New hyperkähler manifolds by fixing monopoles

Conor J. Houghton

Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Silver St., Cambridge CB3 9EW, England.
Email C.J.Houghton@damtp.cam.ac.uk

February 1997

Abstract

The construction of new hyperkähler manifolds by taking the infinite monopole mass limit of certain BPS monopole moduli spaces is considered. The one-parameter family of hyperkähler manifolds due to Dancer is shown to be an example of such manifolds. A new family of fixed monopole spaces is constructed. They are the moduli spaces of four SU_4 monopoles, in the infinite mass limit of two of the monopoles. These manifolds are shown to be non-singular when the fixed monopole positions are distinct.

1   Introduction

The moduli spaces of Bogomolny-Prasad-Sommerfield (BPS) monopoles are hyperkähler manifolds. For charge two SU_2 monopoles, the moduli space is the Atiyah-Hitchin manifold [1]. For two distinct monopoles in the maximally broken SU_3 theory, the moduli space is Taub-NUT space [3, 7, 15]. Since monopole moduli spaces have an isometric SO_3 action corresponding to rotations of the monopoles in space, these hyperkähler manifolds are the only possible non-trivial four dimensional monopole moduli spaces [9, 1]. In this paper other four dimensional hyperkähler manifolds are derived from monopole moduli spaces by taking the infinite mass limit of some of the monopole masses, thus fixing the monopole positions. Fixing monopole positions generally breaks the SO_3 isometry.

A one-parameter deformation of the Atiyah-Hitchin manifold is known, [4, 5]. It was constructed using the hyperkähler quotient. These hyperkähler manifolds will be reinterpreted as BPS monopole moduli spaces, with one monopole fixed. A moduli space of BPS monopoles with two fixed monopoles is then considered. By constructing these moduli spaces via a hyperkähler quotient, they are proven to be non-singular when the fixed monopoles are fixed at different points in space.

2 Nahm data and BPS monopoles

A BPS monopole is a pair (Φ, A_i) satisfying the Bogomolny equation. The Higgs field Φ is an su_n valued scalar field and A_i is the gauge potential. There is an SU_n gauge action on these fields, broken by the asymptotic Higgs field. If SU_n is broken to the maximal torus U_1^{n-1} the Higgs field at infinity is required to lie in the gauge orbit of

$$\Phi_\infty = i\text{diag}(t_1, t_2, \ldots, t_n). \tag{1}$$

By convention $t_1 < t_2 < \ldots < t_n$ and, since Φ is traceless, $t_1 + t_2 + \ldots + t_n = 0$. Because of the asymptotic condition on Φ, it gives a map from the large sphere at infinity into the quotient space $\text{orbit}_{SU_n}\Phi_\infty = SU_n/U_1^{n-1}$. Since $\pi_2(SU_n/U_1^{n-1}) = \mathbb{Z}^{n-1}$ the moduli space of monopoles is divided, topologically, into sectors labelled by $n-1$ integers, k_i, called topological charges. The maximal torus of SU_n is generated by the Cartan space and the matrix Φ_∞ defines a direction in this Cartan space. This direction picks out a unique set of simple roots in the Cartan space; those whose inner product with Φ_∞ is positive. Each U_1 in the maximal torus is generated by one of these simple roots. The k_i are then ordered by the requirement that adjacent k_i's correspond to non-orthogonal roots. A monopole with topological charge $(k_1, k_2, \ldots, k_{n-1})$ will be called a $(k_1, k_2, \ldots, k_{n-1})$-monopole.

The moduli spaces of BPS monopoles are diffeomorphic to spaces of Nahm data [17, 13, 18]. It will be assumed that these diffeomorphisms are isometries. This assumption is known to hold for many examples and is believed to be true generally. The Nahm data corresponding to a $(k_1, k_2, \ldots, k_{n-1})$-monopole are a triplet of skew-hermitian matrix functions defined over the interval $[t_1, t_n]$. The $t_1 < t_2 < \ldots < t_n$ subdivide the interval into $n-1$ abutting subintervals. Corresponding to a $(k_1, k_2, \ldots, k_{n-1})$-monopole a skyline diagram is drawn: a step function over the interval whose height on the i'th subinterval is k_i. For example, a $(3, 1, 2)$-monopole in an SU_4 theory has diagram


$$\tag{2}$$
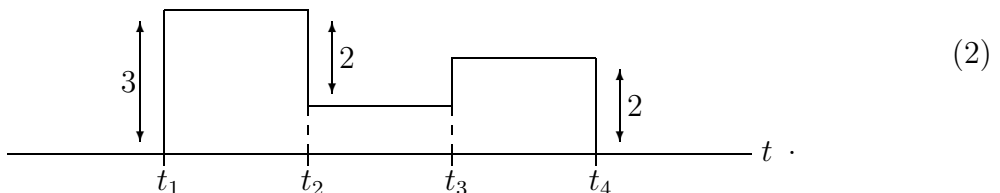
The Nahm triplet is a triplet of square matrix functions of t of different size over different subintervals. The size of the Nahm matrices in a subinterval is given by the height of the skyline in that interval. The matrices must satisfy the Nahm equations in each subinterval. The Nahm equations are

$$\frac{dT_1}{dt} = [T_2, T_3] \tag{3}$$

and two others by cyclic permutations of 1, 2 and 3.

There are boundary conditions relating the Nahm matrices in abutting subintervals. For the purpose of explaining these conditions let us consider the skyline diagram

$$(4)$$
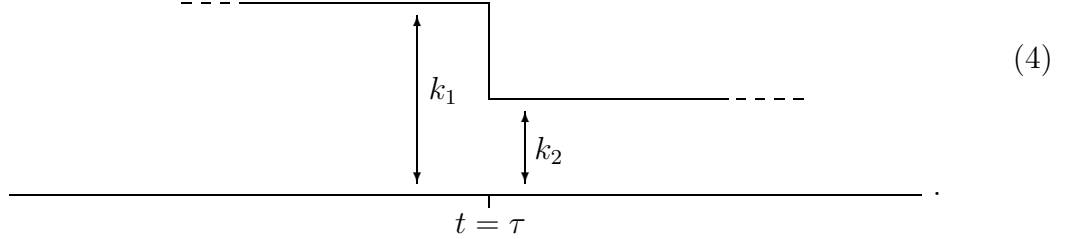
The skyline is k_1 high to the left of τ and $k_2 < k_1$ high to the right of it. Thus, the Nahm triplet, $(T_1(t), T_2(t), T_3(t))$, is a triplet of $k_1 \times k_1$ matrices over the left interval and of $k_2 \times k_2$ matrices over the right interval. As t approaches τ from the left, it is required that

$$
T_i(s) = \left(\begin{array}{c|c}
R_i/s + O(1) & O(s^{(m-1)/2}) \\
\hline
O(s^{(m-1)/2}) & T_i' + O(s)
\end{array} \right) \begin{array}{c} m \\ \\ k_2 \end{array}
\tag{5}
$$

where $s = \tau - t$, $m = k_1 - k_2$ and T_i' is the non-singular limit of the right interval Nahm data at $t = \tau$. The $m \times m$ residue matrices R_i in (5) must form the irreducible m dimensional representation of su_2. Since the one dimensional representation is trivial, there is no singularity when $m = 1$. When k_1 is less than k_2, the conditions are almost the same, again there is a pole with residue matrices forming the $m = k_2 - k_1$ dimensional representation of su_2 and the $k_1 \times k_1$ data are submatrices of the $k_2 \times k_2$ data at the boundary. The situation when $k_1 = k_2$ is very different, but that case will not be required in this paper.

When some of the t_i's in the asymptotic Higgs field are coincident, the residual gauge symmetry is enhanced. If two coincide, one U_1 factor is replaced by an SU_2 factor. If three coincide, two U_1's are lost and an SU_3 gained. Generally the unbroken group is $U_1^r \times K$ where K is a rank $n - r - 1$ semi-simple Lie group. Since $\pi_2(SU_n/(U_1^k \times K)) = \mathbb{Z}^r$ monopole solutions in theories with non-abelian residual theories have fewer topological charges. However, the monopole solutions still have $n - 1$ integer labels. Some of these integers are the usual topological charges. The rest are what are known as holomorphic charges.

The role of the holomorphic charges is subtle. If two t_i's are coincident, there is a zero thickness subinterval in the Nahm interval. The boundary conditions for Nahm data in this situation can be described in terms of those explained above, by formally imagining the zero thickness subinterval as the zero thickness limit of a subinterval of finite thickness. The Nahm data on this subinterval become irrelevant in the limit, but the height of the skyline on vanishing subintervals affects the matching condition between the Nahm matrices over the subintervals on either side.

3

As an example, let us consider SU_3 broken to U_1^2. A $(2,1)$-monopole has skyline

$$2 \quad \updownarrow \qquad \qquad \updownarrow 1 \qquad \qquad (6)$$

$$t = \tau$$
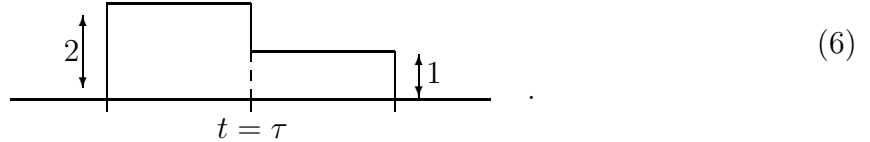
The Nahm data are 2×2 in the left interval and 1×1 in the right interval. The Nahm equations (3) dictate that 1×1 data are constant. Therefore, the right interval triplet is a triplet of imaginary numbers. These numbers are i times the cartesian coordinates of the (,1) part of the $(2,1)$-monopole. The boundary conditions imply that the 2×2 data are non-singular at the boundary, $t = \tau$, between the two intervals and, further, that their entries $T_i(\tau)_{2,2}$ are the 1×1 data. The 2×2 data are singular on the left boundary of the interval and the residues there form an irreducible representation of su_2. Letting the right hand interval vanish, a SU_3 monopole with topological charge two and holomorphic charge one is obtained. Holomorphic charges will be distinguished from topological charges by square bracketing them. Thus, this monopole is a $(2,[1])$-monopole and it has skyline

$$2 \quad \updownarrow \qquad \qquad \qquad (7)$$

$$t_1 \qquad t_2 = t_3$$

The Nahm data are 2×2 matrices with a pole on the left boundary but not on the right one.

In contrast, a $(2,0)$-monopole has skyline

$$2 \quad \updownarrow \qquad \qquad \qquad (8)$$

$$t_1 \qquad \quad t_2 \qquad \quad t_3$$
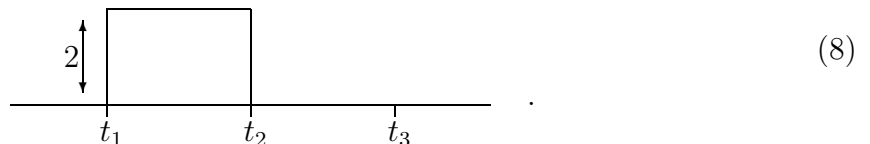
The Nahm data are 2×2 matrices over the left hand subinterval and have poles at t_1 and t_2. There is no data over the right hand subinterval. These data are identical to SU_2 2-monopole data and correspond to the embedding of an SU_2 2-monopole in SU_3. The length of the right hand subinterval does not affect the Nahm data, there is a pole at both t_1 and t_2 irrespective of whether $t_2 = t_3$ or not. If $t_2 = t_3$ the Nahm data correspond to a $(2,[0])$-monopole.

These examples demonstrate how the holomorphic charges determine the boundary conditions and how these boundary conditions can be derived by imagining the non-abelian case as the zero interval thickness limit of the abelian case. It should be noted that different holomorphic charges do not necessarily correspond to different monopoles or to different Nahm data. For example, $(3,[1])$-monopoles could equally well be called $(3,[2])$-monopoles. This ambiguity is discussed, for example, by Weinberg in [21].

4

3 Dancer's family of hyperkähler manifolds

In [4], the moduli space of centred $(2, [1])$-monopoles is constructed. These monopoles are charge $(2, [1])$ SU_3 monopoles. They have the skyline diagram (7). They are called centred because their Nahm data are traceless. The moduli space, M_0^8, is eight dimensional. The Nahm data for such monopoles are a triplet of 2×2 traceless skew-hermitian matrix function over the interval $[-2, 1]$. There is a simple pole at $t = -2$ and the residues there form the irreducible two dimensional representation of su_2. The space of such Nahm triplets, M_0^5, is five dimensional. The whole of M_0^8 is generated by the action of SU_2 on these Nahm data.

In the last Section, for simplicity, the Nahm data described are gauge fixed. While the gauge fixed Nahm data are a triplet of matrix functions, to form the required SU_2 orbit of M_0^5 the quadruplet of matrix functions (T_0, T_1, T_2, T_3) is introduced. This quadruplet is required to satisfy the Nahm equations

$$\frac{dT_1}{dt} + [T_0, T_1] = [T_2, T_3] \tag{9}$$

and two others by cyclic permutations of 1, 2 and 3.

The introduction of T_0 allows a group action to be defined on the space of $(2, [1])$ Nahm data

$$_0\mathcal{G} = \{g \in C^w([-2, 1], U_2) : g(-2) = \mathbb{1}\} \tag{10}$$

and its subgroup

$$_0\mathcal{G}_0 = \{g \in C^w([-2, 1], U_2) : g(-2) = g(1) = \mathbb{1}\}. \tag{11}$$

An action is then defined on (T_0, T_1, T_2, T_3) by

$$T_0 \mapsto gT_0g^{-1} - \frac{dg}{dt}g^{-1}, \tag{12}$$
$$T_i \mapsto gT_ig^{-1}, \qquad (i = 1, 2, 3).$$

If $g \in {}_0\mathcal{G}_0$ then the action is a gauge action. The moduli space of uncentred Nahm data, M^{12}, is the space of gauge inequivalent data. Furthermore, $U_2 = {}_0\mathcal{G}/{}_0\mathcal{G}_0$ and a U_2 action on the data is given by (12) with $g \in {}_0\mathcal{G}$. A hyperkähler quotient by the centre of this U_2 on M^{12} centres the Nahm data, giving M_0^8. The remaining SU_2 action can be fixed by setting T_0 to zero, reducing (9) to (3) and M_0^8 to M_0^5.

There is also SO_3 action. It both rotates the Nahm triplet as a vector and gauge transforms the four Nahm matrices. This action is not tri-holomorphic, it rotates the complex structures.

The SU_2 action on M_0^8 is tri-holomorphic and isometric. This means that there is an induced moment map, μ, from M_0^8 to $\mathbb{R}^3$ formed by the action of a U_1 subgroup of SU_2. Dancer's family of hyperkähler manifolds is the family of hyperkähler four-manifolds

$$M(\boldsymbol{\lambda}) = \mu^{-1}(\boldsymbol{\lambda})/U_1, \tag{13}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^3$. The SO_3 action on M_0^8 is not an isometry of $M(\boldsymbol{\lambda})$, rather, it acts on $\boldsymbol{\lambda}$ to give an isometry between $M(\boldsymbol{\lambda})$ and $M(R\boldsymbol{\lambda})$ where R is an SO_3 matrix. $M(\mathbf{0})$ is a double cover of the Atiyah-Hitchin manifold.

The hyperkähler manifolds $M(\boldsymbol{\lambda})$ are hyperkähler quotients of a monopole moduli space. It will now be shown that they are the infinite mass limit of another monopole space. The moment map μ is known explicitly. If the U_1 subgroup is the subgroup which fixes $i\sigma_3$ when SU_2 acts on su_2 in the adjoint representation, the moment map $\mu : M_0^8 \to \mathbb{R}^3$ given by this U_1 action is

$$\mu : (T_0, T_1, T_2, T_3) \mapsto (-\text{trace}(T_1(1)i\sigma_3), -\text{trace}(T_2(1)i\sigma_3), -\text{trace}(T_3(1)i\sigma_3)). \qquad (14)$$

The level set $\mu^{-1}(\boldsymbol{\lambda})$ consists of Nahm data whose entries $T_i(1)_{2,2}$ are $i\lambda_i/2$ at $t = 1$. For $(2,1)$-monopoles, (6), the data in the right hand interval are given by the $T_i(1)_{2,2}$ entries of the left hand Nahm data at the boundary. Thus, the hyperkähler manifolds $M(\boldsymbol{\lambda})$ are the moduli spaces of $(2,1)$-monopoles with the $(\,,1)$-monopole fixed. The $(\,,1)$-monopole can be fixed by taking its infinite mass limit. The monopole mass is proportional to the length of the corresponding interval, so this limit is

$$(15)$$
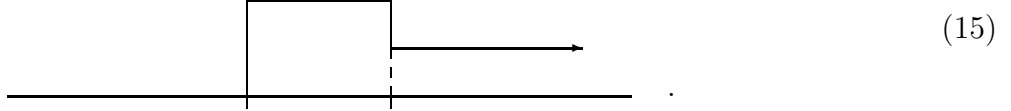
The vector $\boldsymbol{\lambda}$ is now related to the position of the $(\,,1)$-monopole: the monopole whose position is fixed. When the position of the $(\,,1)$-monopole is fixed in the centre, the relative metric of the $(2,\,)$-monopole is Atiyah-Hitchin. That is not surprising. It has been noted, [2, 14], that if three SU_2 monopoles are lined up, with suitable relative phases, the metric is Atiyah-Hitchin.

An advantage of this description of $M(\boldsymbol{\lambda})$ is that its asymptotic behaviour may be calculated using the methods of [19, 8, 16], that is by approximating the monopoles by point particles and calculating their long range interactions. By calculating the asymptotic metric for $(2,1)$-monopoles and then taking the infinite mass limit, it is found that on $M(\boldsymbol{\lambda})$ the asymptotic metric is

$$
\begin{aligned}
ds^2 &= V_1 d\mathbf{r} \cdot d\mathbf{r} + V_2^{-1}(d\chi + \mathbf{W} \cdot d\mathbf{r})^2 \qquad (16) \\
V_1 &= \frac{1}{2} - \frac{1}{r} + \frac{1}{8r_1} + \frac{1}{8r_2} \\
V_2 &= 1 - \frac{1}{r} + \frac{1 - 4r_1r_2}{8r_1r_2 + 2r_1 + 2r_2} \\
\mathbf{W} &= -\mathbf{w} + \frac{1}{8}\mathbf{w}_1 + \frac{1}{8}\mathbf{w}_2
\end{aligned}
$$

where $\mathbf{r}$ is the separation vector for the two $(2,)$-monopoles, r is its length, χ is the relative phase, r_1 and r_2 are the distances between each of the two $(2,)$-monopoles and the $(, 1)$-monopole. These distances are well defined in the point particle approximation.

$$(17)$$

The Dirac monopole potential $\mathbf{w}$ is defined so that $\mathrm{curl}(\mathbf{w}) = \mathrm{grad}(1/r)$. For $\mathbf{w}_1$ and $\mathbf{w}_2$ r is replaced respectively by r_1 and r_2 and the curl and grad performed with respect to the pertinent separation vector. This metric is obviously singular as $r \to 0$. The form (16) is valid only for large r.

4   A new family of hyperkähler manifolds

Another advantage of this description is that it immediately suggests a new family of four dimensional hyperkähler manifolds, $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$. In Section 3, it is shown that $M(\boldsymbol{\lambda})$ is a fixed monopole space derived from the moduli space of charge $(2, 1)$ SU_3 monopoles. This suggests that a new family of hyperkähler manifolds could be constructed by fixing monopoles in the moduli space of charge $(1, 2, 1)$ SU_4 monopoles. A $(1, 2, 1)$-monopole has skyline

$$(18)$$
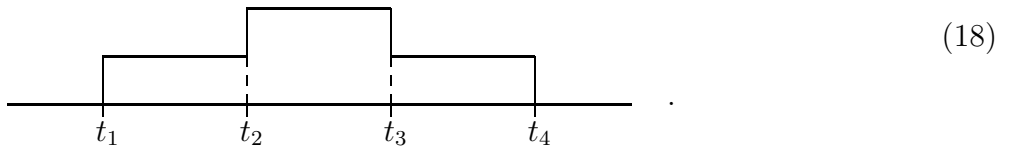
The corresponding Nahm data are 2×2 matrices in the middle subinterval and 1×1 matrices in the left and right subintervals. The Nahm data in the left subinterval are equal to the entries $T_i(t_2)_{2,2}$ of the 2×2 data, the Nahm data in the right subinterval are equal to the entries $T_i(t_3)_{2,2}$. All the Nahm data are analytic.

The limit where the subintervals $[t_1, t_2]$ and $[t_3, t_4]$ become infinitely long gives the $(1, 2, 1)$ fixed monopole spaces:


$$(19)$$
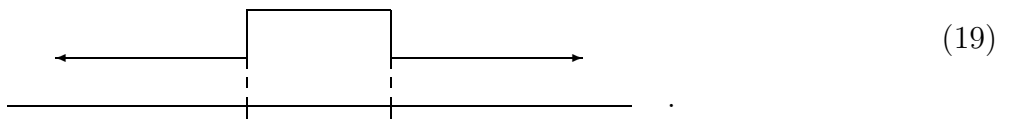
They are labelled by two vectors, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the positions of the two fixed monopoles: the $(1, ,)$-monopole and the $(, , 1)$-monopole. These spaces will be denoted $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$. The

SO_3 action on the charge $(1,2,1)$ moduli space is isometric and rotates the two vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. In the infinite mass limit of the $(1, \ , \ )$-monopole and the $(\ , \ , 1)$-monopole, the action of some $R \in SO_3$ gives an isomorphism between $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $N(R\boldsymbol{\lambda}, R\boldsymbol{\mu})$. Thus, $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a three-parameter family of hyperkähler manifolds. If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are parallel then a U_1 subgroup of the SO_3 action fixes $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and so $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ has a U_1 isometry.

Using the same methods as before, the asymptotic form of the $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ metric can be calculated. It is

$$
\begin{aligned}
ds^2 &= V_1 d\mathbf{r} \cdot d\mathbf{r} + V_2^{-1}(d\chi + \mathbf{W} \cdot d\mathbf{r})^2 \\
V_1 &= \frac{1}{2} - \frac{1}{r} + \frac{1}{8r_1^1} + \frac{1}{8r_2^1} + \frac{1}{8r_1^2} + \frac{1}{8r_2^2} \\
V_2 &= 1 - \frac{1}{r} + \frac{1}{2}\frac{r_1^2 r_2^2 + r_1^2 r_2^1 + r_1^1 r_2^1 + r_1^1 r_2^2 - 4r_1^1 r_1^2 r_2^1 r_2^2}{4r_1^1 r_1^2 r_2^1 r_2^2 + r_1^1 r_1^2 r_2^1 + r_1^1 r_1^2 r_2^2 + r_1^1 r_2^1 r_2^2 + r_1^2 r_2^1 r_2^2} \\
\mathbf{W} &= -\mathbf{w} + \frac{1}{8}\mathbf{w}_1^1 + \frac{1}{8}\mathbf{w}_2^1 + \frac{1}{8}\mathbf{w}_1^2 + \frac{1}{8}\mathbf{w}_2^2
\end{aligned} \tag{20}
$$

where everything is defined as before, except that now there are two fixed monopoles and the distances from the two $(\ , 2, \ )$-monopoles to the first of these have been denoted by r_1^1 and r_2^1 and the distances to the second by r_1^2 and r_2^2.

$$\tag{21}$$
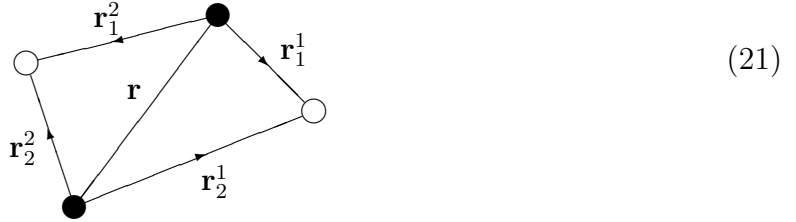
Examining the asymptotic formula, it is interesting to see how flat the $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ metrics are. All the metrics are flat up to the second order in $1/r$.

In the limit where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have infinite length, one would expect $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ to approach the Atiyah-Hitchin manifold. This is because, in the limit where the two fixed monopoles are infinitely far away, one expects the $(\ , 2, \ )$-monopole to interact like ordinary SU_2 2-monopoles. Equivalently, if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have infinite length, the Nahm data are now singular at the boundaries. Similarly, in the limit where one of the vectors, say $\boldsymbol{\mu}$, becomes infinitely long, one expects $N(\boldsymbol{\lambda}, \boldsymbol{\mu}) = M(\boldsymbol{\lambda})$.

5   Non-singularity of the new hyperkähler manifolds

It is not clear from the discussion in Section 4 that the $(1, 2, 1)$-moduli space remains non-singular as the masses of the $(1, ,)$-monopole and the $(, , 1)$-monopole become infinite. Dancer's family; $M(\boldsymbol{\lambda})$, is known to be non-singular because it can be constructed using a hyperkähler quotient. Let us now imitate this by constructing $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ using a hyperkähler quotient. Let us consider the moduli space of $([1], 2, [1])$-monopoles. These monopoles are topological charge two SU_4 monopoles in the theory where the residual symmetry is $SU_2 \times U_1 \times SU_2$. The skyline diagram is

$$(22)$$

$$t = -2 \qquad t = 2$$

The Nahm data are 2×2 matrices analytic over the whole interval $[2, 2]$. There are two commuting SU_2 actions, one at each boundary. These data correspond to $([1], 2, [1])$-monopoles. In this Section, the moduli space of $([1], 2, [1])$-monopoles is used to construct $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ in the same way as the moduli space of $(2, [1])$-monopoles is used to construct $M(\boldsymbol{\lambda})$. It is found that the manifold $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is free of singularities as long as $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

The charge $([1],2,[1])$ Nahm data are quadruplets (T_0, T_1, T_2, T_3) satisfying the Nahm equations (9) and acted on by the gauge group

$$_0\mathcal{G}_0 = \{g \in C^w([-2, 2], U_2) : g(-2) = g(2) = \mathbb{1}\}. \tag{23}$$

The two larger groups,

$$_0\mathcal{G} = \{g \in C^w([-2, 2], U_2) : g(-2) = \mathbb{1}\}, \tag{24}$$
$$\mathcal{G}_0 = \{g \in C^w([-2, 2], U_2) : g(2) = \mathbb{1}\} \tag{25}$$

are defined. These are subgroups of $\mathcal{G} = \{g \in C^w([-2, 2], U_2)\}$.

Two U_2 actions are given by $_0\mathcal{G}/_0\mathcal{G}_0$ and $\mathcal{G}_0/_0\mathcal{G}_0$. These actions commute. The whole $U_2 \times U_2$ action could be described as the $\mathcal{G}/_0\mathcal{G}_0$ action. The centre is $U_1 \times U_1$. The Nahm data are fixed under the central element represented by the constant function $g(t) = e^{i\theta}\mathbb{1}_2$. The element represented by $g(t) = e^{i\theta t}\mathbb{1}_2$ sends (T_0, T_1, T_2, T_3) to $(T_0 - i\theta\mathbb{1}_2, T_1, T_2, T_3)$ and generates the vector field $(-i\mathbb{1}_2, 0, 0, 0)$. The hyperkähler quotient by this action centres the Nahm data. This space of centred data will be called N_0^{12}. It is twelve dimensional. It has an isometric tri-holomorphic $SU_2 \times SU_2$ action. There is also an SO_3 action, which rotates (T_1, T_2, T_3) as a three-vector and commutes with the $SU_2 \times SU_2$ action.

A $U_1 \times U_1$ subgroup of the $SU_2 \times SU_2$ is represented by the elements

$$\alpha(t) = e^{\frac{i\theta}{4}(t+2)\sigma_3}, \qquad \beta(t) = e^{\frac{i\theta}{4}(2-t)\sigma_3}. \tag{26}$$

The moment map, $\mu : N_0^{12} \to \mathbb{R}^3 \times \mathbb{R}^3$ for the action of this subgroup is

$$\mu : (T_0, T_1, T_2, T_3) \mapsto (\boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{27}$$

9

where
$$\boldsymbol{\lambda} = (-\mathrm{trace}(T_1(-2)i\sigma_3), -\mathrm{trace}(T_2(-2)i\sigma_3), -\mathrm{trace}(T_3(-2)i\sigma_3)) \qquad (28)$$

and
$$\boldsymbol{\mu} = (-\mathrm{trace}(T_1(2)i\sigma_3), -\mathrm{trace}(T_2(2)i\sigma_3), -\mathrm{trace}(T_3(2)i\sigma_3)). \qquad (29)$$

By the same argument as in Section 3, N_0^{12} reduces to $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ under the hyperkähler quotient:
$$N(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \mu^{-1}(\boldsymbol{\lambda}, \boldsymbol{\mu})/U_1 \times U_1 \qquad (30)$$

The condition that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ must satisfy, in order for the $U_1 \times U_1$ action to be free, are now needed. These are the conditions for the non-singularity of the $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Let us consider the solutions of the Nahm equations. Using the $\mathcal{G}$ action, T_0 is gauged to zero. This leaves an eight dimensional space acted on by constant elements of $\mathcal{G}$ and by the SO_3 action. By acting with the SO_3 the t invariants: $\mathrm{trace}(T_1 T_2)$, $\mathrm{trace}(T_2 T_3)$ and $\mathrm{trace}(T_3 T_1)$, can be set to zero. This means that if the T_i are written as

$$T_i = \frac{1}{2}i f_i \mathbf{n}_i \cdot \boldsymbol{\sigma} \qquad (i = 1, 2, 3), \qquad (31)$$

the $\mathbf{n}_i$ are constant orthonormal vectors and so the functions f_1, f_2 and f_3 satisfy

$$\frac{df_1}{dt} = f_2 f_3 \qquad (32)$$

and two others given by cyclic permutations of 1, 2 and 3. The SO_3 action can be completely fixed by requiring that
$$f_1^2 \leq f_2^2 \leq f_3^2. \qquad (33)$$

The remaining group action is that of constant elements of $\mathcal{G}$. It is fixed by setting $\mathbf{n}_1 = (1, 0, 0)$, $\mathbf{n}_2 = (0, 1, 0)$ and $\mathbf{n}_3 = (0, 0, 1)$. The resulting subspace of the moduli space N_0^{12} will be called N^3. Since the SO_3 action on N_0^{12} is not free, N^3 is not a manifold. Equations (32) are the well known Euler top equations and can be solved in terms of Jacobi elliptic functions as

$$f_1(t) = \pm\frac{D cn_k(D(t+\tau))}{sn_k(D(t+\tau))}, \quad f_2(t) = \pm\frac{D dn_k(D(t+\tau))}{sn_k(D(t+\tau))}, \quad f_3(t) = \pm\frac{D}{sn_k(D(t+\tau))}, \quad (34)$$

where $0 \leq k \leq 1$ is the elliptic modulus, D and τ are arbitrary real constants and the signs are all minus or exactly two of them are plus. Analyticity of the data requires that $\tau > 2$ and $D(\tau + 2) < 2K(k)$ where $4K(k)$ is the period of sn_k. Further solutions can be found by changing the sign of all three f_i's and sending t to $-t$. The analyticity requirements on these further solutions are that $\tau < 2$ and $D(\tau + 2) < 2K(k)$. This exhausts all the solutions consistent with the various conditions which have been imposed.

Let us now consider the action of the $\alpha(t)$ and $\beta(t)$ given by (26). Since T_0 is zero on N^3 the only element in the group generated by $\alpha(t)$ and $\beta(t)$ which could have a fixed point in N^3 is the constant one $\alpha\beta(t) = e^{i\theta\sigma_3}$. For $\alpha\beta$ to have a fixed point in N^3 it is necessary and

sufficient that $f_1(0) = f_2(0) = 0$. This only occurs if $k = 1$ and $\tau = \infty$. The solutions (34) are then $f_1(t) = 0$, $f_2(t) = 0$ and $f_3(t) = D$ and the hyperkähler quotient gives the space $N((0, 0, D), (0, 0, D))$. This means $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with $\boldsymbol{\lambda} = \boldsymbol{\mu} = (0, 0, D)$ may have a singularity. By considering the action on N^3 of $SU_2 \times SU_2$, it is seen that the only points in N_0^{12} where the action of $\alpha(t)$ or $\beta(t)$ is not free are those points in the $SO_3 \times$ (constant elements of $\mathcal{G}$) orbit of the fixed points occuring in N^3. Therefore, the only potentially singular $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ manifolds are $N(\boldsymbol{\lambda}, \boldsymbol{\lambda})$. In the fixed monopole description, these are the manifolds of coincident fixed monopoles.

6 Other fixed monopole spaces

Following the example of $M(\boldsymbol{\lambda})$ and $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ it is natural to ask whether further non-singular fixed monopole spaces might be constructed by fixing larger numbers of monopoles. For example, a large class of four dimensional hyperkähler manifolds might be derived from the $(k_1, 2, k_2)$-monopole moduli spaces. One might conjecture that, as long as the $(k_1, \ , \ )$-monopoles and the $(\ , \ , k_2)$-monopoles are not fixed in coincident positions, new multi-parameter families of four dimensional hyperkähler manifolds could result.

More complicated mixtures of fixed and unfixed monopoles could be used to give fixed monopole spaces of dimensions higher than four. Fixed charges are distinguished from other charges by enclosing them in curly brackets. It could be conjectured that for $r > 1$ the $(\{k_1\}, l_1, l_2, \ldots, l_r, \{k_2\})$ spaces are non-singular when the $(k_1, \ , \ldots, \ , \ )$-monopoles and the $(\ , \ , \ldots, \ , k_2)$-monopoles are each fixed so they are not coincident with monopoles of the same type. Mixtures of fixed, topological and holomorphic charges might be considered. An example is the space $(1, 2, [1])$:


(35)

It is an interesting space, it has a tri-holomophic SU_2 isometry and an isometric U_1 action which rotates the complex structures.

The asymptotic metrics can always be constructed for fixed monopole spaces using the point monopole methods of [19, 8, 16]. Generally, these asymptotic fixed monopole metrics will be singular. This is not the case for the $(\{k\}, 1)$ space. In the limit of infinite $(k, \ )$-monopole mass the $(k, 1)$-monopole asymptotic metric is the k centre multi-Taub-NUT metric of Hawking [12]. The positions of the k centres are the k fixed monopole positions. Since the multi-Taub-NUT metric is generically non-singular and is the same asymptotically as the $(k, 1)$ metric, it seems likely that they are the same everywhere. Certainly, the $(1, 1)$-monopole metric is known explicitly [3, 7, 15] and the $(\{1\}, 1)$ metric is Taub-NUT. The $(1, 1, 1)$ metric is also known and the infinite mass limit $(\{1\}, 1, \{1\})$ is two centre multi-Taub-NUT.

7 Applications

The $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are gravitational instantons. Gravitational instantons are asymptotically flat solutions of the vacuum Einstein equations. All asymptotically flat four dimensional hyperkähler manifolds are gravitational instantons. As noted earlier, $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ approaches flat space very rapidly.

Fixed monopole spaces are relevant to $(2+1)$ dimensional quantum field theories. In a celebrated recent paper, [11], Hanany and Witten propose a correspondence between three dimensional supersymmetric gauge theories and moduli spaces of magnetic monopoles. In the language of [11] the fixed monopole spaces correspond to brane configurations in which some of the threebranes are infinitely extended in the direction along which the fivebranes are separated. Thus, $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$ corresponds to the configuration

$$\tag{36}$$
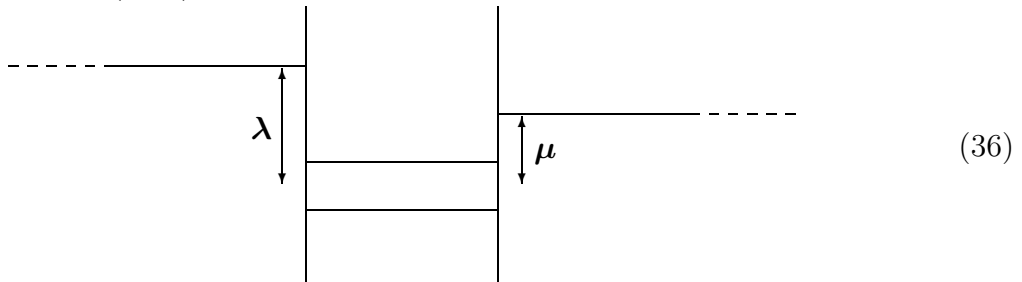
and to quantum field theories with hypermultiplets of masses $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The Dancer space $M(\boldsymbol{\lambda})$ corresponds to

$$\tag{37}$$
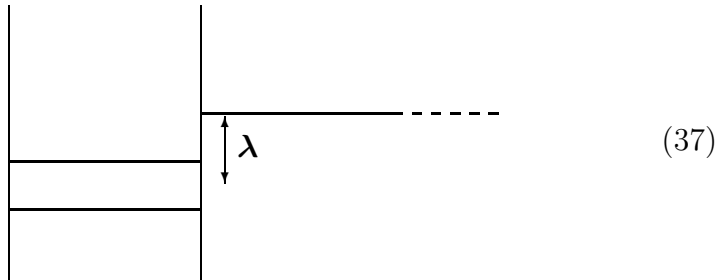
and to quantum field theories with hypermultiplets of mass $\boldsymbol{\lambda}$. These correspondences are described generally in [11]. Our reinterpretation of $M(\boldsymbol{\lambda})$ as a fixed monopole moduli space gives an explanation, in the spirit of [11], of the appearance of $M(\boldsymbol{\lambda})$ in [20].

Appendix: a note on $([1], 2, [1])$-monopoles

The moduli space of $([1], 2, [1])$-monopoles was used in Section 5 to prove the non-singularity of $N(\boldsymbol{\lambda}, \boldsymbol{\mu})$. The discussion in Section 5 would also be useful in studying $([1], 2, [1])$-monopoles *per se*. All $([1], 2, [1])$-monopoles are D_2 symmetric about some axes. The monopole can be orientated by imposing D_2 symmetry about particular axes. By imposing D_2 symmetry about the cartesian axes, the monopoles are restricted to a three

dimensional geodesic submanifold of the moduli space: which will be called X. The space N^3 of Nahm data described above is the quotient of the full moduli space by the full SO_3 action and since this action is not free, N^3 is not a manifold. Instead of quotienting the space of Nahm data by SO_3, D_2 symmetry is imposed on it, giving X. The D_2 symmetry conditions are identical to (31) but without the ordering condition (33). Thus, X is composed of the six copies of N^3 obtained by permuting the inequality (33). These copies are joined at the planes where two of the f_i's are equal. These data, where two of the f_i's are equal, correspond to axially symmetric monopoles. The planes intersect on the lines of spherical symmetry. An example of a line of spherical symmetry is

$$f_1(t) = f_2(t) = f_3(t) = -\frac{1}{t + \tau} \tag{38}$$

where $\tau > 2$.

There are exceptional lines in X given by letting $k = 1$ and taking τ to infinity. These lines are notable in the context of Section 5 as the fixed points of the $U_1 \times U_1$ action. These are the lines where one f_i is constant and the other two are zero. They meet at the point where all three f_i are zero. These lines correspond to the exceptional $([1], 2, [1])$-monopoles produced by embedding two SU_2 1-monopoles.

In their paper [6], Dancer and Leese studied the head on collision of $(2, [1])$-monopoles. These collisions are described by geodesics on a two dimensional manifold that they call Y. Our X is the analog of Y for $([1], 2, [1])$-monopoles. The boundaries of X occur when (D, τ) attain the bounds imposed by analyticity. When (D, τ) attain these bounds, the Nahm data has a pole at one or other end. This means these boundaries are actually copies of the space Y. In fact, the whole of X has eight copies of Y at its boundaries.

We can picture X. Take the $\mathbb{R}^3$ cartesian axes and thicken them. Divide the surfaces of these thickened axes by tracing their intersections with the xy, yz and zx planes. The eight surface elements bounded by these lines are the eight copies of Y. The interior of the thickened axes is X. The cartesian axes themselves are the lines of embedded monopoles. The origin is the spherical embedded monopole. The intersections of the six planes $x = \pm y$, $y = \pm z$ and $z = \pm x$ with X are the planes of axially symmetric monopoles. The lines $x = \pm y = \pm z$ are the lines of spherically symmetric monopoles. This picture of X is not metrically correct.

Acknowledgements

References

[1] M.F. Atiyah and N.J. Hitchin, '*The geometry and dynamics of magnetic monopoles*', Princeton University Press, 1988.

[2] R. Bielawski, Nonlinearity, 9, 1463 (1996).

[3] S.A. Connell, '*The dynamics of the $SU(3)$ charge (1,1) magnetic monopoles*', preprint, `<ftp://maths.adelaide.edu.au/pure/mmurray/oneone.tex>` (1994).

[4] A.S. Dancer, Commun. Math. Phys. 158, 545 (1993).

[5] A.S. Dancer, Q. J. Math. 45, 463 (1994).

[6] A.S. Dancer and R.A. Leese, Proc. R. Soc. (London) A440, 421 (1993).

[7] J.P. Gauntlett and D.A. Lowe, Nucl. Phys. B472, 194-206 (1996).

[8] G.W. Gibbons and N.S. Manton, Phys. Lett. B356, 32 (1995).

[9] G.W. Gibbons and C. Pope, Commun. Math. Phys. 66, 267 (1979).

[10] G.W. Gibbons and P. Rychenkova, '*Hyperkähler quotient construction of BPS monopole moduli spaces*', `hep-th/9608085`, to appear in Commun. Math. Phys.

[11] A. Hanany and E. Witten, '*Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*', `hep-th/9611230`.

[12] S.W. Hawking, Phys. Lett. A60, 81 (1977).

[13] N.J. Hitchin, Commun. Math. Phys. 89, 145 (1983).

[14] C.J. Houghton and P.M. Sutcliffe, Nonlinearity 9, 1609 (1996).

[15] K. Lee, E.J. Weinberg and P. Yi, Phys. Lett. B376, 97 (1996).

[16] K. Lee, E.J. Weinberg and P. Yi, Phys. Rev. D45, 1633 (1996).

[17] W. Nahm, '*The construction of all self-dual multi-monopoles by the ADHM method*', in Monopoles in quantum field theory, eds. N.S. Craigie, P. Goddard and W. Nahm, World Scientific, 1982.

[18] H. Nakajima, '*Monopoles and Nahm's equations*', Sanda 1990, Proceedings, Einstein metrics and Yang-Mills connections.

[19] N.S. Manton, Phys. Lett. 154B, 397 (1985) (E) Phys. Lett. 157B, 475 (1985).

[20] N. Seiberg and E. Witten, '*Gauge dynamics and compactification to three dimensions*', `hep-th/9607163`.

[21] E.J. Weinberg, Nucl. Phys. B167, 500 (1980).