A Note on ”Fuzzy Initial Value Problem for Nth-Order Fuzzy Linear Differential Equations”

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Abstract

Buckley and Feuring [1] solved a fuzzy linear differential equation and on the basis of the obtained solution, they introduced a new result. The main aim of this note is to show that the solution of the fuzzy linear differential equation, obtained by Buckley and Feuring [1], is incorrect and due to which the result, introduced by Buckley and Feuring [1], is also incorrect.

Keywords: Fuzzy numbers; Simultaneous differential equations; Fuzzy linear differential equations.

1 Existing method to find the solution of fuzzy initial value problem for \(n^{th}\) order fuzzy linear differential equation

Buckley and Feuring [1] proposed a new method to find the solution of the following fuzzy differential equation

\[
y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{(1)} + a_0(x)y = g(x),
\]

where the \(a_i(x), 0 \leq i \leq n - 1\), and \(g(x)\) are continuous on some interval \(I\), subject to initial conditions \(y(0) = \bar{\gamma}_0, y^{(1)}(0) = \bar{\gamma}_1, \ldots, y^{(n-1)}(0) = \bar{\gamma}_{n-1}\), for fuzzy numbers \(\bar{\gamma}_i, 0 \leq i \leq n - 1\). The interval \(I\) can be [0, \(T\]) for some \(T > 0\) or \(I = [0, \infty)\).

To find the solution of fuzzy linear differential Eq. (1.1), first solve the fuzzy initial value problem and then check to see if it defines a fuzzy function for \(x\) in \(I\).

Let \(\bar{Y}(x)\) denote the fuzzy subset of \(\mathbb{R}\) for each \(x \in I\) so that its \(\alpha\)-cuts are closed, bounded, intervals for all \(x\). Set \(\bar{Y}(x) = [y_1(x, \alpha), y_2(x, \alpha)], x \in I, \alpha \in [0, 1]\). We substitute

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the $\alpha$-cuts of $\bar{Y}(x)$ into the differential equation and then solve for $y_1(x, \alpha)$ and $y_2(x, \alpha)$. Then $y_i(x, \alpha)$ are assumed to have continuous derivatives on $x$ of order $n$ for all $\alpha$. From Eq. (1.1), Buckley and Feuring [1] obtained

$$\left[ y^n_1(x, \alpha), y^n_2(x, \alpha) \right] + a_{n-1}(x)[y^{n-1}_1(x, \alpha), y^{n-1}_2(x, \alpha)] + \cdots + a_0(x)[y_1(x, \alpha), y_2(x, \alpha)] = \left[ g(x), g(x) \right]$$

subject to the initial conditions:

1. $y_{1}(0, \alpha) = \gamma_{01}(\alpha), \ldots, y^{(n-1)}_{1}(0, \alpha) = \gamma_{n-1,1}(\alpha)$
2. $y_{2}(0, \alpha) = \gamma_{02}(\alpha), \ldots, y^{(n-1)}_{2}(0, \alpha) = \gamma_{n-1,2}(\alpha),$

to be solved for the $y_i(x, \alpha), i = 1, 2$.

The symbol $y^k_i(x, \alpha)$ is the $k$th derivative on $x$ for fixed $\alpha \in [0, 1], i = 1, 2$. One does interval arithmetic in Eq. (1.2) to obtain two equations to solve simultaneously for $y_{1}(x, \alpha)$ and $y_{2}(x, \alpha)$.

2 Existing Results

Buckley and Feuring [1] pointed out that for $a < 0, b \geq 0$ with roots $r_1 < r_2$, Eq. (1.2) can be converted into the following differential equations

$$y''_{1}(x, \alpha) + ay'_{2}(x, \alpha) + by_{1}(x, \alpha) = g(x) \quad (2.3)$$
$$y''_{2}(x, \alpha) + ay'_{1}(x, \alpha) + by_{2}(x, \alpha) = g(x) \quad (2.4)$$

with $y_{1}(0, \alpha) = \gamma_{01}(\alpha), y'_{1}(0, \alpha) = \gamma_{11}(\alpha), y_{2}(0, \alpha) = \gamma_{02}(\alpha)$ and $y'_{2}(0, \alpha) = \gamma_{12}(\alpha)$.

Buckley and Feuring [1] also claimed that on solving Eq. (2.3) and Eq. (2.4), the following solution is obtained ([1], equation (55), (56), pp. 252)

$$y_{2} = c_{1}e^{r_{1}x} + c_{2}e^{r_{2}x} + G(x) \quad (2.5)$$
$$y_{1} = c_{3}e^{r_{1}x} + c_{4}e^{r_{2}x} + G(x) \quad (2.6)$$

where, $c_{1} = c_{3}$ and $c_{2} = c_{4}$, i.e., $y_{1} = y_{2}$.

On the basis of the obtained result, Buckley and Feuring [1] pointed out that for $a < 0$ and $b \geq 0$, the set of simultaneous differential Eq. (2.3) and Eq. (2.4) does not have any fuzzy solution.

3 Drawback of the existing result

Using Maple software, the solution of simultaneous differential Eq. (2.3) and Eq. (2.4), is

$$y_{1} = c_{1}e^{r_{1}x} + c_{2}e^{r_{2}x} + c_{3}e^{r_{1}x} + c_{4}e^{r_{2}x} + G(x) \quad (3.7)$$
\[ y_2 = \frac{1}{8bc}(c_1e^{r_1x}(a^2 - 4b)^{3/2} - c_2e^{r_2x}(a^2 - 4b)^{3/2} + c_3e^{r_3x}(a^2 - 4b)^{3/2} - c_4e^{r_4x}(a^2 - 4b)^{3/2} \\
- a^2c_1e^{r_1x}\sqrt{a^2 - 4b} + a^2c_2e^{r_2x}\sqrt{a^2 - 4b} - a^2c_3e^{r_3x}\sqrt{a^2 - 4b} + a^2c_4e^{r_4x}\sqrt{a^2 - 4b} \\
- 8c_1e^{r_1x}ab + 4c_1e^{r_1x}b\sqrt{a^2 - 4b} - 8c_2e^{r_2x}ab - 4c_2e^{r_2x}b\sqrt{a^2 - 4b} + 8c_3e^{r_3x}ab \\
+ 4c_3e^{r_3x}b\sqrt{a^2 - 4b} + 8c_4e^{r_4x}ab - 4c_4e^{r_4x}b\sqrt{a^2 - 4b} + G(x) \tag{3.8} \]

where,
\[ r_1 = \frac{1}{2}(a + \sqrt{a^2 - 4b}), \quad r_2 = -\frac{1}{2}(-a + \sqrt{a^2 - 4b}), \quad r_3 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \quad r_4 = -\frac{1}{2}(a + \sqrt{a^2 - 4b}) \]

It is obvious from equations Eq. (3.7) and Eq. (3.8) that \( y_1 \neq y_2 \) i.e., for \( a < 0, b \geq 0 \) the fuzzy solution of Eq. (2.3) and Eq. (2.4) may or may not exist which shows that the result claimed by Buckley and Feuring is incorrect.

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**References**

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