Convergence Properties of Constrained Linear System under MPC Control Law using Affine Disturbance Feedback

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Abstract

This paper shows new convergence properties of constrained linear discrete time system with bounded disturbances under Model Predictive Control (MPC) law. The MPC control law is obtained using an affine disturbance feedback parametrization with an additional linear state feedback term. This parametrization has the same representative ability as some recent disturbance feedback parametrization, but its choice together with an appropriate cost function results in a different closed-loop convergence property. More exactly, the state of the closed-loop system converges to a minimal invariant set with probability one. Deterministic convergence to the same minimal invariant set is also possible if a less intuitive cost function is used. Numerical experiments are provided that validate the results.
1 Introduction

This paper considers the system:

\[ x_{t+1} = Ax_t + Bu_t + w_t, \]
\[ (x_t, u_t) \in Y, w_t \in W, \forall t \geq 0 \]  

where \( x_t \in \mathbb{R}^n \), \( u_t \in \mathbb{R}^m \) and \( w_t \in W \subset \mathbb{R}^n \) are the state, control and disturbance of the system at time \( t \) respectively, and \( Y \) represents the joint constraint set on \( x_t \) and \( u_t \). The study of such a system under the Model Predictive Control (MPC) framework has been an active area of research. See, for example, [3, 8, 23, 15, 18, 1, 19] and the references cited therein. An important research issue remains the choice of control parametrization within the control horizon. Several choices have been proposed in the literature [23, 8, 17, 18] and a popular choice is \( u_t = Kx_t + c_t \) where \( K \) is a fixed feedback gain and \( c_t \) is the new optimization variable. However, such a choice is known to be conservative [5, 17, 12, 24, 25] and its use will result in a relatively small domain of attraction.

A natural extension of the fixed-gain parametrization is the time-varying affine state feedback \( u_t = K_t x_t + c_t \) where both \( K_t \) and \( c_t \) are the optimization variables within the horizon. However, such a parametrization is not computationally amiable as the resulting optimization problem is not convex. More recently, control parametrization based on affine function of disturbances have appeared [17, 4, 12, 24, 25]. This parametrization is appealing as the resulting problem is convex and solvable via standard numerical routines. Specifically, [17] proposes the control parametrization

\[ u^L_t = \sum_{j=1}^{i} M^j_t w_{i-j} + v_i, \quad i = 0, \ldots, N - 1 \]  

where \( M^j_t \) and \( v_i \) are the optimization variables and \( N \) is the length of the horizon. [12] show that (3) is equivalent, in terms of the set of states reachable within the horizon, to that of time-varying affine state feedback. They also show that, under mild assumptions, the origin of the closed-loop system is input-to-state stable (ISS) with respect to the disturbance input under the MPC control law derived using (3) and a cost function that corresponds to the Linear Quadratic (LQ) cost for system (1) with \( w_t \equiv 0 \). Recently, [24] 2008 propose an extended disturbance feedback parametrization

\[ u^W_t = K_f x_t + c_t + \sum_{j=1}^{N-1} C^j_t w_{i-j}, \quad i = 0, \ldots, N - 1 \]  

where \( K_f \) is a fixed feedback gain, \( c_t \) and \( C^j_t \) are the optimization variables. They show that parametrization (4) under the MPC framework has the same domain of attraction as that of using (3). Using an appropriate cost function, they also show a stronger stability result: state of the closed-loop system converges to the minimal disturbance invariance set, \( F_\infty \), of the system \( x_{t+1} = (A + BK_f)x_t + w_t \). Unlike (3), index \( j \) runs from 1 to \( N - 1 \) in (4) and therefore \( i - j \) can be negative in \( w_{i-j} \). When this happens, \( w_{i-j} \) refers to past realized disturbances. This also means that the resulting MPC control law derived from (4) requires the values of \( x_t \) and \( w_{t-1}, \ldots, w_{N+1} \) for its evaluation at time \( t \).
This work proposes a new control parametrization based on (4) and a new cost function. The use of which results in an MPC control law requiring only the measurement of $x_t$ for its evaluation at time $t$. The resulting closed-loop system has the same domain of attraction as [17] and [12] but with a stronger convergence result: the closed-loop system state converges to $F_\infty$ with probability one; and deterministic convergence to the same set if a less intuitive cost function is used.

The rest of this paper is organized as follows. This section ends with notations used, assumptions needed and a brief review of standard results. Section 2 states the control parametrization, the finite horizon (FH) optimization problem and the cost function used. Section 3 discusses the computation of the MPC problem. The probabilistic convergence of the state of the closed-loop system is given in section 4. Section 5 shows a formulation that strengthens the convergence result under a weaker set of assumptions. This, however, requires the use of a somewhat less intuitive cost function. Numerical examples are the contents of section 6 and they are followed by the conclusions.

The following notations are used. $Z_k^+ := \{1, \cdots, k\}$ and $Z_k := \{0, 1, \cdots, k\}$ are the respective sets of positive and non-negative integers up to $k$. $\| \cdot \|$ is the standard 2-norm for matrices and vectors. Given matrices $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{p \times q}$: $\|A\|_F$ is the Frobenious norm, $\text{vec}(A) = [A_1^T \cdots A_m^T]^T \in \mathbb{R}^{nm}$ where $A_i$ is the $i^{th}$ column of $A$ is the stacked vector of columns of $A$ and $A \otimes B \in \mathbb{R}^{np \times qm}$ is the Kronecker product of $A$ and $B$. $A \succ 0$ means that square matrix $A$ is positive definite (semi-definite). For any $A \succ 0$, $\|x\|_A^2 = x^T A x$. $1_r$ is a $r$-element column vector with all elements being 1 and $I_n$ is the $n \times n$ identity matrix. For any set $X, Y \subset \mathbb{R}^n$, $X + Y := \{x + y : x \in X, y \in Y\}$ is the Minkowski sum of $X$ and $Y$.

The system (1)-(2) is assumed to satisfy the following assumptions:

(A1) system $(A, B)$ is stabilizable;

(A2) the set $Y$ is a polytope having a characterization

$$Y := \{(x, u) | Y_x x + Y_u u \leq 1_q\} \subset \mathbb{R}^{n+m} \quad (5)$$

for some $Y_x \in \mathbb{R}^{q \times n}$ and $Y_u \in \mathbb{R}^{q \times m}$;

(A3) the disturbance $w_t, t \geq 0$ are independent and identically distributed (i.i.d.) with zero mean and $W$ is a polytope characterized by

$$W := \{w | Hw \leq 1_r\} \subset \mathbb{R}^n, \quad (6)$$

for some $H \in \mathbb{R}^{r \times n}$;

(A4) a compact and non-empty constraint-admissible disturbance invariant set exists for system (1)-(2) under the feedback law $u = K_f x$ and takes the form

$$X_f := \{x | Gx \leq 1_y\} \subset \mathbb{R}^n \quad (7)$$

for some $G \in \mathbb{R}^{g \times n}$ and that $X_f$ contains the minimal disturbance invariant set of $x_{t+1} = (A + BK_f)x_t + w_t$ in its interior.
The above assumptions can be rationalized in the following ways. (A1) is standard. The characterization of \( Y \) in (A2) is made out of the need for a concrete computational representation. (A3) is a typical assumption on the disturbance and has been used in several prior works [11, 22]. That \( W \) is a polytope and contains the origin in its interior is an assumption made out of convenience for the presentation of the main result. Relaxation of (A3) is possible and the details are given in section 5 and illustrated via a numerical example in section 6. The existence of \( X_f \) in (A4) has been shown by [13] 1998 provided that \( W \) is sufficiently small. More exactly, for any feedback gain \( K_f \in \mathbb{R}^{m \times n} \) such that \( \Phi := A + BK_f \) is strictly stable and sufficiently small \( W \), \( X_f \) is the maximal, constraint admissible and disturbance invariant in the sense that \( \Phi x + w \in X_f, (x, K_f x) \in Y \) for all \( x \in X_f \) and for all \( w \in W \). It is also known [14, 9] that the state of the system \( x_{t+1} = \Phi x_t + w_t \) converges to the minimal disturbance invariant set, \( F_\infty \), given by

\[
F_\infty(K_f) := W + (A + BK_f)W + (A + BK_f)^2W + \cdots \tag{8}
\]

and that \( F_\infty \) is compact. The assumption that \( F_\infty(K_f) \subset X_f \) is also a natural consequence when \( W \) is not too large.

2 Control parametrization

MPC formulation solves an \( N \)-stage finite horizon (FH) optimization problem. Let \( x_i \) and \( u_i \), \( i \in \mathbb{Z}_{N-1} \) denote the predicted state and predicted control at the \( i \)th stage respectively within the horizon. The proposed control parametrization within the FH optimization problem takes the form

\[
u_i = K_f x_i + d_i + \sum_{j=1}^{i} D_j^i w_{i-j} \quad \text{for all } i \in \mathbb{Z}_{N-1} \tag{9}
\]

where \( d_i \in \mathbb{R}^m, D_j^i \in \mathbb{R}^{m \times n}, j \in \mathbb{Z}^+, i \in \mathbb{Z}_{N-1} \) are the variables of the FH problem and \( K_f \) is the given feedback gain in (A4). Since \( i - j \geq 0 \), \( w_{i-j} \) is the \((i - j)\)th predicted disturbance at each stage \( i \). In this regard, (9) is similar to (3) in that only predicted disturbances are used in the parametrization. In addition, \( u_i \) is equivalent to \( u_i^L \) and \( u_i^W \) in terms of the family of functions that they represent. To state this precisely, let

\[
\mathbf{d} := [d_0^T, d_1^T, \ldots, d_{N-1}^T]^T \in \mathbb{R}^{Nm}, \quad \mathbf{D} := \begin{bmatrix}
0 & \cdots & 0 & 0 \\
D_1^1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
D_{N-1}^{N-1} & \cdots & D_{N-1}^1 & 0
\end{bmatrix} \in \mathbb{R}^{Nm \times Nn} \tag{10}
\]

and \( \mathbf{v} \) and \( \mathbf{M} \) to be similarly defined as \( \mathbf{d} \) and \( \mathbf{D} \) in structure but having entries \( v_i \) instead of \( d_i \) and \( M_i^j \) instead of \( D_i^j \) respectively. The equivalence of \( u_i, u_i^L \) and \( u_i^W \) are stated in the following theorem.

**Theorem 1.** Suppose \( x_0 \) and \( K_f \) are given. Then (i) the set \( \{u_i\}_{i=0}^{N-1} \) of (9) is equivalent to \( \{u_i^L\}_{i=0}^{N-1} \) of (3) in the sense that for any \((\mathbf{v}, \mathbf{M})\), there exists a unique \((\mathbf{d}, \mathbf{D})\) such that \( \{u_i\}_{i=0}^{N-1} = \{u_i^L\}_{i=0}^{N-1} \) and vice versa. (ii) \( \{u_i\}_{i=0}^{N-1} \) is equivalent to \( \{u_i^W\}_{i=0}^{N-1} \) of (4).
Proof. See Appendix A.

The FH optimization problem under parametrization (9), referred hereafter as \( P_N(x) \), is

\[
\min_{d, D} J(d, D)
\]

\[
\text{s.t. } x_0 = x,
\]

\[
x_{i+1} = Ax_i + Bu_i + w_i, \quad i \in \mathbb{Z}_{N-1}
\]

\[
u_i = K_f x_i + d_i + \sum_{j=1}^{i} D_i^j w_{i-j}, \quad i \in \mathbb{Z}_{N-1}
\]

\[(x_i, u_i) \in Y, \quad \forall w_i \in W, \quad i \in \mathbb{Z}_{N-1}
\]

\[x_N \in X_f, \quad \forall w_i \in W, \quad i \in \mathbb{Z}_{N-1}
\]

where \( Y \) and \( X_f \) are the corresponding sets given by (5) and (7) respectively. The cost function \( J(d, D) \) takes the form

\[
J(d, D) := \sum_{i=0}^{N-1} \left[ \|d_i\|_Q^2 + \sum_{j=1}^{i} \|\text{vec}(D_i^j)\|_A^2 \right]
\]

for any choice of \( \Psi \in \mathbb{R}^{m \times m} \) and \( \Lambda \in \mathbb{R}^{mn \times mn} \) that satisfy

\[
\Psi = \Psi^T > 0, \quad \Lambda \succeq \Sigma_w \otimes \Psi
\]

where \( \Sigma_w \) is the covariance matrix of \( w_t \) and \( \text{vec}(\cdot) \) is stacking operator defined in Section 1. Clearly, \( J(d, D) \) is a measure of the deviation of \( u_i \) of (14) from the linear control law \( K_f x_i \) and the motivation for it as the objective function is clear: penalizing the use of non-zero \( (d, D) \) forces the asymptotic behavior of the closed-loop system to approach that of \( x_{i+1} = (A + BK_f)x_i \). The technical condition (18) is to ensure convergence of the closed-loop states and its exact role will become clear in the proof of Theorem 3. Several comments on \( J(d, D) \) are in order.

Remark 1. A connection between \( J(d, D) \) and the standard LQ cost can be established. Specifically, suppose \( Q \succeq 0 \) and \( R > 0 \) are given and let \( P = A^TPA - A^TPB(R+B^TPB)^{-1}B^TPA + Q \), the solution of the algebraic Riccati equation, \( \Psi = R + B^TPB, \quad \Lambda = \Sigma_w \otimes \Psi \) and \( K_f = -(R + B^TPB)^{-1}B^TPA \). It is shown [10, 24, 25] that

\[
E_w \left[ \sum_{i=0}^{N-1} \left( \|x_i\|_Q^2 + \|u_i\|_R^2 + ||x_N||_F^2 \right) \right] = x_0^T P x_0 + N \text{trace}(\Sigma_w P) + J(d, D)
\]

where \( E_w \) is the expectation taken over \( \{w_0, \ldots, w_{N-1}\} \) within the horizon. Since the first two terms on the right hand side of (19) are independent of \( (d, D) \), minimizing \( J(d, D) \) is equivalent to minimizing the expected infinite horizon LQ cost over the disturbance input.

Remark 2. From (18) and (19), it may appear that \( \Sigma_w \) is needed for the determination of \( \Lambda \). This is not true. The choice of \( \Lambda \) can be made to satisfy (18) even when \( \Sigma_w \) is not known accurately. One simple choice is to let \( \Lambda = \alpha^2 I_n \otimes \Psi \) where \( \alpha := \max_{w \in W} \|w\| \). Then it follows that \( \Lambda \succeq \Sigma_w \otimes \Psi \) because \( \alpha^2 I_n \succeq w w^T \) for all \( w \in W \) and \( \alpha^2 I_n \otimes \Psi \succeq E[ww^T] \otimes \Psi \). Consequently, (A3) provides for conditions that guarantee the computability of \( \max_{w \in W} \|w\| \).
Further discussion on the choice of $\Psi$ and $\Lambda$ and their influence on closed-loop system trajectories are discussed in section 4. Several associated sets, needed to facilitate the discussions in the sequel, are introduced. Let the feasible set of optimization problem $\mathcal{P}_N(x)$ be

$$T_N := \{(x, d, D)| (d, D) \text{ is feasible to } \mathcal{P}_N(x)\}$$

and the set of admissible initial states, or equivalently, the domain of attraction of the MPC controller is

$$\mathcal{X}_N := \{x| \exists (d, D) \text{ such that } (x, d, D) \in T_N\}.$$  \hspace{1cm} (21)

The rest of the MPC formulation is standard: $\mathcal{P}_N(x_t)$ is solved at each time $t$ to obtain the optimizer $(d_t^*, D_t^*) := (d^*(x_t), D^*(x_t))$ and the first control, $u_0^*$, is applied to (1) at time $t$ resulting in the MPC control law,

$$u_t = u_0^* = K_f x_t + d_0^*.$$ \hspace{1cm} (22)

Remark 3. From (20) and (21), it is easy to see that $T_N$ and $\mathcal{X}_N$ depend only on the constraints (12)-(16). If $u_t^W$ or $u_t^W$ replaces $u_t$ in (14), it follows from Theorem 1 that the corresponding domain of attraction, $\mathcal{X}_N^W$ and $\mathcal{X}_N^W$ is the same as $\mathcal{X}_N$. However, the stability of the corresponding closed-loop systems can differ. See Remark 6.

3 The Computation of $\mathcal{P}_N(x)$

Following (5), (7), (12)-(14), inequalities (15) and (16) can be collectively restated as

$$\bar{A}x + \bar{B}d + \max_{w \in W^N} [\bar{B}D + \bar{G}] w \leq 1_s$$

where $s = Nq + g$, $w := [w_0^T \ w_1^T \ \cdots \ w_{N-1}^T]^T$, $W^N := \{w| \bar{H}w \leq 1\}$ with $\bar{H} = I_N \otimes H$, $\ell = N\tau$ following (6). $\bar{A}$, $\bar{B}$ and $\bar{G}$ are appropriate matrices given in Appendix B and the max operator is meant to be taken row-wise. Correspondingly, the $i$th row of (23) can be rewritten as $\max\{e_i^T w| \bar{H}w \leq 1\} \leq b_i$ for some $e_i \in R^{Nm}$, $b_i \in R$ that depend on $x, d$ and $D$. Let $z_i \in R^\ell$ be the Lagrange multiplier corresponding to the rows of $\bar{H}$. Then, it follows by duality that $\max\{e_i^T w| \bar{H}w \leq 1\} = \min\{z_i^T 1_\ell| \bar{H}^T z_i = e_i, z_i \geq 0\}$. Collecting over all the rows of (23), the $\mathcal{P}_N(x)$ can be equivalently stated as

$$\min_{d, D, Z} \quad J(d, D)$$

s.t. $$\bar{A}x + \bar{B}d + Z^T 1_\ell \leq 1_s$$

$$Z^T \bar{H} = \bar{B}D + \bar{G}$$

$$z_i \geq 0, \ i = 1, \ldots, s$$

where $Z = [z_1 \cdots z_s] \in R^{\ell \times s}$ and the minimization of $Z$ is relaxed since the existence of any one feasible $Z$ is enough to guarantee that $(x, d, D) \in T_N$.

Remark 4. The above duality results can be extended to $W$ sets that are non-polyhedral. See, for example, treatments of such sets in [6, 20]. If $W$ is a second-order cone [16, 2] representable
bounded set with non-empty interior such that 
\[ W^N = \{ w \mid \| L_i w - l_i \| \leq \lambda_i^T w - \theta_i, \ i \in \mathbb{Z}_k^+ \} \]
for some matrices \( L_i, l_i, \lambda_i \) and \( \theta_i, \ i \in \mathbb{Z}_k^+ \), then it follows from duality that 
\[ \max \{ e^T w \mid w \in W^N \} = \min_{(\mu_i, \eta_i)} \left\{ \sum_{i=1}^k (\mu_i^T l_i - \eta_i \theta_i) \mid \right. \]
\[ \left. \sum_{i=1}^k (L_i^T \mu_i - \eta_i \lambda_i) = e, \ \| \mu_i \| \leq \eta_i, \ i \in \mathbb{Z}_k^+ \} \]. Similarly, if \( W \) is a bounded semi-definite cone representable set with non-empty interior such that 
\[ W^N = \{ \Omega \in R^{N_n} \mid \sum_{i=1}^{N_n} \Omega_i C_i - F \succ 0, \ i \in \mathbb{Z}_{N_n}^+ \} \]
where \( C_i \) and \( F \) are symmetrical matrices of appropriate dimension, then 
\[ \max \{ e^T w \mid w \in W^N \} = \min_Y \{ \text{Trace}(FT) \mid \text{Trace}(C_i Y) = e_i, \ i \in \mathbb{Z}_{N_n}^+, Y \preceq 0 \} \].

**Remark 5.** While the duality result is available for \( W \) being a second-order or semi-definite cone representable set, the availability of \( X_f \) satisfying (A4) deserves some clarifications. When \( W \) is non-polyhedral, computation of a constraint-admissible disturbance invariant set \( X_f \) may not be easy. A simple approach is to construct a polytope \( W_p \) such that \( W_p \supset W \) and \( W_p \approx W \). In that case, \( X_f \) satisfying (A4) can be constructed using \( W_p \) following existing computational methods [14]. Using this \( X_f \) in (16) and Remark 4, \( \mathcal{P}_N(x) \) becomes either a second-order cone or a semi-definite cone programming problem. It is worthy to note that the use of such an \( X_f \) in (16) and with \( w_t \in W \) for all \( i \in \mathbb{Z}_{N-1} \) in both (15) and (16) is less conservative than replacing \( W \) by \( W_p \) throughout (12)-(16). An example using such an approach is illustrated in Section 6.

4 Feasibility and Probabilistic Convergence

The feasibility of \( \mathcal{P}_N(x_t) \) at different time instants and stability of the closed-loop system under the feedback law (22) are addressed in this section.

**Theorem 2.** Suppose (A1)-(A4) are satisfied, the FH optimization problem \( \mathcal{P}_N(x) \) has the following properties: (i) \( T_N \) is convex and compact. (ii) If \( x \in X_N \), the optimal solution of \( \mathcal{P}_N(x) \) exists. (iii) If \( \mathcal{P}_N(x_t) \) admits an optimal solution, so does \( \mathcal{P}_N(x_{t+1}) \) under the feedback law (22) for all \( w_t \in W \).

**Proof.** See Appendix C.

The main result of probabilistic convergence of the state of the closed-loop system is stated in the next theorem.

**Theorem 3.** Suppose \( x_0 \in X_N \) and (A1)-(A4) are satisfied. System (1) under MPC control law (22) obtained from the solution of \( \mathcal{P}_N(x) \) using cost function (17) with condition (18) satisfied has the following properties: (i) \( x_t, u_t \in Y \) for all \( t \geq 0 \), (ii) \( x_t \to F_\infty(K_f) \) with probability one as \( t \to \infty \) (iii) \( x_t \) enters \( X_f \) in finite time with probability one.

**Proof.** See Appendix D.

**Remark 6.** It is of interest to know if the results of Theorem 3 can be extended to the case where \( u_t^f \) is used in (14) in view of Theorem 1. As seen in the proof of Theorem 3, the convergence property depends on the choices of the cost function and the control parametrization. Since (9) and (3) are equivalent when condition (36) is satisfied, the results of Theorem 3 is applicable under the following conditions: \( u_t^f \) is used in (14); a new cost function \( J^f(M, v) := J((I + \]
$K\varphi B)^{-1}(v - K\varphi Ax), (I + K\varphi B)^{-1}(M - K\varphi G)$ is used in (11) where $A, B, K, G$ and $\varphi$ are those given in (32) and (33); and $(M, v)$ becomes the optimization variables for $P_N(x)$. The approach by [12] uses the nominal LQ cost as the cost function and it is not clear if the results of Theorem 3 remains true under that situation.

One associated issue in the formulation of $P_N(x)$ is the choices of $\Psi$ and $\Lambda$ in $J(d, D)$. How should $\Psi$ and $\Lambda$ be chosen and how do these choices affect the closed-loop system trajectories? As $x_t \rightarrow F_\infty$ with probability one from result (ii) of Theorem 3, it implies that $x_t$ enters $X_f$ with probability one and stays within thereafter since $F_\infty \subset X_f$. When this happens, the optimal $(d, D)$ are zero in $P_N(x)$ and the MPC control law becomes $u_t = K_f x_t$ for all $t$ thereafter. The closed-loop system behavior then corresponds to that of the system $x_{t+1} = (A + BK_f)x_t$. Clearly, the choices of $\Lambda$ and $\Psi$ does not affect the asymptotic behavior of the system but only the transient when $x_t \notin X_f$.

Suppose $\Lambda = \Sigma_w \otimes \Psi$. Then admissible changes in $\Psi$ will not result in changes in the system behavior since $\Sigma_w \otimes \Psi$ is linear in $\Psi$. On the other hand, if $\Psi$ is fixed, $\Lambda$ can be chosen to be increasingly "larger" than $\Sigma_w \otimes \Psi$. In loose terms, a "larger" choice of $\Lambda$ penalizes the use of $D$ versus the use of $d$ in $J(d, D)$. Such a preference would mitigate the effect of the disturbance feedback component in the control parametrization, resulting in a parametrization that is closer in spirit to $u_t = K_f x_t + d_t$ of [8]. When this happens, the transient response for the system may become slower even though the domain of attraction $X_N$ remains unaffected. This observation together with the associated details used in the experimental study are discussed in section 6.

5 Deterministic Convergence

While the assumption of $W$ being a compact set is reasonable, the assumption of $w_t$ being zero mean and i.i.d. is harder to verify in practice. This section is concerned with the relaxation of assumption (A3) while achieving a stronger convergence result than that of Theorem 3. Consider

(A3a) $w_t \in W$ and $W$ is a polytope characterized by $W := \{w | Hw \leq 1_r \} \subset R^n$ for some $H \in R^{r \times n}$.

and define the cost function

$$V(d, D) := \sum_{i=0}^{N-1} \left( \|d_i\|^2_\Psi + \sum_{j=1}^{i} (\gamma_1 \|\text{vec}(D_j^i)\|^2 + \gamma_2 \|\text{vec}(D_j^i)\|) \right)$$

for some constants $\gamma_1$ and $\gamma_2$ satisfying

$$\gamma_1 \geq \alpha^2 \|\Psi\|, \quad \gamma_2 \geq 2\alpha\beta \|\Psi\|$$

where $\alpha := \max_{w \in W} \|w\|$ and $\beta := \max_{(x, d, D) \in T_N, i \in \mathbb{Z}_{N-1}} \|d_i\|$. The existence of $\alpha$ and $\beta$ are guaranteed by compactness of the $W$ and $T_N$ sets, provided for in (A3a) and part (i) of Theorem 2 respectively.
Theorem 4. Suppose \( x_0 \in X_N \) and \((A1\text{-}A2),(A3a)\) and \((A4)\) are satisfied and \( J(d,D) \) is replaced by \( V(d,D) \) in \( P_N(x) \) satisfying condition (29), then system (1) under the MPC control law \((22)\) satisfies (i) \((x_1,u_1) \in Y \) for all \( t \geq 0 \), (ii) \( x_t \to F_{\infty}(K_f) \) as \( t \to \infty \) (iii) \( x_t \) enters \( X_f \) in finite time.

Proof. See Appendix E

Remark 7. Several choices of the cost function of \((28)\) are possible. For example, the results of Theorem 4 remain true if \( \| \text{vec}(D^2) \| \) is replaced by \( \| D^2 \| \). This may be more appealing as less conservative bounds on \( \gamma_1 \) and \( \gamma_2 \) can be found to ensure the non-negativity of \( p(w_t) \). However, its use will result in a semi-definite programming problem for \( P_N(x) \) and is less desirable computationally. The use of \( \| \text{vec}(D^2) \| \) or \( \| D^2 \| \text{F} \) results in a second-order cone programming for \( P_N(x) \) and is computationally more amiable.

Remark 8. The computation of \( \beta \) can be simplified to \( \beta = \max_{(x,d,D) \in T_N} \| d_0 \| \), see Appendix F for details. Note that any upper bound of \( \beta \) can be used to guarantee the results of Theorem 4. One such upper bound is \( \beta := \| \sigma \| \) where \( \sigma_t := \max_{(x,d,D) \in T_N} |d_0(i)| \) and \( d_0(i) \) is the \( i \)th element of \( d_0 \).

6 Numerical Examples

Four experiments are conducted on a system to validate the results of the previous sections. The parameters and constraints of the system are: \( A = [1.1 1; 0 1.3], B = [1 1]^T, K_f = [-0.7434 - 1.0922], Y = \{(x,u) \mid u \leq 1, \|x\|_\infty \leq 8\} \) and \( W = \{H\hat{w} \mid \|\hat{w}\|_\infty \leq 0.2\} \) where \( H = [1 - 0.2; 0 1] \) and \( \hat{w} \in R^2 \) is a random vector uniformly distributed over \([-0.2,0.2] \times [-0.2,0.2] \) with covariance matrix \( \Sigma_{\hat{w}} = 0.0133I_2 \). Expressed in the form of \((5)\), the set \( Y_x = [0 0; 0 0; 0 1/8; 0 -1/8; 1/8 0; -1/8 0] \) and \( Y_u = [1;-1;0;0;0] \). The set \( X_f \) is the corresponding maximal constraint-admissible disturbance invariant set of \((1)\) under \( u_t = K_f x_t \) given by \( X_f = \{ x \mid Gx \leq 1_4 \} \), where \( G = [-0.7434 -1.0922; 0.7434 1.0922; 0.8252 -0.2391; -0.0282 0.2391] \).

The first experiment, Experiment I, uses the cost function of \((17)\) with \( \Psi = 1, \Lambda = \Lambda_{\text{op}} := \Sigma_w \otimes \Psi = [0.0139 -0.0027; -0.0027 0.0133], N = 8 \) and \( x_0 = [-4 2]^T \). The simulation results over 15 different disturbance realizations shown in Fig. 1 to 4 by solid lines. It is clear from Fig. 1 and 2 that the constraints are satisfied by all trajectories, in accordance to property (i) of Theorem 3. In addition, Figure 1 shows the convergence of \( x_t \) into \( F_{\infty}(K_f) \), a tight outer bound of \( F_{\infty}(K_f) \) obtained using procedures given in \([21]\). This convergence is further verified in Fig. 3 and 4 where the plots of \( \text{dis}(x_t,F_{\infty}) := \min_{x \in F_{\infty}} \| x - x_t \| \), the minimum distance to \( F_{\infty} \), and \( d_t := d_{0t}^* \) against increasing \( t \) are shown respectively.

The case where \( W \) is non-polyhedral is shown in Experiment II, in connection to Remarks 4 and 5. A different disturbance characteristic is used here: \( w \) is uniformly distributed over \( W := \{ \hat{w} \mid \|S_1\hat{w}\| \leq 1, \|S_2\hat{w}\| \leq 1 \} \) where \( S_1 = [5 1; 0 2.5] \) and \( S_2 = [2.5 0.5; 0 5] \). Note that a tight bounding polytope, \( W_p \), such that \( W_p \supset W \), is needed for the computation of \( X_f \) satisfying \((A4)\) and it corresponds to the \( W \) set of Experiment I (see Fig. 5). Also, \( \Psi \) and \( \Lambda \) of the first experiment are used and it is easy to verify that condition \((18)\) remains true.
because $\Sigma w \succ \Sigma \bar{w}$. In this case, $\bar{W}$ is a second-order cone representable set and the conversion of (15) and (16) for all $w_i \in \bar{W}$ follows the expression in Remarks 4, resulting in $\mathcal{P}_N(x)$ being a second-order cone programming problem. The simulation results with $N = 8$ and $x_0 = [2 -1]^T$ for 15 different realizations of $\{w_i\}$ are plotted in Fig. 1 to 4 using dash-dot lines.

Experiment III is designed to understand the influence of $\Lambda$ and $\Psi$ of (17) on the performance of the closed-loop system. As stated in section 4, choices of these matrices affect only the transient behavior when $x_t \notin X_f$ and not the asymptotic behavior of the closed-loop system. To quantify the transient, the average number of time step, $t_f(x_0)$, taken to enter $X_f$ from a given $x_0$ is reported. Here, the average is taken over different realizations of the disturbances. Without loss of generality, values of $\Lambda$ is increased from $\Lambda_{op}$ (see discussion in section 4). Table 1 shows the values of $t_f(x_0)$ and the associated standard deviations over 20 disturbance realizations for several choices of $x_0$, $N$ and $\Lambda$. For each $x_0$, the same 20 disturbance realizations are used for the different $\Lambda$ in computing $t_f(x_0)$ and the standard deviations. From the table, $t_f(x_0)$ generally increases when $\Lambda$ increases. For comparison purpose, the corresponding trajectories of the system under same settings as the Experiment I except for $\Lambda = 10^4\Lambda_{op}$ are plotted in Fig. 1 to 4 using dash lines. From Fig. 3 and 4, the slower convergence of the state and control trajectories are clearly evident.

The last experiment, Experiment IV, considers the case discussed in Section 5. The system parameters are the same as those in the first experiment except that the distribution of $\bar{w}$ is assumed to be unknown. The parameters of (29) are: $\alpha = 0.3124$ and $\beta = 2.7307$ (when $N = 8$) and $\beta = 3.5425$ (when $N = 10$). Correspondingly, the weight matrices of (28) are $\Psi = 1$, $\gamma_1^{op} := \alpha^2\|\Psi\| = 0.0976$, $\gamma_2^{op} := 2\alpha\beta\|\Psi\| = 1.7059$ (2.213 when $N = 10$). The values of $\gamma_1$ and $\gamma_2$ are increased separately and jointly to assess their influence on the system behavior.

The general effect of increasing values of $\gamma_1$ and $\gamma_2$ appears to have similar trend on the system as the increase in $\Lambda$. The time taken to reach $X_f$ from any given $x_0$ increases, although to a lesser percentage than that by $\Lambda$, with increasing values of $\gamma_1$ and $\gamma_2$ with $\gamma_2$ having a heavier influence.
Figure 2: Control trajectories of the first three experiments: solid line for first experiment, dashdot line for the second and dash line for the third.

Figure 3: Distance between states and $F_\infty(K_f)$ of the first three experiments: solid line for first experiment, dashdot line for the second and dash line for the third.
Figure 4: Values of $d_t$ of the first three experiments: solid line for first experiment, dashdot line for the second and dash line for the third.

Figure 5: $W_p$ set and $\bar{W}$ set.
Table 1: Average time step, \( t_f(x_0) \), and its standard deviation

<table>
<thead>
<tr>
<th>Initial Condition</th>
<th>( x_0 )</th>
<th>( N )</th>
<th>( \Lambda_{op} )</th>
<th>( 10^2 \Lambda_{op} )</th>
<th>( 10^4 \Lambda_{op} )</th>
<th>( (\gamma_1^{op}, \gamma_2^{op}) )</th>
<th>( (10\gamma_1^{op}, \gamma_2^{op}) )</th>
<th>( (\gamma_1^{op}, 10\gamma_2^{op}) )</th>
<th>( (10\gamma_1^{op}, 10\gamma_2^{op}) )</th>
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</thead>
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<td>4.2</td>
<td>(0.4104)</td>
<td>4.9</td>
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<td>5.2</td>
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<tr>
<td>([-2.5 - 1.2]^T]</td>
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<td>4.9</td>
<td>(0.5525)</td>
<td>5.15</td>
<td>5.15</td>
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<td>5.15</td>
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</tr>
<tr>
<td>([-4 - 1]^T]</td>
<td>10</td>
<td>6.3</td>
<td>(0.7327)</td>
<td>6.75</td>
<td>7.45</td>
<td>7.2</td>
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<td>([-6 2]^T]</td>
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<td>8.6</td>
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</table>
7 Conclusions

Convergence results for constrained linear system under MPC control law using a new control parametrization and a new cost function are presented. The parametrization uses affine disturbance feedback together with a linear state feedback term, $K_F x$, and is a modification of the parametrization by [25]. Such a parametrization is similar to existing disturbance parameterizations in the literature in terms of the set of state reachable. Using the proposed cost function and the new parametrization, the closed-loop system state converges to the minimal robust invariant set $F_{\infty}(K_f)$ with probability one. Deterministic convergence to $F_{\infty}(K_f)$ is also possible using a less intuitive cost function. The asymptotic behavior of the closed-loop system is determined by the choice of $K_f$ so long as the weight matrices of the cost function satisfy some mild conditions.

References


Proof. (i) Suppose the predicted states, predicted controls and predicted disturbances within the horizon are $\mathbf{x} := [x^T_0 \, x^T_1 \, \cdots \, x^T_N]^T \in \mathbb{R}^{(N+1)n}$, $\mathbf{u} := [u^T_0 \, u^T_1 \, \cdots \, u^T_{N-1}]^T \in \mathbb{R}^{Nm}$, and $\mathbf{w} := [w^T_0 \, w^T_1 \, \cdots \, w^T_{N-1}]^T \in \mathbb{R}^{Nn}$. The state $\mathbf{x}$ and the and the control sequence, $\{u_i\}_{i=0}^{N-1}$, defined by (9) can be equivalently stated as

$$\begin{align*}
\mathbf{x} &= A\mathbf{x}_0 + B\mathbf{u} + G\mathbf{w} \\
\mathbf{u} &= \mathbf{K}\mathbf{x} + \mathbf{d} + \mathbf{D}\mathbf{w}
\end{align*}$$

where

$$A := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \quad G := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I \end{bmatrix}, \quad \mathbf{K} = [I_N \otimes K_f \, 0]$$

and $\mathbf{d}$ and $\mathbf{D}$ are those given by (10). Using (31) in (30), we get $\mathbf{x} = \varphi A\mathbf{x}_0 + \varphi B\mathbf{d} + (\varphi BD + \varphi G)\mathbf{w}$ where

$$\varphi = (I - BK)^{-1}$$

and $\mathbf{u}$ becomes

$$\mathbf{u} = \mathbf{K}\varphi A\mathbf{x}_0 + (I + \mathbf{K}\varphi \mathbf{B})\mathbf{d} + [(I + \mathbf{K}\varphi \mathbf{B})\mathbf{D} + \mathbf{K}\varphi G]\mathbf{w}$$

Consider the parametrization (3) and the sequence $\{u^L_i\}_{i=0}^{N-1}$ expressed by the variables $(\mathbf{M}, \mathbf{v})$ of (10). It follows that

$$\mathbf{u}^L = \mathbf{v} + \mathbf{M}\mathbf{w}$$

Comparing (34) and (35), $\mathbf{u} = \mathbf{u}^L$ if and only if

$$\begin{cases} \\
K\varphi A\mathbf{x}_0 + (I + K\varphi \mathbf{B})\mathbf{d} = \mathbf{v} \\
(I + K\varphi \mathbf{B})\mathbf{D} + K\varphi \mathbf{G} = \mathbf{M}.
\end{cases}$$

Note that $(I + K\varphi \mathbf{B})$ is a lower triangular matrix with all diagonal elements being 1 and is always invertible. In addition, $K\varphi \mathbf{G}$ is a strict lower triangular block matrix like $\mathbf{M}$ and its multiplication by $(I + K\varphi \mathbf{B})^{-1}$ on the left also results in a strict lower triangular block matrix. Hence, the mapping between $(\mathbf{M}, \mathbf{v})$ and $(\mathbf{D}, \mathbf{d})$ by (36) is unique or one-to-one for all choices of $\mathbf{K}$ and $\mathbf{x}_0$. This establishes the equivalence of $\{u_i\}_{i=0}^{N-1}$ and $\{u^L_i\}_{i=0}^{N-1}$.


(ii) Set \( C_i^j = 0 \) for all \( j > i \) in (4) and it follows that \( u_i \) is a special case of \( u_i^W \). To show the converse, let

\[
\begin{align*}
    d_i &= c_i + \sum_{j=i+1}^{N-1} C_i^j w_{i-j}, \quad i \in \mathbb{Z}_{N-1} \\
    D_i^j &= C_i^j \\
\end{align*}
\]

for any \( c_i, C_i^j \) that defines \( u_i^W \). This establishes the equivalence of \( \{u_i\}_{i=0}^{N-1} \) and \( \{u_i^W\}_{i=0}^{N-1} \). □

\[ \text{B Expressions of Matrices in (23)} \]

The \( \bar{A}, \bar{B} \) and \( \bar{G} \) in (23) are

\[
\bar{A} = \mathcal{Y} \left[ \begin{array}{c} \varphi A \\ K \varphi A \end{array} \right], \quad \bar{B} = \mathcal{Y} \left[ \begin{array}{c} \varphi B \\ I + K \varphi B \end{array} \right], \quad \bar{G} = \mathcal{Y} \left[ \begin{array}{c} \varphi G \\ K \varphi G \end{array} \right], \quad \mathcal{Y} = \left[ \begin{array}{ccc} I_N \otimes Y_x & 0 & I_N \otimes Y_u \\ 0 & G & 0 \end{array} \right]
\]

where \( A, B, G, K \) and \( \varphi \) are defined in (32) and (33).

\[ \text{C Proof of Theorem 2} \]

Proof. (i) Since \( Y \) is compact from (A2), the projection of \( Y \) onto \( x \) and \( u \) space, denoted by \( X_Y \) and \( U_Y \) respectively, are bounded. From (9) and the fact that \( W \) is a polytope in \( R^n \), \( D_i^j \) must be bounded in order for \( U_i \in U_Y \). Since the origin is inside \( W \), \( K_i x_i + d_i \) must be inside \( U_Y \). Therefore, \( d_i \) is bounded as \( x_i \) and \( U_Y \) are bounded. This, together with \( T_N \) being closed and convex from (25)-(27) leads to the desired result.(ii) Since \( x \in X_N \), \( P_N(x) \) is feasible. From (i), this means that \( \Pi_N(x) := \{(d, D) | (x, d, D) \in T_N\} \) is compact. This, together with the fact that \( J(d, D) \) is continuous with respect to \( (d, D) \) means that the optimal solution exists by Weierstrass’ Theorem [7]. (iii) The proof follows standard arguments but the details are given for their relevance to Theorem 3. For clarity, additional subscripts “\( t \)” and “\( t+1 \)” are used to denote the variables at the different times. Let \((\mathbf{d}_t^*, \mathbf{D}_t^*)\) denote the optimal solution of \( \mathcal{P}_N(x_t) \). At time \( t+1 \), \( w_t \) is realized and \((\mathbf{d}_{t+1}, \mathbf{D}_{t+1})\) is chosen as

\[
\begin{align*}
    \hat{d}_{i[t+1]} &= \begin{cases} 
    d_{i+1[t]} + (D_{i+1[t]}^*)^\top w_t & i \in \mathbb{Z}_{N-2} \\
    0 & i = N - 1 
    \end{cases} \\
    \hat{D}_{i[t+1]} &= \begin{cases} 
    (D_{i+1[t]}^*)^\top & j \in \mathbb{Z}_i^+, \ i \in \mathbb{Z}_{N-2}^+ \\
    0 & j \in \mathbb{Z}_{N-1}^+, \ i = N - 1 
    \end{cases}
\end{align*}
\]

This choice of \((\mathbf{d}_{t+1}, \mathbf{D}_{t+1})\) is feasible to \( \mathcal{P}_N(x_{t+1}) \) for all possible \( w_t \in W \) due to the disturbance invariance of \( X_f \) for system (1) under control law \( u_t = K_f x_t \) and that \((\mathbf{d}_t^*, \mathbf{D}_t^*)\) is the optimal solution at time \( t \). That the the optimum of \( \mathcal{P}_N(x_{t+1}) \) exists follows from the compactness of \( \Pi_N(x_{t+1}) \) and the Weierstrass’ theorem [7]. □
D Proof of Theorem 3

Proof. (i) The stated result follows directly from Theorem 2. (ii) Let $J_t^* := J(d_t^*, D_t^*)$ and $\tilde{J}_{t+1}(w_t) := J(d_{t+1}(w_t), D_{t+1})$ where $(d_{t+1}(w_t), D_{t+1})$ are given by (38)-(39). Then it follows that

$$J_t^* - \tilde{J}_{t+1}(w_t) = \sum_{i=0}^{N-1} (\|d_{it}^*\|^2 - \|\hat{d}_{it+1}\|^2) + \sum_{i=1}^{N-1} \|\text{vec}(D_{it}^*)\|^2_\Lambda$$

$$= d_{0t}^* + \sum_{i=1}^{N-1} (\|d_{it}^*\|^2 - \|\hat{d}_{i-1}\|^2) + \sum_{i=1}^{N-1} \|\text{vec}(D_{it}^*)\|^2_\Lambda$$

$$= d_{0t}^* + \sum_{i=1}^{N-1} (\|d_{it}^*\|^2 + (D_{it}^*)^* w_t \|\|^2_\Psi) + \sum_{i=1}^{N-1} \|\text{vec}(D_{it}^*)\|^2_\Lambda$$

$$= d_{0t}^* + g(w_t)$$

(40)

where

$$g(w_t) = \sum_{i=1}^{N-1} (\|\text{vec}(D_{it}^*)\|^2_\Lambda - 2(d_{it}^*)^T \Psi(D_{it}^*)^* w_t - \|D_{it}^* w_t\|^2_\Psi).$$

(41)

Taking the expectation of (40) over $w_t$, it follows that

$$J_t^* - d_{0t}^* = E_{w_t} \left[ \tilde{J}_{t+1}(w_t) \right] + E_{w_t} [g(w_t)]$$

$$\geq E_{w_t} \left[ \tilde{J}_{t+1}(w_t) \right]$$

$$\geq E_{w_t} \left[ J_{t+1}^*(w_t) \right] = E_t \left[ J_{t+1}^*(w_t) \right].$$

(42)

where $E_t$ in (43) is the expectation taken over $w_i, i \geq t$. Inequality (42) follows from the fact that $E_{w_t} [g(w_t)] \geq 0$. This is so because by taking the expectation of (41), one gets

$$E_{w_t} [g(w_t)] = \sum_{i=1}^{N-1} (\|\text{vec}(D_{it}^*)\|^2_\Lambda - \|\text{vec}(D_{it}^*)\|^2_\Lambda - 2(d_{it}^*)^T \Psi(D_{it}^*)^* E[w_t])$$

(44)

where the last term is zero due to (A3) and the rest is non-negative due to (18). The inequality in (43) follows from the fact that $J_{t+1}(w_t) \geq J_{t+1}^*(w_t)$ for every $w_t \in W$ which implies that $E_{w_t} [\tilde{J}_{t+1}(w_t)] \geq E_{w_t} [J_{t+1}^*(w_t)]$. Equality (43) follows from the fact that $J_{t+1}^*(w_t)$ depends on $w_t$ only and not on any $w_i, i > t$.

Repeating the inequality of (43) for increasing $t$, one gets

$$J_{t+1}^*(x_{t+1}) - d_{0t}^*(x_{t+1}) \|\|^2_\Psi \geq E_{w_{t+1}} [J_{t+2}^*(x_{t+1}, w_{t+2})]$$

where the dependence of the various quantities on $x_{t+1}$ are added for clarity. Since $x_{t+1}$ depends on $x_t$ and $w_t$, the above can be equivalently written as

$$J_{t+1}^*(w_t) - d_{0t}^*(w_t) \|\|^2_\Psi \geq E_{w_{t+1}} [J_{t+2}^*(w_t, w_{t+1})].$$

(45)
The above inequality holds true for all possible $w_t$, hence

$$E_{w_t}[J_{t+1}^*(w_t)] - E_{w_t}[\|d_{0|^t+1}(w_t)\|_\Psi^2] \geq E_{w_t}[E_{w_{t+1}}[J_{t+2}^*(w_t, w_{t+1})]] = E_t[J_{t+2}^*(w_t, w_{t+1})]$$

(46)

or

$$E_t[J_{t+1}^*(w_t)] - E_t[\|d_{0|^t+1}(w_t)\|_\Psi^2] \geq E_t[J_{t+2}^*(w_t, w_{t+1})]$$

(47)

The equality in (46) follows from the i.i.d. assumption in (A3), particularly,

$$E_t[E_{w_{t+1}}[J_{t+2}^*(w_t, w_{t+1})]] = \int \int J_{t+2}^*(w_t, w_{t+1})f_{w_{t+1}}(w_{t+1})dw_{t+1}f_{w_t}(w_t)dw_t$$

$$= \int \int J_{t+2}^*(w_t, w_{t+1})f_{w_{t+1}, w_{t+1}}(w_t, w_{t+1})dw_{t+1}dw_t$$

$$= E_{w_{t+1}, w_{t+1}}[J_{t+2}^*(w_t, w_{t+1})] = E_t[J_{t+2}^*(w_t, w_{t+1})]$$

where $f_{w_t}(\cdot)$, $f_{w_{t+1}}(\cdot)$ and $f_{w_{t+1}, w_{t+1}}(\cdot, \cdot)$ are density functions of $w_t$, $w_{t+1}$ and their joint density function respectively and $f_{w_{t+1}, w_{t+1}}(\cdot, \cdot) = f_{w_t}(\cdot)f_{w_{t+1}}(\cdot)$ follows from assumption (A3). Summing (43) and (47) leads to

$$J_t^* \geq \|d_{0|t}^*\|_\Psi^2 + E_t[\|d_{0|^t+1}(w_t)\|_\Psi^2] + E_t[J_{t+2}^*(w_t, w_{t+1})]$$

Repeating the above procedure infinite times leads to

$$\infty > J_t^* \geq \|d_{0|t}^*\|_\Psi^2 + \sum_{i=t+1}^{\infty} E_t[\|d_{0|i}(w_t, \cdots, w_{t-1})\|_\Psi^2] + \lim_{r \to \infty} E_t[J_{t+r}^*(w_t, \cdots, w_{t+r-1})]$$

where the left inequality follows from Theorem 7(ii). Using the fact that $\lim_{r \to \infty} E_t[J_{t+r}^*(w_t, \cdots, w_{t+r-1})] > 0$ and $\|d_{0|t}^*\|_\Psi^2$ is finite, we have

$$\infty > \sum_{i=t+1}^{\infty} E_t[\|d_{0|i}(w_t, \cdots, w_{t-1})\|_\Psi^2]$$

By applying Markov bound (given non-negative random variable $\eta$ and any $\epsilon \geq 0$, $E[\eta] \geq \epsilon Pr\{\eta \geq \epsilon\}$) and considering $\|d_{0|t}^*\|_\Psi^2$ as a random number, we have

$$\infty > \epsilon \sum_{i=t+1}^{\infty} Pr(\|d_{0|i}^*\|_\Psi^2 \geq \epsilon)$$

(48)

for any arbitrary small $\epsilon > 0$. From the First Borel-Cantelli Lemma [26], this implies that $\lim_{t \to \infty} Pr(\|d_{0|i}^*\|_\Psi^2 \geq \epsilon) = 0$. Hence $d_{0|i} \to 0$ with probability one as $t$ increases. Consequently, the MPC control law (22) converges to $K_{\infty}x_t$ with probability one. When this happens, the closed-loop system converges to $x_{t+1} = \Phi x_t + w_t$ and, hence, $x_t$ converges to $F_{\infty}(K_f)$ with probability one. (iii) follows directly from (ii) and assumption (A4) that $F_{\infty}(K_f) \subset \text{int}(X_f)$. □ □
E  Proof of Theorem 4

Proof. (i) The replacement of cost function $J(d,D)$ by $V(d,D)$ does not affect the feasibility of problem $P_N(x)$. This means that part (i) of Theorem 3 remains valid. (ii) Let $V_t^*$ and $\hat{V}_{t+1}$ be defined in the same manner as $J_t^*$ and $\hat{J}_{t+1}$ in the statement of proofs of Theorem 3. Following the same reasoning as in (40), it can be shown that

$$V_t^* - \hat{V}_{t+1}(w_t) = \|d_{0|t}\|^2_P + p(w_t)$$

where

$$p(w_t) = \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_{i|t})^*\|^2 + \gamma_2 \|\text{vec}(D_{i|t})^*\| - 2(d_{i|t})^T \Psi (D_{i|t})^* w_t - \|(D_{i|t})^* w_t\|^2_P).$$

Hence

$$p(w_t) \geq \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_{i|t})^*\|^2 + \gamma_2 \|\text{vec}(D_{i|t})^*\| - 2\alpha \beta \|\Psi\| \|\text{vec}(D_{i|t})^*\|_F - \alpha^2 \|\Psi\| \|\text{vec}(D_{i|t})^*\|^2)$$

$$\geq \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_{i|t})^*\|^2 + \gamma_2 \|\text{vec}(D_{i|t})^*\| - 2\alpha \beta \|\Psi\| \|\text{vec}(D_{i|t})^*\|_F) (51)$$

$$= \sum_{i=1}^{N-1} (\gamma_1 \|\text{vec}(D_{i|t})^*\|^2 + \gamma_2 \|\text{vec}(D_{i|t})^*\| - 2\alpha \beta \|\Psi\| \|\text{vec}(D_{i|t})^*\|_F)$$

where the facts $\|(D_{i|t})^*\|_F \leq \|(D_{i|t})^*\|_F$ and $\|(D_{i|t})^*\|_F = \|\text{vec}(D_{i|t})^*\|$ are used. Hence, $p(w_t) \geq 0$ under (29). As a consequence, equation (49) implies

$$V_t^* - \|d_{0|t}\|^2_P \geq V_{t+1}^* \geq 0$$

(52)

Hence, $\{V_t^*\}$ is a monotonic non-increasing sequence and is bounded from below by zero. This means that $V_\infty := \lim_{t \to \infty} V_t^* \geq 0$ exists. Repeating (52) for $t$ from 0 to $\infty$ and summing them up, it follows that

$$\infty > V_0^* - V_\infty \geq \sum_{t=0}^\infty \|d_{0|t}\|^2_P$$

(53)

Since $\Psi$ is positive definite, this implies that $\lim_{t \to \infty} d_{0|t}^* = 0$ and $\lim_{t \to \infty} u_t = K \tilde{x}_t$. Therefore, the stated result follows. (iii) follows (ii) and assumption (A4) that $F_\infty(K_f) \subset \text{int}(X_f)$. □ □

F  Computation of $\beta$

$\beta := \max_{(x,d,D) \in T_N} \|d_i\| = \max_{(x,d,D) \in T_N} \|d_0\|$ is due to the fact that for any $(x,d,D) \in T_N$ and integer $i \in \mathbb{Z}_{N-1}^+$, a set of $(\bar{x},\bar{d},\bar{D}) \in T_N$ can be found such that $\tilde{d}_0 = d_i$. Specifically, given $(x,d,D) \in T_N$ and let the correspondingly defined state and control sequence be
\{x_0, \ldots, x_N\} and \{u_0, \ldots, u_{N-1}\}. According to (16) \(x_N \in X_f\) for all possible disturbances. Then for any \(i \in \mathbb{Z}_{N-1}^+\), \((\bar{x}, \bar{d}, \bar{D})\) can be defined by

\[
\bar{x} = \Phi^i x + \sum_{j=0}^{i-1} \Phi^{i-j} B d_j, \quad \bar{d}_j = \begin{cases} d_{j+i} & j \in \mathbb{Z}_{N-1-i} \hspace{1cm} 0 \leq j \leq N-1, \\ 0 & N - i \leq j \leq N - 1 \end{cases}, \quad \bar{D}_j^k = \begin{cases} D_{j+i}^k & j \in \mathbb{Z}_{N-1-i}^+ \hspace{1cm} k \in \mathbb{Z}_{N-1-i}^+, \\ 0 & N - i \leq j \leq N - 1 \end{cases}
\]

where \(\bar{x}\) is the nominal state of \(x_i\) defined by \((x, d, D)\) and \((\bar{d}, \bar{D})\) define the control sequence \(\{u_i, \ldots, u_{N-1}\}\). According to (A4) under controller \(u_t = K_f x_t\) all the constraints are satisfied and \(x_t \in X_f\) for \(t \geq N\) since \(x_N \in X_f\). Therefore, \((\bar{x}, \bar{d}, \bar{D})\) satisfies (12)-(16), namely \((\bar{x}, \bar{d}, \bar{D}) \in T_N\). As a result, \(\max_{(\bar{x}, \bar{d}, \bar{D}) \in T_N} \|d_0\| \geq \max_{(x, d, D) \in T_N} \|d_i\|\), for any \(i \in \mathbb{Z}_{N-1}\) and \(\beta = \max_{(x, d, D) \in T_N} \|d_0\|\).