String theory extensions of Einstein-Maxwell fields: the stationary case

Alfredo Herrera-Aguilar[1], Oleg V. Kechkin[2 3]

Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio C–3, Ciudad Universitaria, CP 58040. Morelia, Mich., México

Abstract

We present a new approach for generating solutions in heterotic string theory compactified down to three dimensions on a torus with $d+n>2$, where d and n stand for the number of compactified space–time dimensions and Abelian gauge fields, respectively. It is shown that in the case when $d=2k+1$ and n is arbitrary, one can apply a solution–generating procedure starting from solutions of the stationary Einstein theory with k Maxwell fields; our approach leads to classes of solutions which are invariant with respect to the total group of three-dimensional charging symmetries. We consider a particular extension of the stationary Einstein–multi-Maxwell theory obtained on the basis of the Kerr–multi-Newman–NUT special class of solutions and establish the conditions under which the resulting multi–dimensional metric is free of Dirac string peculiarities.

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[1] E-mail address: herrera@itzel.ifm.umich.mx

[2] E-mail address: kechkin@ginette.ifm.umich.mx, kechkin@depni.npi.msu.su

[3] On leave from Institute of Nuclear Physics of M.V. Lomonosov Moscow State University, Vorob'jovy Gory, 119899 Moscow, Russia

1 Introduction

Symmetry based approaches used for the construction of solutions in the framework of effective field (low energy) limits of string theories play an important role [1], [2], [3]. In this paper we develop a new approach which allows one to extend the solution space of the stationary Einstein–multi-Maxwell (EmM) theory to the realm of heterotic string theory compactified down to three spatial dimensions on a torus. Namely, we show that a new charging symmetry invariant subspace of solutions of heterotic string theories (with $d = 2k+1$ toroidally compactified space–time dimensions and arbitrary number n of original Abelian fields) can be generated from the solution spectrum of the stationary Einstein theory with k Maxwell fields. In particular, for the critical cases of heterotic ($d = 7$) and bosonic ($d = 23$) string theories one must start from the EmM theory with $k = 3$ and $k = 11$ Maxwell fields, respectively.

The toroidal compactification of heterotic string theory with arbitrary values of d and n was originally performed in [4]–[5], whereas the special case when the resulting theory is three–dimensional was originally studied in [6]–[7]. There, the corresponding symmetric space model was identified and an explicit representation in terms of a null–curvature matrix was given (see [8] for such models and their classification). In this paper we exploit the general formalism developed in [9], [10] and [11] as a natural matrix generalization of the stationary Einstein–Maxwell theory written down in terms of potentials which are closely related to the Ernst ones (see [12], [13] and [14]). In the framework of this formalism, the subgroup of charging symmetry transformations (which preserve the property of asymptotic flatness of the solutions) acts as a linear and homogeneous map; this fact allows one to work with the solution spectrum of the theory in a transparently charging symmetry invariant form. In particular, all the results of this paper are automatically invariant with respect to the action of the total three-dimensional subgroup of charging symmetries of the heterotic string theory compactified on a torus.

In this paper we continue our investigation on string theory extensions of Einstein–Maxwell fields. In a previous work [15] we have studied two theories with $d+n = 2$; here we deal with theories with $d+n > 2$. Such a split of the effective theories with arbitrary d and n, which arises in the low–energy limit of heterotic string theory, follows from the study of the general Israel–Wilson–Perjés class of solutions of heterotic string theory performed in [11]. The new formalism allows one to construct, in particular, a continuos generalization of the extremal Israel–Wilson–Perjés families of solutions in the corresponding string theories to the field of non–extremal ones. In [15] it was shown that the *static* Einstein–Maxwell theory plays the role of starting system for two theories with $d+n = 2$; in this paper we show that for the theories with $d + n > 2$ such starting systems can be related to the *stationary* Einstein–

multi-Maxwell theory. Here we illustrate the developed general approach by considering an extension of the Ker–Newman–NUT solution to the realm of heterotic string theory.

2 New Formalism for 3D Heterotic String Theory

In this section we review the necessary elements of the new formalism developed in [11] for the D–dimensional ($D = d + 3$) heterotic string theory with n Abelian gauge fields.

We start with the action for the bosonic sector of the low-energy heterotic string theory [1],[2]:

$$\mathcal{S}_D = \int d^D X |\text{det} G_{MN}|^{\frac{1}{2}} e^{-\Phi} \left(R_D + \Phi_{,M}\Phi^{,M} - \frac{1}{12} H_{MNK}H^{MNK} - \frac{1}{4} F^I_{MN} F^{I\,MN} \right), \quad (2.1)$$

where $H_{MNK} = \partial_M B_{NK} - \frac{1}{2} A^I_M F^I_{NK} + \text{ cyclic}\,\{M, N, K\}$ and $F^I_{MN} = \partial_M A^I_N - \partial_N A^I_M$. Here X^M is the M-th ($M = 1, ..., D$) coordinate of the physical space-time of signature $(-, +, ..., +)$, G_{MN} is the metric, whereas Φ, B_{MN} and A^I_M $\quad (I = 1, ..., n)$ are the dilaton, Kalb-Ramond and Abelian gauge fields, respectively. To determine the result of the toroidal compactification to three dimensions, let us put $D = d + 3$, $X^M = (Y^m, x^\mu)$ with $Y^M = X^m$ $(m = 1, ..., d)$ and $x^\mu = X^{d+\mu}$ $\quad (\mu = 1, 2, 3)$ and introduce the $d \times d$ matrix $G_0 = \text{diag}\,(-1; 1, ..., 1)$, the $(d + 1) \times (d + 1)$ and $(d + 1 + n) \times (d + 1 + n)$ matrices Σ and Ξ of the form $\text{diag}\,(-1, -1; 1, ..., 1)$, respectively, and the $(d + 1) \times (d + 1 + n)$ matrix field $\mathcal{Z} = \mathcal{Z}(x^\lambda)$ together with the three-metric $h_{\mu\nu} = h_{\mu\nu}(x^\lambda)$. In [11] it was shown that the resulting theory after the toroidal compactification of the first d dimensions Y^m can be expressed in terms of the pair $(\mathcal{Z}, h_{\mu\nu})$; its effective dynamics is given by the action

$$\mathcal{S}_3 = \int d^3 x h^{\frac{1}{2}} \left(-R_3 + L_3 \right), \quad (2.2)$$

where $R_3 = R_3(h_{\mu\nu)}$ is the curvature scalar for the three-dimensional line element $ds_3^2 = h_{\mu\nu} dx^\mu dx^\nu$ and

$$L_3 = \text{Tr} \left[\nabla \mathcal{Z} \left(\Xi - \mathcal{Z}^T \Sigma \mathcal{Z} \right)^{-1} \nabla \mathcal{Z}^T \left(\Sigma - \mathcal{Z} \Xi \mathcal{Z}^T \right)^{-1} \right]. \quad (2.3)$$

In order to translate this (σ-model) description into the language of the field components of the heterotic string theory, let us introduce three doublets of $(\mathcal{Z}, h_{\mu\nu})$-related potentials $(\mathcal{M}_\alpha, \vec{\Omega}_\alpha)$ $\quad (\alpha = 1, 2, 3)$ according to the relations

$$\begin{aligned}
&\mathcal{M}_1 = \mathcal{H}^{-1}, \quad \nabla \times \vec{\Omega}_1 = \vec{J}, \\
&\mathcal{M}_2 = \mathcal{H}^{-1} \mathcal{Z}, \quad \nabla \times \vec{\Omega}_2 = \mathcal{H}^{-1} \nabla \mathcal{Z} - \vec{J} \mathcal{Z}, \\
&\mathcal{M}_3 = \mathcal{Z}^T \mathcal{H}^{-1} \mathcal{Z}, \quad \nabla \times \vec{\Omega}_3 = \nabla \mathcal{Z}^T \mathcal{H}^{-1} \mathcal{Z} - \mathcal{Z}^T \mathcal{H}^{-1} \nabla \mathcal{Z} + \mathcal{Z}^T \vec{J} \mathcal{Z},
\end{aligned} \quad (2.4)$$

where $\mathcal{H} = \Sigma - \mathcal{Z}\Xi\mathcal{Z}^T$ and $\vec{J} = \mathcal{H}^{-1}\left(\mathcal{Z}\Xi\nabla\mathcal{Z}^T - \nabla\mathcal{Z}\Xi\mathcal{Z}^T\right)\mathcal{H}^{-1}$. In Eq. (2.4) the scalars $\mathcal{M}_\alpha$ are off-shell defined magnitudes, whereas the vectors $\vec{\Omega}_\alpha$ are defined on-shell. The scalar and vector potentials forming any doublet have the same matrix dimensionalities; let us represent them in the following block form

$$\begin{pmatrix} 1\times 1 & 1\times d \\ d\times 1 & d\times d \end{pmatrix}, \quad \begin{pmatrix} 1\times 1 & 1\times d & 1\times n \\ d\times 1 & d\times d & d\times n \end{pmatrix}, \quad \begin{pmatrix} 1\times 1 & 1\times d & 1\times n \\ d\times 1 & d\times d & d\times n \\ n\times 1 & n\times d & n\times n \end{pmatrix} \tag{2.5}$$

for $\alpha = 1, 2, 3$ respectively, where, for example, the '13' block components of the potentials $\mathcal{M}_2$ and $\vec{\Omega}_2$ are $1\times n$ matrices. Afterwards let us define the following set of scalar magnitudes:

$$S_\alpha = U_\alpha + S_0^{-1} W_1^T W_\alpha, \tag{2.6}$$

where

$$\begin{aligned}
S_0 &= -\mathcal{M}_{1,11} + 2\mathcal{M}_{2,11} - \mathcal{M}_{3,11}, \\
U_1 &= G_0\mathcal{M}_{1,22}G_0 + G_0\mathcal{M}_{2,22} + (\mathcal{M}_{2,22})^T G_0 + \mathcal{M}_{3,22}, \\
U_2 &= G_0\mathcal{M}_{1,22} - G_0\mathcal{M}_{2,22}G_0 + (\mathcal{M}_{2,22})^T - \mathcal{M}_{3,22}G_0, \\
U_3 &= \sqrt{2}\left(G_0\mathcal{M}_{2,23} + \mathcal{M}_{3,23}\right), \\
W_1 &= -\mathcal{M}_{1,12}G_0 - \mathcal{M}_{2,12} + (\mathcal{M}_{2,21})^T G_0 + \mathcal{M}_{3,12}, \\
W_2 &= \mathcal{M}_{1,12} - \mathcal{M}_{2,12}G_0 + (\mathcal{M}_{2,21})^T - \mathcal{M}_{3,12}G_0, \\
W_3 &= \sqrt{2}\left(\mathcal{M}_{2,13} + \mathcal{M}_{3,13}\right),
\end{aligned} \tag{2.7}$$

and vector fields

$$\begin{aligned}
\vec{V}_1 &= \left[-\vec{\Omega}_{1,12}G_0 + \vec{\Omega}_{2,12} + (\vec{\Omega}_{2,21})^T G_0 + \vec{\Omega}_{3,12}\right]^T, \\
\vec{V}_2 &= \left[-\vec{\Omega}_{1,12} - \vec{\Omega}_{2,12}G_0 + (\vec{\Omega}_{2,21})^T - \vec{\Omega}_{3,12}G_0\right]^T, \\
\vec{V}_3 &= \sqrt{2}\left(\vec{\Omega}_{2,13} + \vec{\Omega}_{3,13}\right)^T.
\end{aligned} \tag{2.8}$$

In terms of them the heterotic string theory fields read:

$$\begin{aligned}
ds_D^2 &= ds_{d+3}^2 = (dY + V_{1\mu}dx^\mu)^T S_1^{-1}(dY + V_{1\nu}dx^\nu) + S_0 ds_3^2, \\
e^\Phi &= |S_0 \det S_1|^{\frac{1}{2}}, \\
B_{mk} &= \frac{1}{2}\left(S_1^{-1}S_2 - S_2^T S_1^{-1}\right)_{mk},
\end{aligned}$$

$$
\begin{aligned}
B_{m\,d+\nu} &= \left\{ V_{2\nu} + \frac{1}{2}\left(S_1^{-1}S_2 - S_2^T S_1^{-1}\right) V_{1\nu} - S_1^{-1}S_3 V_{3\nu} \right\}_m , \\
B_{d+\mu\,d+\nu} &= \frac{1}{2}\left[V_{1\mu}^T \left(S_1^{-1}S_2 - S_2^T S_1^{-1}\right) V_{1\nu} + V_{1\mu}^T V_{2\nu} - V_{1\nu}^T V_{2\mu} \right] , \\
A_m^I &= \left(S_1^{-1}S_3\right)_{mI} , \\
A_{d+\mu}^I &= \left(-V_{3\mu} + S_3^T S_1^{-1} V_{1\mu}\right)_I . \qquad (2.9)
\end{aligned}
$$

¿From Eq. (2.9) it follows that, apart from the magnitudes S_0, S_α and $\vec{V}_\alpha$, we also must compute $\det S_1$ and S_1^{-1} in order to obtain the field components of the heterotic string theory (2.1). Therefore, after some algebraic calculations it can be proved that

$$
\begin{aligned}
\det S_1 &= \left(1 + S_0 W_1 U_1^{-1} W_1^T\right) \det U_1, \\
S_1^{-1} &= U_1^{-1} - \frac{S_0^{-1} U_1^{-1} W_1^T W_1 U_1^{-1}}{1 + S_0^{-1} W_1 U_1^{-1} W_1^T}. \qquad (2.10)
\end{aligned}
$$

Finally let us point out that the magnitudes S_0, U_α, W_α and $\vec{V}_\alpha$ can be used to explicitly write down any solution of the theory under consideration. In Section 4 we shal calculate them for an extension of the stationary EmM theory to the realm of the low–energy heterotic string theory.

At the end of this Section let us notice that the transformation

$$
\mathcal{Z} \longrightarrow \mathcal{C}_1 \mathcal{Z} \mathcal{C}_2 \qquad (2.11)
$$

with $\mathcal{C}_1^T \Sigma \mathcal{C}_1 = \Sigma$ and $\mathcal{C}_2^T \Xi \mathcal{C}_2 = \Xi$ is a transparent symmetry of the theory under consideration (see Eq. (2.3)). In [10] and [11] it was shown that this symmetry coincides with the total group of three dimensional charging symmetries. The above reviewed formalism, based on the use of the matrix potential $\mathcal{Z}$, possesses the lowest matrix dimensionality ($\mathcal{Z}$ is a $(d+1)\times(d+n+1)$ matrix field and the theory (2.3) is, in fact, a $O(d+1,d+n+1)/O[(d+1)\times(d+n+1)]$ symmteric space model of dimension $(d+1)(d+n+1)$ [8]). From Eq. (2.11) it also follows that the general transformation of the charging symmetry subgroup acts as a linear and homogeneous map – a fact that was just discussed in the Introduction. It is clear that the new formalism is specially convenient for the study of asymptotically flat solutions of heterotic string theory toroidally compactified to three dimensions because all the results can be obtained in a transparent charging symmtery invariant form.

3 String theories from stationary Einstein–multi-Maxwell system

In this section we show how to map solutions of the stationary Einstein theory with k Maxwell fields into solutions to the three–dimensional heterotic string theory with $d = 2k+1$ toroidally compactified space–time dimensions and arbitrary number n of Abelian gauge fields.

First of all let us formulate the main idea of our approach; it is related to the heterotic string/Einstein–Maxwell theory correspondence and the explicite form of the Isreal–Wilson–Perjés class of solutions in both of these theories [11]. In order to achieve this aim, let us represent the stationary Einstein–Maxwell (EM) theory in a very similar form to Eq. (2.3). Namely, it is well known that the effective three–dimensional Lagrangian of the stationary EM theory reads:

$$L_3 = L_{EM} = \frac{1}{2f^2}\left|\nabla E - \bar{F}\nabla F\right|^2 - \frac{1}{f}\left|\nabla F\right|^2, \tag{3.1}$$

where $f = \frac{1}{2}(E + \bar{E} - |F|^2)$, and E and F are the conventional Ernst potentials. Let us introduce the 1×2 matrix potential

$$z = (z_1 \; z_2) \tag{3.2}$$

with

$$z_1 = \frac{1-E}{1+E}, \qquad z_2 = \frac{\sqrt{2}F}{1+E}. \tag{3.3}$$

Then

$$L_{EM} = 2\frac{\nabla z(\sigma_3 - z\dagger z)^{-1}\nabla z\dagger}{1 - z\sigma_3 z\dagger}, \tag{3.4}$$

where $\sigma_3 = \mathrm{diag}\,(1\; -1)$. By comparing Eqs. (2.3) and (3.4) it follows that the map

$$\mathcal{Z} \longleftrightarrow z, \quad \Xi \longleftrightarrow \sigma_3, \quad \Sigma \longleftrightarrow 1, \tag{3.5}$$

together with the interchange of operations $T \longleftrightarrow \dagger$, relates three–dimensional heterotic string and stationary Einstein-Maxwell theories; the factor ‘2’ in (3.4) can be understood as a consequence of the exact matrix representation of complex magnitudes (see below and [11] as well). Therefore, the Israel–Wilson-Perjés (IWP) class of solutions of the Einstein-Maxwell theory [16] can be rewritten in terms of the z–potential as $z = \lambda q$, where $\lambda = \lambda(x^\mu)$

is a complex harmonic function ($\nabla^2\lambda = 0$), q is a 1×2–matrix constant parameter and the parameter $\kappa = q\sigma_3 q\dagger$ vanishes; in this case, the corresponding three–dimensional metric $h_{\mu\nu}$ is flat. It is clear that, in view of the correspondence (3.5), the IWP class of solutions of the heterotic string theory arises in the framework of the ansatz

$$\mathcal{Z} = \Lambda \mathcal{Q}, \tag{3.6}$$

where $\Lambda = \Lambda(x^\mu)$ is a real harmonic matrix function and $\mathcal{Q}$ is a constant matrix parameter. In [11] it was shown that this fact actually takes place if the parameter

$$\kappa = \mathcal{Q}\Xi\mathcal{Q}^T \tag{3.7}$$

vanishes and the three–metric $h_{\mu\nu}$ is flat again, in complete accordance with the correspondence (3.5). There, it also was shown that the restriction $\kappa = 0$ completely fixes the dimensionality of the matrices Λ and $\mathcal{Q}$: for two theories with $d+n=2$, such matrices have dimensions $(d+1) \times 1$ and 1×3, respectively, whereas for the theories with $d+n>2$ the dimensions are $(d+1) \times 2$ and $2 \times (d+n+1)$, respectively.

The main idea of our approach is to remove the $\kappa = 0$ restriction and, therefore, to consider the resulting generalization of the IWP class of solutions of heterotic string theory to the subspace of the non–extremal solutions. This means that we shall preserve the form of the ansatz (3.6) and the dimensions of the matrices Λ and $\mathcal{Q}$, but we shall allow arbitrary values of the parameter κ defined by Eq. (3.7). Such a procedure can be applied in a very natural way in the framework of the stationary EM theory, where it defines, for instance, a continuous extension of the extremal Kerr–multi-Newman–NUT solution to the corresponding non–extremal one [14]. This extension is really interesting from the point of view of physical applications; the example given above concerns black hole physics in EM theory [17]. In view of the correspondence (3.5) the same motivation for the study of the ansatz (3.6) with $\kappa \neq 0$ must be valid for the low–energy heterotic string theory; thus, the study of such an ansatz is also interesting in the framework of black holes in string theory [3], [18].

In this paper we consider string theories with $d+n>2$, when κ is a symmetric 2×2–matrix. By straightforwardly substituting the ansatz (3.6) into the equations of motion derived from Eqs. (2.2) and (2.3), one obtains

$$\begin{gathered}\nabla^2\Lambda + 2\nabla\Lambda\kappa\Lambda^T\left(\Sigma - \Lambda\kappa\Lambda^T\right)^{-1}\nabla\Lambda = 0,\\ R_{3\ \mu\nu} = Tr\left[\Lambda_{,(\mu}\kappa\left(1 - \Lambda^T\Sigma\Lambda\kappa\right)^{-1}\Lambda^T_{,\nu)}\left(\Sigma - \Lambda\kappa\Lambda^T\right)^{-1}\right].\end{gathered} \tag{3.8}$$

It is obvious that in the case $\kappa = 0$ we recover the extremal case studied in [11], whereas for $\kappa \neq 0$ we have the above announced continuos extension of the formalism to the non–extremal case. Below we study the situation when κ is nonzero and, moreover, nondegenerate matrix with signature $\tilde{\Sigma} = \mathrm{diag}(-1,-1)$. The reason for considering such a particularization of the ansatz (3.6) is that, in this case, the effective system defined by Eqs. (3.8) corresponds to some new heterotic string theory by itself. Actually, The Eqs. (3.8) are the equations of motion for the action (2.2) with the matter Lagrangian (2.3) replaced by

$$\tilde{\mathcal{L}}_3 = Tr\left[\nabla\Lambda\kappa\left(1-\Lambda^T\Sigma\Lambda\kappa\right)^{-1}\nabla\Lambda^T\left(\Sigma-\Lambda\kappa\Lambda^T\right)^{-1}\right]. \tag{3.9}$$

Then, as an algebraic fact it follows that there exists a nondegenerate matrix K such that

$$\kappa = K\tilde{\Sigma}K^T. \tag{3.10}$$

Let us introduce the new matrix potential

$$\tilde{\mathcal{Z}} = K^T\Lambda^T \tag{3.11}$$

and set $\tilde{\Xi} = \Sigma$. We claim that it is possible to rewrite the effective Lagrangian $\tilde{\mathcal{L}}_3$ (3.9) in terms of $\tilde{\mathcal{Z}}$, $\tilde{\Sigma}$ and $\tilde{\Xi}$; the resulting Lagrangian exactly coincides with the relation (2.3) up to the tilde. Thus, the effective system (3.9) is nothing else than the heterotic string theory with $\tilde{d} = 1$ compactified dimensions and $\tilde{n} = d-1$ Abelian gauge fields. From Eqs. (3.6) and (3.11) it follows that

$$\mathcal{Z} = \tilde{\mathcal{Z}}^T T. \tag{3.12}$$

where $T = K^{-1}\mathcal{Q}$. Eq. (3.12) maps the space of solutions of the theory in terms of $\tilde{\mathcal{Z}}$ into that of the theory in terms of the potential $\mathcal{Z}$, so that the matrix T plays the role of a symmetry operator. Let us now calculate the general explicit form of such symmetry operator using Eqs. (3.6) and (3.10). Without loss of generality (see [11] for details), the matrix $\mathcal{Q}$ can be parametrized in the form

$$\mathcal{Q} = \left(\begin{array}{ccc} 1 & 0 & n_1^T \\ 0 & 1 & n_2^T \end{array}\right), \tag{3.13}$$

where n_a $(a = 1, 2)$ are two $(d+n-1)\times 1$ columns. Thus, the extremal case corresponds to the restriction $n_a^T n_b = \delta_{ab}$, i.e. it is realized by the unit orthogonal columns n_a. Our generalization of the extremal ansatz corresponds, in this geometric language, to the case of columns with arbitrary length and arbitrary angle between them which is compatible with the signature $\tilde{\Sigma}$ of the matrix κ.

Now we are able to compute the matrix κ and to determine the quantity K using, for example, the orthogonalization procedure of the theory of quadratic forms. A special solutions reads:

$$K = \left(\begin{array}{cc} \sqrt{1-n_1^T n_1} & 0 \\ -\frac{n_1^T n_2}{\sqrt{1-n_1^T n_1}} & \sqrt{\frac{1-n_1^T n_1 - n_2^T n_2 + (n_1^T n_1)(n_2^T n_2) - (n_1^T n_2)^2}{1-n_1^T n_1}} \end{array} \right), \tag{3.14}$$

In order to obtain a general solution K to the quadratic equation (3.10) one must generalize the special solution (3.14) through the map $K \longrightarrow K\mathcal{C}$ where $\mathcal{C}^T\tilde{\Sigma}\mathcal{C} = \tilde{\Sigma}$, i.e. $\mathcal{C} \in O(2)$. However, this map is effectively equivalent to the transformation $\tilde{\mathcal{Z}} \longrightarrow \mathcal{C}^{T\ -1}\tilde{\mathcal{Z}}$ as it follows from Eq. (3.12). It is clear that $\mathcal{C}^{T\ -1}$ is nothing more than an alternative notation for the 'left' subgroup $\mathcal{C}_1$ of the charging symmetry transformation (see Eq. (2.11)), thus, it can be omitted for the charging symmetry invariant classes of solutions repersented by $\tilde{\mathcal{Z}}$. Thus, without loss of generality one can take $\mathcal{Q}$ and K in the form given by Eqs. (3.13) and (3.14) when constructing a symmetry operator according to Eq. (3.12). It is worth noticing that the definition of K is consistent with the assumed signature of κ.

Now let us consider a special situation with $d = 2k+1$, when the potential $\tilde{\mathcal{Z}}$ can be splitted into $k+1$ 2×2–matrix blocks:

$$\tilde{\mathcal{Z}} = \left(\begin{array}{cccc} \tilde{\mathcal{Z}}_1, & \tilde{\mathcal{Z}}_2, & ..., & \tilde{\mathcal{Z}}_{k+1} \end{array} \right). \tag{3.15}$$

Let us consider a consistent ansatz with

$$\tilde{\mathcal{Z}}_{\mathcal{P}} = \left(\begin{array}{cc} z'_{\mathcal{P}} & -z''_{\mathcal{P}} \\ z''_{\mathcal{P}} & z'_{\mathcal{P}} \end{array} \right), \tag{3.16}$$

where $\mathcal{P} = 1, 2, ..., k+1$. Let us introduce $k+1$ complex functions $z_{\mathcal{P}} = z'_{\mathcal{P}} + iz''_{\mathcal{P}}$. Our statement is that the heterotic string theory field equations corresponding to the special subsystem (3.15)–(3.16) can be derived from the effective Lagrangian

$$L_3 = L_{EmM} = 2\frac{\nabla\tilde{z}(\tilde{\sigma}_3 - \tilde{z}\dagger\tilde{z})^{-1}\nabla\tilde{z}\dagger}{1-\tilde{z}\tilde{\sigma}_3\tilde{z}\dagger}, \tag{3.17}$$

where $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, ...\ \tilde{z}_{k+1})$ and $\tilde{\sigma}_3 = \mathrm{diag}(1, -1, -1, ..., -1)$. It is clear that in the case $k = 1$ one recovers the Eq. (3.4), i.e. one deals with the stationary Einstein–Maxwell theory. It is natural to suppose that in the case of arbitrary k one obtains the stationary Einstein theory with k Maxwell fields. In order to verify this statement it is convenient to introduce new variables (compare to Eq. (3.3))

$$E = \frac{1-\tilde{z}_1}{1+\tilde{z}_1}, \qquad F_p = \frac{\sqrt{2}\tilde{z}_{p+1}}{1+\tilde{z}_1}, \tag{3.18}$$

where $p = 1, 2, ..., k$. In terms of these variables the effective Lagrangian adopts the form

$$L_{EmM} = \frac{1}{2f^2}\left|\nabla E - \bar{F}_p \nabla F_p\right|^2 - \frac{1}{f}\left|\nabla F_p\right|^2, \tag{3.19}$$

where $f = \frac{1}{2}(E+\bar{E}-F_p\bar{F}_p)$, thus, they can be interpreted as the conventional Ernst potentials of the classical four–dimensional Einstein theory with k Maxwell fields in the stationary case.

The theory (3.17) can be studied in a form which is very close to that of the heterotic string theory (see Section 2) by using the correspondence (3.5) modified to the case of k Maxwell fields, i.e., by exchanging

$$z \longrightarrow \tilde{z}, \qquad \sigma_3 \longrightarrow \tilde{\sigma}_3, \tag{3.20}$$

(see Eqs. (3.4) and (3.7)). Namely, it is convenient to introduce the doublets $(\tilde{m}_\alpha, \tilde{\vec{\omega}}_\alpha)$ with

$$\begin{aligned}
&\tilde{m}_1 = \tilde{h}^{-1}, \qquad \nabla \times \tilde{\vec{\omega}}_1 = \tilde{\vec{j}}, \\
&\tilde{m}_2 = \tilde{h}^{-1}\tilde{z}, \qquad \nabla \times \tilde{\vec{\omega}}_2 = \tilde{h}^{-1}\nabla\tilde{z} - \tilde{\vec{j}}\tilde{z}, \\
&\tilde{m}_3 = \tilde{h}^{-1}\tilde{z}\dagger\tilde{z}, \qquad \nabla \times \tilde{\vec{\omega}}_3 = \tilde{h}^{-1}(\nabla\tilde{z}\dagger\tilde{z} - \tilde{z}\dagger\nabla\tilde{z}) + \tilde{\vec{j}}\tilde{z}\dagger\tilde{z},
\end{aligned} \tag{3.21}$$

where $\tilde{h} = -(1 - \tilde{z}\tilde{\sigma}_3\tilde{z}\dagger)$ and $\tilde{\vec{j}} = -\tilde{h}^{-2}\left(\tilde{z}\tilde{\sigma}_3\nabla\tilde{z}\dagger - \nabla\tilde{z}\tilde{\sigma}_3\tilde{z}\dagger\right)$, (compare with Eq. (2.4)). It is clear that the doublet $(\tilde{m}_1, \tilde{\vec{\omega}}_1)$ consists of complex functions, whereas the doublets $(\tilde{m}_2, \tilde{\vec{\omega}}_2)$ and $(\tilde{m}_3, \tilde{\vec{\omega}}_3)$, of $1 \times (k+1)$– and $(k+1) \times (k+1)$–matrices, respectively. From Eqs. (3.15)–(3.16) we extract a rule for reconstructing the heterotic string theory described by the potential $\tilde{\mathcal{Z}}$ of dimension $2 \times [2(k+1)]$. Further, from Eqs. (3.12)–(3.14) one obtains the explicit form of the symmetry map $\tilde{\mathcal{Z}} \longrightarrow \mathcal{Z}$. It is interesting that this map is nonholomorphic due to the transposition of $\tilde{\mathcal{Z}}$ in Eq. (3.12), which is equivalent to the Hermitean conjugation of $\tilde{z}$ in view of the correspondence (3.20) discussed above.

Thus, a symmetry transformation that maps the space of solutions of the stationary Einstein theory with k Maxwell fields into the corresponding subspace of solutions of the heterotic string with $d = 2k+1$ toroidally compactified dimensions and n arbitrary Abelian gauge fields is established by the following procedure. First of all, one must calculate in explicit form three doublets of potentials $(\tilde{m}_\alpha, \tilde{\vec{\omega}}_\alpha)$ for the stationary EmM theory. After that one must rewrite them in the form $(\tilde{M}_\alpha, \tilde{\vec{\Omega}}_\alpha)$ using the exact matrix representation of complex magnitudes (Eqs. (3.15)–(3.16) give, in fact, an example of such a representation of the complex potential $\tilde{z}$ in terms of the real potential $\tilde{\mathcal{Z}}$). The next step consists of calculating the matrix potentials $(M_\alpha, \vec{\Omega}_\alpha)$ for the heterotic string theory which is an image of the EmM system according to the map (3.12). Finally, one must obtain the explicit form of the magnitudes S_0, W_α, U_α and $\vec{V}_\alpha$, using the found potentials $(M_\alpha, \vec{\Omega}_\alpha)$.

At the end of this Section let us compute the doublets $(M_\alpha, \vec{\Omega}_\alpha)$. By using Eqs. (2.4) and (3.12), after some algebraic calculations, one obtains

$$
\begin{aligned}
&M_1 = \Sigma + \Sigma\tilde{M}_3\Sigma, \qquad \vec{\Omega}_1 = -\Sigma\tilde{\vec{\Omega}}_3\Sigma,\\
&M_2 = -\Sigma\tilde{M}_2^T T, \qquad \vec{\Omega}_2 = -\Sigma\tilde{\vec{\Omega}}_2^T T,\\
&M_3 = T^T(\tilde{M}_1 + 1)T, \qquad \vec{\Omega}_3 = -T^T\tilde{\vec{\Omega}}_1 T.
\end{aligned} \tag{3.22}
$$

In the next Section we shall exploit these formulae in order to construct a subspace of solutions for the heterotic string theory with $d = 2k + 1$ and arbitrary n starting from the stationary Einstein theory with k Maxwell fields.

4 Solutions via Kerr–multi-Newman–NUT family

In order to calculate the potentials S_0, W_α, U_α and $\vec{V}_\alpha$ for the heterotic string fields, which correspond to the Einstein–multi-Maxwell ones, according to the scheme developed in the previous Section, let us parametrize the potential $\tilde{\mathcal{Z}}$ and the symmetry operator T in the appropriate form. For $\tilde{\mathcal{Z}}$ is convenient to set

$$
\tilde{\mathcal{Z}} = \left(\tilde{\mathcal{Z}}_I \; \tilde{\mathcal{Z}}_{II}\right), \tag{4.1}
$$

where $\tilde{\mathcal{Z}}_I$ is a 2×1–column and $\tilde{\mathcal{Z}}_{II}$ is a $2 \times (2k + 1)$–matrix that read

$$
\tilde{\mathcal{Z}}_I = \begin{pmatrix} \tilde{z}_1' \\ \tilde{z}_1'' \end{pmatrix}, \qquad \tilde{\mathcal{Z}}_{II} = \begin{pmatrix} -\tilde{z}_1'' & \tilde{z}_{1+p}' & -\tilde{z}_{1+p}'' \\ \tilde{z}_1' & \tilde{z}_{1+p}'' & \tilde{z}_{1+p}' \end{pmatrix}. \tag{4.2}
$$

Therefore, for T we choose the following segmentation

$$
T = (T_I \; T_{II} \; T_{III}) \tag{4.3}
$$

where T_I is a 2×1–column, T_{II} is a $2\times(2k+1)$–matrix and T_{III} is a $2\times n$–matrix, i.e.

$$T_I = \mathcal{K}^{-1}\mathcal{Q}_I, \qquad \tilde{\mathcal{Q}}_I = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$T_{II} = \mathcal{K}^{-1}\mathcal{Q}_{II}, \qquad Q_{II} = \begin{pmatrix} 0 & r_1^T \\ 1 & r_2^T \end{pmatrix},$$
$$T_{III} = \mathcal{K}^{-1}\mathcal{Q}_{III}, \qquad \tilde{\mathcal{Q}}_{III} = \begin{pmatrix} l_1^T \\ l_2^T \end{pmatrix}, \tag{4.4}$$

where we have naturally decomposed $n_a^T = \left(r_a^T \; l_a^T\right)$, into the rows r_a^T and l_a^T of dimension $1\times 2k$ and $1\times n$, respectively. It is worth noticing that the block representation (4.1) and (4.3) of the magnitudes $\tilde{\mathcal{Z}}$ and T is directly related to the number of compactified dimensions and Abelian vector fields ($2k+1$ and n, respectively) and, thus, is actually fruitful for the application of Eqs (2.9). Thus, after some algebraic computations, Eqs. (2.9) yield the following expression for the scalar S_0

$$S_0 = 1 - T_I^T T_I - \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_I - T_I\right)^T\left(\tilde{\mathcal{Z}}_I - T_I\right), \tag{4.5}$$

whereas for the columns W_α one obtains:

$$W_1 = T_I^T T_{II} + \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_I - T_I\right)^T\left(\tilde{\mathcal{Z}}_{II} - T_{II}\right),$$
$$W_2 = -\left[T_I^T T_{II} + \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_I + T_I\right)^T\left(\tilde{\mathcal{Z}}_{II} + T_{II}\right)\right]G_0,$$
$$W_3 = \sqrt{2}\left[T_I + \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_I + T_I\right)\right]^T T_{III}; \tag{4.6}$$

finally, for the matrix potentials U_α one gets

$$U_1 = G_0 + T_{II}^T T_{II} + \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_{II} - T_{II}\right)^T\left(\tilde{\mathcal{Z}}_{II} - T_{II}\right),$$
$$U_2 = -\left[T_{II}^T T_{II} - \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_{II} - T_{II}\right)^T\left(\tilde{\mathcal{Z}}_{II} + T_{II}\right)\right]G_0,$$
$$U_3 = \sqrt{2}\left[T_{II} - \tilde{h}^{-1}\left(\tilde{\mathcal{Z}}_{II} - T_{II}\right)\right]^T T_{III}. \tag{4.7}$$

On the other hand, the vector matrices $\tilde{\vec{\Omega}}_\alpha$ possess the following parametrization

$$\tilde{\vec{\Omega}}_1 = \vec{\omega}\epsilon, \qquad \tilde{\vec{\Omega}}_2 = \left(\tilde{\vec{\Omega}}_{2,I} \; \tilde{\vec{\Omega}}_{2,II}\right), \qquad \tilde{\vec{\Omega}}_3 = \begin{pmatrix} 0 & \tilde{\vec{\Omega}}_{3,I\,II} \\ \tilde{\vec{\Omega}}_{3,II\,I} & \tilde{\vec{\Omega}}_{3,II\,II} \end{pmatrix}, \tag{4.8}$$

where $\vec{\omega}=\tilde{\vec{\omega}}_1^{''}$ (the magnitude $\tilde{\vec{\omega}}_1$ is purely imaginary, i.e. $\tilde{\vec{\omega}}_1^{'}=0$), ϵ is the antisymmetric 2×2–matrix with $\epsilon_{12}=-1$, the block componenets of $\tilde{\vec{\Omega}}_2$ read

$$\tilde{\vec{\Omega}}_{2,I}=\left(\begin{array}{c}\tilde{\vec{\omega}}_{2,1}^{'}\\ \\ \tilde{\vec{\omega}}_{2,1}^{''}\end{array}\right),\qquad \tilde{\vec{\Omega}}_{2,II}=\left(\begin{array}{ccc}-\tilde{\vec{\omega}}_{2,1}^{''} & \tilde{\vec{\omega}}_{2,1+p}^{'} & -\tilde{\vec{\omega}}_{2,1+p}^{''}\\ \\ \tilde{\vec{\omega}}_{2,1}^{'} & \tilde{\vec{\omega}}_{2,1+p}^{''} & \tilde{\vec{\omega}}_{2,1+p}^{'}\end{array}\right), \tag{4.9}$$

whereas $\tilde{\vec{\Omega}}_{3,II\;I}=-\tilde{\vec{\Omega}}_{3,I\;II}^{T}$, $\tilde{\vec{\Omega}}_{3,II\;II}=-\tilde{\vec{\Omega}}_{3,II\;II}^{T}$ ($\tilde{\vec{\Omega}}_3=-\tilde{\vec{\Omega}}_3^{T}$) and

$$\tilde{\vec{\Omega}}_{3,I\;II}=\left(\begin{array}{ccc}-\tilde{\vec{\omega}}_{3,1\;1}^{''} & \tilde{\vec{\omega}}_{3,1\;1+p}^{'} & -\tilde{\vec{\omega}}_{3,1\;1+p}^{''}\end{array}\right). \tag{4.10}$$

Note that, in view of Eqs. (2.11) and (3.22), only the magnitudes $\tilde{\vec{\omega}}$, $\tilde{\vec{\Omega}}_{2,I}$, $\tilde{\vec{\Omega}}_{2,II}$ and $\tilde{\vec{\Omega}}_{3,I\;II}$ are necessary for the explicit construction of the potential $\vec{V}_\alpha$. Finally, the explicite expressions for these vectors read

$$\begin{aligned}\vec{V}_1&=\tilde{\vec{\omega}}T_{II}^T\epsilon T_I-\tilde{\vec{\Omega}}_{2,II}^{T}T_I+T_{II}^T\tilde{\vec{\Omega}}_{2,I}-\tilde{\vec{\Omega}}_{3,I\;II}^{T},\\ \vec{V}_2&=-\mathcal{G}_0\left(\tilde{\vec{\omega}}T_{II}^T\epsilon T_I+\tilde{\vec{\Omega}}_{2,II}^{T}T_I+T_{II}^T\tilde{\vec{\Omega}}_{2,I}+\tilde{\vec{\Omega}}_{3,I\;II}^{T}\right),\\ \vec{V}_3&=\sqrt{2}\left(\tilde{\vec{\omega}}\epsilon T_I+\tilde{\vec{\Omega}}_{2,I}\right)^T T_{III}.\end{aligned} \tag{4.11}$$

At this stage some remarks are in order: Eqs. (4.5), (4.7) and (4.11) also possess another parametrization which is based on the identities $\mathcal{K}\mathcal{K}^T=-\kappa$ and $\mathcal{K}\epsilon\mathcal{K}^T=\sqrt{\det\kappa}\epsilon$. In both representations the potentials S_0, W_α, U_α and $\vec{V}_\alpha$ become trivial ($S_0=1$, $U_1=G_0$, other fields vanish) for a starting Einstein–multi-Maxwell solution corresponding to $\tilde{\mathcal{Z}}=0$. This fact reflects the underlying property of the primordial symmetry map (3.12) in the language of the potentials which define the components of the physical fields of string theory. As a last remark let us point out that the appearance of the $(2k+1)\times(2k+1)$–matrix $G_0=\text{diag}(-1,1,1,...,1)$, which describes the flat metric corresponding to extra dimensions, in Eq. (4.7) is very natural.

As a matter of fact, our generating–procedure, based on the use of Eq. (3.12) and the special choice of the starting matrix potential $\tilde{\mathcal{Z}}$ in an Einstein–multi-Maxwell form (see Eqs. (4.1) and (4.2)), brakes the complex structure of the starting theory. Actually, in the general case, the $2\times[2(k+1)+n]$–dimensional symmetry operator T does not represent any complex magnitude t of dimension $1\times(k+1+n/2)$. In particular, the number of Abelian gauge fields n can be even. However, if n is odd, i.e., $n = 2J$, and also $r_{2,2p} = r_{1,2p-1} \equiv r_p^{'}$, $r_{2,2p-1} = -r_{1,2p} \equiv r_p^{''}$, $l_{2,2j} = l_{1,2j-1} \equiv l_j^{'}$, $l_{2,2j-1} = -l_{1,2j} \equiv l_j^{''}$, $(p = 1, 2, ..., k; j = 1, 2, ..., J)$, then

$$T = \frac{1}{\sqrt{1-\mathcal{N}^2}} \left(\begin{array}{cccccc} 1 & 0 & r_p^{'} & -r_p^{''} & l_j^{'} & -l_j^{''} \\ & & & & & \\ 0 & 1 & r_p^{''} & r_p^{'} & l_j^{''} & l_j^{'} \end{array} \right), \tag{4.12}$$

where $\mathcal{N}^2 = n_1^T n_1 = n_2^T n_2$. In this special case, vectors n_1 and n_2 have the same length and are mutually orthogonal ($n_1^T n_2 = 0$). Therefore, from Eq. (4.12) it immediately follows that the operator T is a real matrix representation of the complex $1 \times (k+1+J)$ row $t = |1-\mathcal{N}^2|^{-1/2} \, (1 \; r_p \; l_j)$ where $r_p = r_p^{'} + i r_p^{''}$ and $l_j = l_j^{'} + i l_j^{''}$. Notice that in the special case under consideration it is possible to express Eq. (3.22) in a complex form by substituting $M_\alpha \longrightarrow m_\alpha$, $\tilde{M}_\alpha \longrightarrow \tilde{m}_\alpha$, $T \longrightarrow t$ and $\Sigma \longrightarrow \tilde{\sigma}_3$. Thus, in this special case it is possible to keep pure complex notations.

As an example of a concrete class of solutions of the Einstein–multi-Maxwell theory one can consider the solution which arises in the framework of the ansatz

$$\tilde{z} = \lambda \tilde{q}, \tag{4.13}$$

where λ is a complex function and $\tilde{q}$ is a $1 \times (k+1)$ constant complex row. The corresponding effective system is related to the Lagrangian

$$L_3 = 2\tilde{\kappa} \frac{|\nabla \lambda|^2}{(1 - \tilde{\kappa}|\lambda|^2)}, \tag{4.14}$$

in the case of $\tilde{\kappa} \neq 0$, where

$$\tilde{\kappa} = \tilde{q} \tilde{\sigma}_3 \tilde{q} \, \dagger \, . \tag{4.15}$$

When $\tilde{\kappa} = 0$ one obtains a decoupled three–dimensional flat space and a harmonic field λ. The parameter $\tilde{\kappa}$ plays the role of a coupling constant between three–dimensional gravity and the complex scalar field λ. We claim that the following concrete choice of the scalar field and the three metric

$$\lambda = \frac{1}{R - ia\cos\theta},$$

$$ds_3^2 = \Delta \left(\frac{dR^2}{R^2 + a^2 - \tilde{\kappa}} + d\theta^2 \right) + \left(R^2 + a^2 - \tilde{\kappa} \right) \sin^2\theta d\varphi^2, \tag{4.16}$$

where $\Delta = R^2 + a^2\cos^2\theta - \tilde{\kappa}$ and a is a constant, gives a solution of the corresponding equations of motion. Note that in Eq. (4.16) the value of the parameter $\tilde{\kappa}$ is arbitrary. In what follows, this concrete class of solutions will be considered as the typical starting one in the framework of the developed solution–generating scheme.

To start with, we need explicit expressions for the magnitudes $\tilde{m}_\alpha$ and $\tilde{\vec{\omega}}_\alpha$. For the scalar sector one immediately gets

$$\begin{aligned}
\tilde{m}_1 &= -\left(1 + \frac{\tilde{\kappa}}{\Delta}\right), \\
\tilde{m}_2 &= -\tilde{q}\frac{R + ia\cos\theta}{\Delta}, \\
\tilde{m}_3 &= -\tilde{q}^{\dagger}\tilde{q}\frac{1}{\Delta},
\end{aligned} \tag{4.17}$$

whereas for the vector one, after the corresponding integration, one finds that

$$\begin{aligned}
\tilde{\vec{\omega}}_{1\,\varphi} &= -i\frac{a\tilde{\kappa}\sin^2\theta}{\Delta}, \\
\tilde{\vec{\omega}}_{2\,\varphi} &= \tilde{q}\left(-\cos\theta + ia\sin^2\theta\frac{R + ia\cos\theta}{\Delta}\right), \\
\tilde{\vec{\omega}}_{3\,\varphi} &= -i\tilde{q}^{\dagger}\tilde{q}\frac{a\sin^2\theta}{\Delta},
\end{aligned} \tag{4.18}$$

and other vector components vanish. These relations defines the extension of the solution (4.16) to the realm of the heterotic string theory according to the relations (2.6), (**??**), (4.5)–(4.7) and (4.11). Let us discuss on both, the starting Einstein–multi-Maxwell family of solutions and the resulting heterotic string theory fields.

First of all, let us compute the Ernst potentials (3.18) corresponding to the solution (4.16):

$$\begin{aligned}
\mathcal{E} &= 1 - \frac{2(M + iN)}{r + i(N - a\cos\theta)}, \\
\mathcal{F}_p &= \sqrt{2}\frac{(e_p + ig_p)}{r + i(N - a\cos\theta)},
\end{aligned} \tag{4.19}$$

where

$$\tilde{q}_1 = M + iN, \qquad \tilde{q}_{1+p} = e_p + ig_p \tag{4.20}$$

and $r = R + M$ and $\tilde{\kappa} = -M^2 - N^2 + \sum_p \left(e_p^2 + g_p^2\right)$. It is clear that our starting solution is precisely the Kerr–multi-Newman family of solutions with non–trivial NUT parameter. Thus, (r, θ, φ) stand for conventional oblate spheroidal coordinates, whereas the parameters M, N, e_p and g_p are the mass, NUT, electric and magnetic charges, respectively. Another interesting issue concerns the asymptotical flatness of the resulting multidimensional field configuration in the framework of our solution–generating method in the general case. It turns out that the generating field configurations contain the so–called "Dirac strings" and are not asymptotically flat, i.e. a term which is proportional to $\cos\theta$ at spatial infinity $(R \longrightarrow \infty)$. The same situation takes place for the starting four–dimensional metric of the Einstein–multi-Maxwell theory: the corresponding term is proportional to the NUT parameter and it vanishes if $N = 0$. Thus, in the starting solution this Dirac string peculiarity is removable. From Eqs. (4.11) and (4.18) it follows that the Dirac string for the metric (i.e. for the magnitude $\vec{V}_1$) is absent if one imposes the restriction

$$\tilde{\mathcal{Q}}_I^T T_{II} = T_I^T \tilde{\mathcal{Q}}_{II}, \tag{4.21}$$

on the starting charge configuration and the operator of the symmetry transformation. Here $\tilde{\mathcal{Q}}_I$ and $\tilde{\mathcal{Q}}_{II}$ are respectively 2×1 and $2 \times (2k+1)$ block components of the charge matrix $\tilde{\mathcal{Q}} = \left(\tilde{\mathcal{Q}}_I \; \tilde{\mathcal{Q}}_{II}\right)$ which realizes a real matrix representation of the complex charge parameter $\tilde{q}$. Notice that all the relations which involve the matrix $\tilde{\mathcal{Q}}$ can be obtained from the relations for $\tilde{\mathcal{Z}}$ by replacing $z_{\mathcal{P}} \longrightarrow q_{\mathcal{P}}$ in $\tilde{\mathcal{Z}}$; notice that we have used a decomposition of $\tilde{\mathcal{Q}}$ similar to that of the Eqs. (4.1)–(4.2)).

It is possible to solve the algebraic restriction (4.21) for the general case. However, for the special case (4.12), when the symmetry operator can be represented in complex form, this can be done in aN easy and elegant way. Actually, a simple algebraic analysis shows that in this case Eq. (4.21) leads to $N = 0$ and the relations

$$e_p = r_p^{'} M, \qquad g_p = r_p^{''} M, \tag{4.22}$$

i.e., to the NUT–less starting solution and to hard relations between the electromagnetic charges and the non–electromagnetic sector of the symmetry operator T. In this special case, up to construction, the resulting metric is asymptotically flat at spatial infinity. Notice that the resulting matter fields of heterotic string theory also contain Dirac strings. In order to remove them one must impose the corresponding restrictions on the magnitudes $\vec{V}_2$ and $\vec{V}_3$ (to elliminate, in turn, the terms proportional to $\cos\theta$). Here we will not discuss these pure algebraic topics; thus, our final solutions will include, for instance, magnetically charged field configurations.

At the end of this Section let us note that our solution–generating procedure, based on Eq. (3.12), maps the full Einstein–multi-Maxwell theory into the pure bosonic string theory sector in the case $l_a \equiv 0$. Actually, this last restriction can be imposed independently of our generation scheme. This fact, is also reflected in the number of compactified space–time dimensions, which is equal to $2k+1$, where k is precisely the number of starting Maxwell fields. Thus, a surprising fact is that the Abelian gauge field sector of heterotic string theory is not related to the Maxwell sector of the starting Einstein–multi-Maxwell theory: all the string theory gauge fields depend only on the structure of the symmetry operator T. Namely, one obtains n Abelian vector fields $U(1)$ if one chooses the parameter l_a of height n.

5 Conclusion and Discussion

The main result of this paper is the presentation of a new and explicit scheme of generation of heterotic string theory solutions from stationary fields of the Einstein–multi-Maxwell theory. Namely, one can start with an arbitrary stationary solution of Einstein theory coupled to k Maxwell fields and obtain a solution of heterotic string theory with n Abelian gauge fields which lives in $2(k+2)$ dimensions by making use of the procedure developed in this paper. It is worth noticing that our symmetry approach is based on pure algebraic calculations only as far as all the potentials of the starting Einstein–multi-Maxwell theory have been already computed.

Let us make two remarks concerning the properties of the underlying symmetry map (3.12) (or (4.5)–(4.7) and (4.11) in an equivalent and physically meaningful form). First of all, it is interesting to notice that for the case $k=3$, the resulting heterotic string theory becomes ten–dimensional. However, the complete theory with $k=3$ (we refer to a theory with arbitrary potential $\tilde{\mathcal{Z}}$ of dimension 2×8, not to the Einstein theory with 3 Maxwell fields) corresponds to the bosonic sector of $N=4$ supergravity in four dimensions. Thus, when $n=16$, our procedure relates $N=D=4$ and $N=1$, $D=10$ supergravities in a transparent form. Keeping this in mind, it will be interesting to study the problem of supersymmetric, and therefore BPS saturated, solutions in the framework of the established correspondence. Namely, a question that naturally arises is: whether or not supersymmetric solutions of the four dimensional theory map into supersymmetric solutions of the ten–dimensional one. If they do so, how many supersymmetries will preserve under this correspondence? Notice that some classes of four–dimensional supersymmetric solutions have been extensively studied during last several years (see, for instance, [19], [18]); some special classes of ten–dimensional supersymmetric solutions have also been obtained (see [20], [3]) and the topic is still under active investigation till now.

Our second remark is related to the level of generality of the map (3.12). We consider this issue in the framework of asymptotically flat field configurations in the three–dimensional sense. Namely, we consider that the fields which are encoded in the potential $\mathcal{Z}$ vanish at spatial infinity. In this sense our map (3.12) is complete with respect to the total group of three–dimensional charging symmetries, i.e., to the transformations that preserve the asymptotical triviality of the matrix potential $\mathcal{Z}$. Thus, if in Eq. (3.12) the potential $\tilde{\mathcal{Z}}$ and the symmetry operator T have the most general form, our procedure is non–generalizable by making use of hidden symmetries that act in the subspace of three–dimensional asymptotically flat field solutions. However, if one starts with the potential $\tilde{\mathcal{Z}}$ and the symmetry operator T given in the matrix representation which corresponds to the complex $\tilde{z}$ and t, one partially loses the charging symmetry self–invariance of the resulting solutions of the heterotic string theory. In fact, one loses the part of the total charging symmetry group of the heterotic string theory which breaks the special (complex) structure of the matrix potential $\tilde{\mathcal{Z}}$ and the symmetry operator T given by Eqs. (4.2) and (4.12). This lost symmetry sector is evidently non–trivial and can be used for the further generalization of the solutions obtained in the framework of the pure complex generating scheme developed at the end of the previous Section.

In this paper we have constructed as well the string theory extension of the Kerr–multi-Newman solution of the Einstein–multi-Maxwell theory. This extension was presented as some simple and natural application of the developed general formalism. It was also shown how to remove all the Dirac string peculiarities from the resulting multi–dimensional metric field, so that the resulting space–time of heterotic string theory is asymptotically flat. Note that the constructed family of solutions constitutes the first example of extension of the Kerr–multi-Newman solution to the realm of the heterotic string theory. The obtained class of solutions is really interesting from the point of view of black hole physics and can be studied in detail in a conventional manner [3], [17]–[18].

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