_{t-}{}^{L2,1} &= z^{b\alpha_1}(1-z)^{1-b(\alpha_3-b)}F(w-u,w-v,w-u-v+1;1-z). \end{aligned}$$

Similarly, the solutions in the (1,2) case are obtained by replacing $b\to b^{-1}$ and $u,v,w\to\bar{u},\bar{v},\bar{w}$.

7.1.1. *Fusion and braiding.* If a basis for the space of conformal blocks with diagonal monodromy around the origin is chosen then fusion matrix and the phases $\Omega_{s\sigma}^{L2,1}$, $\sigma=+,-$ defined by [10]

$$\mathcal{F}_{s\sigma}^{L2,1}(e^{\pi i}z)=\Omega_{s\sigma}^{L2,1}\mathcal{F}_{s\sigma}^{L2,1}(z)$$

form a complete set of duality data, i.e. completely determine the monodromies of conformal blocks. Here the phases $\Omega_{s\sigma}^{L2,1}$ are read off as

$$\Omega_{s+}^{L2,1}=e^{\pi i b\alpha_1}\qquad \Omega_{s-}^{L2,1}=e^{\pi i(1-b(\alpha_1-b))}$$

The fusion matrix is calculated by using the standard results on analytic continuation of hypergeometric functions:

$$\begin{aligned} \mathcal{G}^{L}{}_{+}^{(0)} &= \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}\mathcal{G}^{L}{}_{+}^{(1)}+\frac{\Gamma(w)\Gamma(u+v-w)}{\Gamma(u)\Gamma(v)}\mathcal{G}^{L}{}_{-}^{(1)} \\ \mathcal{G}^{L}{}_{-}^{(1)} &= \frac{\Gamma(2-w)\Gamma(w-u-v)}{\Gamma(1-u)\Gamma(1-v)}\mathcal{G}^{L}{}_{+}^{(1)}-\frac{\Gamma(2-w)\Gamma(u+v-w)}{\Gamma(u-w+1)\Gamma(v-w+1)}\mathcal{G}^{L}{}_{-}^{(1)} \end{aligned}$$

7.2. Solutions to decoupling equations (2,1). This case is of course well-known [FZ, TK]. For the sake of a coherent presentation, I will nevertheless present the relevant results.

Equation $\partial_x\mathcal{F}=0$ of course implies $\mathcal{F}(x,z)=\mathcal{F}^+(z)+x\mathcal{F}^-(z)$. It straightforward to reduce the system of two ordinary differential equations that follows from the KZ-equations to hypergeometric equations for $\mathcal{F}^+(z)$, $\mathcal{F}^-(z)$. The normalization prescription is

$$\mathcal{F}(x,z)\sim z^{h_{21}-h_2-h_1}x^{j_1+j_2-j_{21}}(1+\mathcal{O}(x)+\mathcal{O}(z))$$

[10] It will be assumed throughout that z^λ is defined by the principal value of the logarithm and that $\arg(z)\in(-\pi,0]$

in the limit of first taking $z \to 0$, then $x \to 0$. A set of normalized solutions for the s- and t-channel are then given by ($t_- = t^{-1}$)

$$\begin{aligned}
\mathcal{F}^{2,1}_{s+} = & \quad z^{t-j_1}(1-z)^{t-j_3}\Big(F(u+1,v,w;z) - x\frac{v}{w}F(u+1,v+1,w+1;z)\Big)\\
\mathcal{F}^{2,1}_{s-} = & \quad z^{-t_-(j_1+1)}(1-z)^{t-j_3}\Big(xF(u-w+1,v-w+1,1-w;z) - \\
& \qquad z\frac{u-w+1}{1-w}F(u-w+2,v-w+1,2-w;z)\Big),\\
\mathcal{F}^{2,1}_{t+} = & \quad (1-z)^{t-j_3}z^{t-j_1}\Big(F(u+1,v,u+v-w+1;1-z) + \\
& \qquad (1-x)\frac{v}{u+v-w+1}F(u+1,v+1,u+v-w+2;1-z)\Big)\\
\mathcal{F}^{2,1}_{t-} = & \quad (1-z)^{-t_-(j_3+1)}z^{t-j_1}\Big((1-x)F(w-u,w-v,w-u-v;1-z) - \\
& \qquad (1-z)\frac{w-v}{w-u-v}F(w-u,w-v+1,w-u-v+1;1-z)\Big).
\end{aligned}$$

7.2.1. *Fusion and braiding.* The $\Omega^{2,1}_{s\sigma}$-factors are

$$\Omega_{s+} = e^{\pi i t_- j_1} \qquad \Omega^{(0)}_{-} = e^{\pi i(1-t_-(j_1+1))}$$

The fusion relations now read

$$\begin{aligned}
\mathcal{F}^{2,1}_{s+} &= \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}\mathcal{F}^{2,1}_{t+} + \frac{\Gamma(w)\Gamma(u+v-w+1)}{\Gamma(u+1)\Gamma(v)}\mathcal{F}^{2,1}_{t-}\\
\mathcal{F}^{2,1}_{s-} &= \frac{\Gamma(1-w)\Gamma(w-u-v)}{\Gamma(-u)\Gamma(1-v)}\mathcal{F}^{2,1}_{t+} - \frac{\Gamma(1-w)\Gamma(u+v+w+1)}{\Gamma(u-w+1)\Gamma(v-w+1)}\mathcal{F}^{2,1}_{t-}
\end{aligned}$$

7.2.2. *Reduction to Liouville conformal blocks.* There are general arguments [FGPP][PRY] that taking the limit $x \to z$ relates the WZNW conformal blocks to their Liouville counterparts. Here one finds

$$\mathcal{F}^{2,1}_{s+}(z,z) = \mathcal{F}^{L2,1}_{s+}(z) \qquad \mathcal{F}^{2,1}_{s-}(z,z) = \frac{u}{u-w+1}\mathcal{F}^{L2,1}_{s-}(z) \tag{85}$$

7.3. **Solutions to decoupling equations (1,2).** It is easy to see that the decoupling equation (58) for the (1,2) degenerate field coincides with the third order ordinary differential equation satisfied by the generalized hypergeometric function $F_1(\alpha,\beta,\beta',\gamma;x,z)$ of Appell and Kampé de Feriet ([AK], Chap. III, eqn. (31)), provided one identifies the parameters as follows:

$$\begin{aligned}
\alpha = -\Delta = j_4 - j_1 - j_3 + t/2 \qquad & \beta = t\\
\beta' = \Delta - 1 + t - 2j_1 - 2j_3 = t/2 - j_1 - j_3 - j_4 - 1 \qquad & \gamma = t - 2j_1
\end{aligned} \tag{86}$$

Equation (II) has three linearly independent solutions, unique up to linear combinations with arbitrary functions of z as coefficients. The z dependence is determined by the KZ-equation: If one sets

$$\mathcal{F}(x,z) = z^{-j_1}(1-z)^{-j_3}F(x,z), \tag{87}$$

then the KZ equation for $\mathcal{F}$ is equivalent to a similar equation for F which may easily seen to be one of the partial differential equations satisfied by F_1, namely the first of eqns. (13), Chap. III in [AK]. The function F can therefore be any linear combination of the three linearly independent solutions of the system of partial differential equations for F_1. As shown in [AK], Chap. III, pp. 55-65, each of them can be expressed in terms of the function F_1 itself, for which one has representations

$$F_1(\alpha,\beta,\beta',\gamma;x,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 dt\; t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}(1-zt)^{-\beta'} \tag{88}$$

or as the power series

$$F_1(\alpha,\beta,\beta',\gamma;x,z) = \sum_{n,m=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{\gamma_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}$$

The task now is to identify solutions to the F_1 equations with conformal blocks corresponding to definite intermediate representations. The s-channel conformal blocks should be of the form [T2]

$$\mathcal{F}_s = \sum_{n=0}^{\infty} z^{s+n} f_n(x), \quad f_0(x) = \sum_{m=0} x^{r+m} g_n \tag{89}$$

It may be shown that for generic (i.e. non-integer) values of $2j_1 - t$ and t there indeed exist three linearly independent solutions of the form (89). These are uniquely specified up to multiplication by (x,z)-independent factors once one has chosen one of the three possible values for r:

$$r_+ = 0 \qquad r_- = -t \qquad r_\times = 2j_1 + 1 - t. \tag{90}$$

The values of s for the corresponding solutions are

$$s_+ = -j_1 \qquad s_- = j_1 + 1 \qquad s_\times = -j_1. \tag{91}$$

In order to identify these solutions with conformal blocks one needs to have $r_i = j_1 + j_2 - j_{21,i}$ ($i \in \{+,-,\times\}$) and $s_i = h_{j_{21,i}} - h_{j_2} - h_{j_1}$ which is satisfied by

$$j_{21,+} = j_1 - t/2 \qquad j_{21,-} = j_1 + t/2 \qquad j_{21,\times} = -j_1 - 1 + t/2. \tag{92}$$

One thereby recovers a special case of the fusion rules derived in [AY]. Note that the values s_+ and $s_\times$ coincide.

The task is now to find the explicit expressions for these solutions in terms of the F_1-functions. The relevant solutions of the F_1-system will be denoted F_{s+}, F_{s-}, $F_{s\times}$ respectively, the corresponding conformal blocks by $F \to \mathcal{F}$, cf. (87). A table of solutions to the system of equations satisfied by F_1 that possess simple integral representations similar to (88) has been given in [AK], Chap. III, Sect. XV. The ones that will be needed below are

$$\begin{aligned}
Z_1 &= F_1(\alpha,\beta,\beta',\gamma;x,z)\\
Z_2 &= F_1(\alpha,\beta,\beta',\alpha+\beta+\beta'+1-\gamma;1-x,1-z)\\
Z_5 &= z^{\beta+1-\gamma}(1-z)^{\gamma-\alpha-1}(x-z)^{-\beta}F_1\big(1-\beta',\beta,\alpha+1-\gamma,2+\beta-\gamma,\tfrac{z}{z-x},\tfrac{z}{z-1}\big)\\
Z_7 &= z^{\beta+\beta'-\gamma}(1-z)^{\gamma-\alpha-\beta'}(x-z)^{-\beta}F_1\big(1-\beta',\beta,\gamma-\beta-\beta',\gamma+1-\alpha-\beta';\tfrac{z-1}{z-x},\tfrac{z-1}{z}\big)\\
Z_8 &= x^{-\alpha}F_1\big(\alpha,\alpha+1-\gamma,\beta',\alpha+1-\beta;\tfrac{1}{x},\tfrac{z}{x}\big)
\end{aligned}$$

In view of above considerations it suffices to first take the limit $z \to 0$ (our variable z corresponds to y in loc. cit.), before taking $x \to 0$ in order to compare the asymptotic behavior of the solutions given in loc. cit., p. 62 with that expected for the conformal blocks. In this way one easily identifies $F_{s+} = Z_1$ and $F_{s-} = Z_5$. Finding the solution corresponding to $F_{s\times}$ is a little more complicated since none of the solutions given in loc. cit. has the required asymptotic behavior. Consider however

$$Z_8 = x^{-\alpha} F_1\left(\alpha, \alpha+1-\gamma, \beta', \alpha+1-\beta; \frac{1}{x}, \frac{z}{x}\right)$$

From the power series expansion of F_1 one easily sees that Z_8 is analytic as function of z in a neighborhood of $z=0$ and

$$Z_8|_{z=0} = x^{-\alpha} F(\alpha, \alpha+1-\gamma; \alpha+1-\beta; 1/x),$$

where $F(\alpha,\beta,\gamma,x)$ is the ordinary hypergeometric function. This is rewritten in terms of functions with simple asymptotics for $x \to 0$ by using standard results on the analytic continuation of hypergeometric functions:

$$\begin{aligned} Z_8|_{z=0} &= e^{\pi i\alpha}\frac{\Gamma(\alpha+1-\beta)\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(1-\beta)}F(\alpha,\beta;\gamma;z) \\ &+ e^{\pi i(\gamma-\alpha-1)}\frac{\Gamma(\alpha+1-\beta)\Gamma(\gamma-1)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}x^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;x) \end{aligned}$$

for $\arg(-1/x) \in (-\pi, 0]$. Since $1-\gamma = 2j_1+1-t$, the second term has the asymptotics required for $F_\times$ which may therefore be represented as

$$F_{s\times} = e^{-\pi i(\alpha+1-\gamma)}\frac{\Gamma(\alpha)\Gamma(\gamma-\beta)}{\Gamma(\alpha+1-\beta)\Gamma(\gamma-1)}(Z_8 - e^{\pi i\alpha}Z_1) \qquad \text{for } \arg(-1/x) \in (-\pi, 0],$$

with a similar expression for $\arg(-1/x) \in (0,\pi]$. In this way one finds the following two bases for solutions:

$$\begin{aligned} F_{s+} =& Z_1 \\ F_{s\times} =& e^{\pi i(\alpha+1-\gamma)}\frac{\Gamma(\alpha)\Gamma(\gamma-\beta)}{\Gamma(\alpha+1-\beta)\Gamma(\gamma-1)}\left(Z_8 - e^{-\pi i\alpha}\frac{\Gamma(\alpha+1-\beta)\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(1-\beta)}Z_1\right) \\ F_{s-} =& Z_5 \\ F_{t+} =& Z_2 \\ F_{t\times} =& \frac{e^{\pi i(\alpha+\beta+\beta'-\gamma)}\Gamma(\alpha)\Gamma(1+\alpha+\beta'-\gamma)}{\Gamma(\alpha+1-\beta)\Gamma(\alpha+\beta+\beta'-\gamma)}\left(Z_8 - \frac{\Gamma(\alpha+1-\beta)\Gamma(\gamma-\alpha-\beta-\beta')}{\Gamma(1-\beta)\Gamma(\gamma-\beta-\beta')}Z_2\right) \\ F_{s-} =& Z_5 \end{aligned}$$

7.3.1. *Fusion and braiding.* The $\Omega^{1,2}_{s\sigma}$-factors are

$$\Omega^{1,2}_{s+} = e^{-\pi i j_1} \qquad \Omega^{1,2}_{s-} = e^{\pi i(j_1+1)} \qquad \Omega^{1,2}_{s\times} = e^{-\pi i j_1}$$

The fusion matrix may be computed from the integral representations of the Z_i-functions by using the technique exemplified in chapter III, section XVI of [AK]. The result is

$$\begin{aligned} \mathcal{F}^{1,2}_{s+} =& \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta')}\mathcal{F}^{1,2}_{t+} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\beta')}\mathcal{F}^{1,2}_{t-} \\ &+ \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\beta)}\mathcal{F}^{1,2}_{t\times} \\ \mathcal{F}^{1,2}_{s-} =& \frac{\Gamma(2+\beta-\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(1-\beta')\Gamma(1-\alpha)}\mathcal{F}^{1,2}_{t+} + e^{\pi i\beta}\frac{\Gamma(2+\beta-\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(1+\beta+\beta'-\gamma)\Gamma(1-\gamma+\alpha)}\mathcal{F}^{1,2}_{t-} \\ &+ e^{\pi i(\beta+\beta'+\alpha-\gamma)}\frac{\Gamma(2+\beta-\gamma)\Gamma(\gamma-\alpha-\beta')\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(1-\beta')\Gamma(1+\beta+\beta'-\gamma)\Gamma(\beta)}\mathcal{F}^{1,2}_{t\times} \end{aligned}$$

$$\mathcal{F}_{s\times}^{1,2} = \frac{\Gamma(\gamma-\beta)}{\Gamma(1-\beta)}\Bigg(\frac{\Gamma(2-\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(\gamma-\beta-\beta')\Gamma(1-\alpha)}\mathcal{F}_{t+}^{1,2} - e^{\pi i\gamma}\frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(\beta')}\mathcal{F}_{t-}^{1,2}$$
$$+ e^{\pi i\gamma}\left(e^{\pi i(\beta+\beta'-\gamma)}\frac{\sin\pi\gamma}{\sin\pi\beta} - \frac{\sin\pi(\gamma-\alpha)}{\sin\pi(\gamma-\alpha-\beta')}\right)\frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(\beta)\Gamma(1+\alpha+\beta'-\gamma)}\mathcal{F}_{t\times}^{1,2}\Bigg)$$

7.3.2. *Alternative choice of basis for conformal blocks.* An important simplification of the fusion relations is achieved by using the following linear combinations of the conformal blocks:

$$\begin{aligned}
\mathcal{G}_{s+}^{1,2} &= \mathcal{F}_{s+}^{1,2} \\
\mathcal{G}_{s-}^{1,2} &= \frac{\Gamma(\beta+\beta')\Gamma(1+\beta-\gamma)}{\Gamma(\beta)\Gamma(\beta+\beta'+1-\gamma)}\mathcal{F}_{s\times}^{1,2} + \frac{\Gamma(\beta+\beta')\Gamma(\gamma-\beta-1)}{\Gamma(\gamma-1)\Gamma(\beta')}\mathcal{F}_{s-}^{1,2} \\
\mathcal{G}_{s\times}^{1,2} &= \frac{\Gamma(1-\beta)\Gamma(1+\beta-\gamma)}{\Gamma(2-\gamma)}\mathcal{F}_{s\times}^{1,2} + \frac{\Gamma(1-\beta')\Gamma(\gamma-\beta-1)}{\Gamma(\gamma-\beta-\beta')}\mathcal{F}_{s-}^{1,2} \\
\mathcal{G}_{t+}^{1,2} &= \mathcal{F}_{t+}^{1,2} \\
\mathcal{G}_{t-}^{1,2} &= \frac{\Gamma(\beta+\beta')\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\beta)}\mathcal{F}_{t\times}^{1,2} + \frac{\Gamma(\beta+\beta')\Gamma(\alpha+\beta'-\gamma)}{\Gamma(\alpha+\beta+\beta'-\gamma)\Gamma(\beta')}\mathcal{F}_{t-}^{1,2} \\
\mathcal{G}_{t\times}^{1,2} &= \frac{\Gamma(1-\beta)\Gamma(\gamma-\alpha-\beta')}{\Gamma(1+\gamma-\alpha-\beta-\beta')}\mathcal{F}_{t\times}^{1,2} + \frac{\Gamma(1-\beta')\Gamma(\alpha+\beta'-\gamma)}{\Gamma(1+\alpha-\gamma)}\mathcal{F}_{t-}^{1,2}
\end{aligned}$$

The fusion relations for this basis read

$$\begin{aligned}
\mathcal{G}_{s+}^{1,2} &= \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta')}\mathcal{G}_{t+}^{1,2} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(\beta+\beta')\Gamma(\alpha)}\mathcal{G}_{t-}^{1,2} \\
\mathcal{G}_{s-}^{1,2} &= \frac{\Gamma(2-\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(1-\alpha)\Gamma(1-\beta-\beta')}\mathcal{G}_{t+}^{1,2} + \frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(1+\beta+\beta'-\gamma)}\mathcal{G}_{t-}^{1,2} \\
\mathcal{G}_{s\times}^{1,2} &= e^{\pi i(\beta+\beta'+1)}\mathcal{G}_{t\times}^{1,2}
\end{aligned}$$

Two remarks will be important:

1. The fusion matrix in the subspace spanned by $\mathcal{G}_{s+}^{1,2}$ $\mathcal{G}_{s}-^{1,2}$ is identical to that of the Liouville conformal blocks $\mathcal{F}_{s+}^{L1,2}$, $\mathcal{F}_{s-}^{1,2}$. This will be further explained in the next subsection.
2. The conformal block $\mathcal{G}_{s\times}^{1,2}$ transforms into itself under fusion.

7.3.3. *Reduction to Liouville conformal blocks.* The analysis of the behavior of the G-basis for the conformal blocks is facilitated by the observation that

$$\begin{aligned}
\mathcal{G}_{s-}^{1,2} &= z^{-j_1}(1-z)^{-j_3}\frac{\Gamma(\alpha)\Gamma(\beta+\beta')}{\Gamma(\gamma-1)\Gamma(\beta')}\left(Z_2 - \frac{\Gamma(\alpha+\beta+\beta'-\gamma)\Gamma(1-\gamma)}{\Gamma(\beta+\beta'+1-\gamma)\Gamma(\alpha+1-\gamma)}Z_1\right) \\
\mathcal{G}_{t-}^{1,2} &= z^{-j_1}(1-z)^{-j_3}\frac{\Gamma(\alpha)\Gamma(\beta+\beta')}{\Gamma(\gamma)\Gamma(\alpha+\beta+\beta'-\gamma)}\left(Z_1 - \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta')}Z_2\right)
\end{aligned}$$

Since $F_1(\alpha,\beta,\beta',\gamma;x,z)$ is analytic in x around $x=z$ it follows that $\mathcal{G}_{s+}^{1,2}$, $\mathcal{G}_{s-}^{1,2}$, $\mathcal{G}_{t+}^{1,2}$, $\mathcal{G}_{t-}^{1,2}$ all share that property. By using

$$F_1(\alpha,\beta,\beta',\gamma;z,z) = F(\alpha,\beta+\beta',\gamma;z),$$

standard relations on analytic continuation of hypergeometric functions and observing that

$$\alpha = \bar{u} \qquad \beta+\beta' = \bar{v} \qquad \gamma = \bar{w}$$

one finds that $\mathcal{G}_{s+}^{1,2}$, $\mathcal{G}_{s-}^{1,2}$, $\mathcal{G}_{t+}^{1,2}$, $\mathcal{G}_{t-}^{1,2}$ reduce to $\mathcal{F}_{s+}^{L1,2}$, $\mathcal{F}_{s-}^{L1,2}$, $\mathcal{F}_{t+}^{L1,2}$, $\mathcal{F}_{t-}^{L1,2}$ respectively for $x \to z$. $\mathcal{G}_\times$ however will not be analytic in x around $x = z$ but rather behave as

$$\mathcal{G}_s \sim (x-z)^{1-\beta-\beta'}(C_1 + \mathcal{O}(x-z))$$

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