Replicator Dynamics with Dynamic Payoff Reallocation Based on the Government’s Payoff

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Abstract—In a population which consists of a large number of players interacting with each other, the payoff of each player often conflicts with the total payoff of the population which he/she belongs to. In such a situation, the “government” which has the comprehensive perspective is introduced for governing the population. When the government collects and reallocates players’ payoffs for governing the population, the evolutions of population states are modeled by replicator-mutator dynamics. In this paper, we propose a model which describes the evolution of the government’s reallocation strategy and investigate stability of its equilibrium points.

1. Introduction

A problem called social dilemma occurs when the purpose of each person conflicts with the total purpose of the community which he/she belongs to [1, 2]. Evolutionary game theory has been used as a powerful mathematical framework to analyze such a social problem [3, 4]. Especially, when the government collects and reallocates players’ payoffs for governing the population, the evolutions of population states are modeled by replicator-mutator dynamics [5].

The social problem is the conflict between the payoff of each player and the total payoff of the population. Therefore, such a problem is unsolvable by personal effort of each player. The “government” which has the comprehensive perspective is required for governing the population. In the real world, it corresponds to the rulers such as the government of countries or cities, and executives of organizations or companies. The government is willing to lead the population state to a desirable state by intervening in the population. In replicator dynamics with reallocation of payoffs [5], we consider that players interact with each other in their own population, and the government intervenes in the interactions by collecting and reallocating payoffs. In this model, the government’s action is the rate of collecting payoffs from players and the rate is independent of the population state. However, in the case that the government can change the rate depending on the state as its policy, it can be modeled as a player.

In this paper, we define the government’s payoff as a sum of benefits which depend on the current state of the population which he/she belongs to. In such a situation, the “government” which has the comprehensive perspective is introduced for governing the population. When the government collects and reallocates players’ payoffs for governing the population, the evolutions of population states are modeled by replicator-mutator dynamics. In this paper, we propose a model which describes the evolution of the government’s reallocation strategy and investigate stability of its equilibrium points.

2. Intervention by Collection and Reallocation of Payoffs

We consider a population which consists of a large number of players and the “government” which intervenes in interactions between players who belong to the population. The intervention can be modeled as a strategy of the government. Suppose that the government changes its strategy depending on the population’s state. In this paper, as the government’s intervention, we deal with the collections and reallocations of players’ payoffs. Figure 1 shows an illustration of the collection and the reallocation of payoffs by the government.

Let \( P \) be the population of players. Suppose that \( \Phi_p = \{1, 2, \cdots, m_p\} \) be a set of pure strategies of \( P \), and \( S_p \) be a set of population states of \( P \). A population state \( s_p = (s_p^1, s_p^2, \cdots, s_p^{m_p})^T \in S_p \) is a distribution of strategies in the population \( P \), where \( s_p^i \) is the proportion of players with a pure strategy \( i \in \Phi_p \). Let \( r_p : S_p \to \mathbb{R} \) be the payoff function for the players of \( P \) with the pure strategy \( i \in \Phi_p \), and \( \bar{r}_p(s_p) \) be the average payoff, i.e., \( \bar{r}_p(s_p) = \sum_{i \in \Phi_p} s_p^i r_p^i(s_p) \). We assume that \( r_p^i(s_p) \geq 0 \) and \( \bar{r}_p(s_p) > 0 \).

Suppose that \( q^j \) is a proportion of payoffs which is collected from players with \( j \in \Phi_p \) and reallocated to players with \( i \in \Phi_p \) to the total payoff of player with \( j \in \Phi_p \). We call the matrix \( Q = (q^j) \) a reallocation matrix. Using the
above definitions, replicator dynamics with collections and reallocations of payoffs is given as follows [5]:

\[ s_i^p = \sum_{j \in \Phi_p} s_i^p r_p(s_j) q^{ij} - s_i^p \bar{r}_g(s_p). \]  

(1)

Equation (1) is known as replicator-mutator dynamics [6].

Suppose that \( \Phi_g = [1, 2] \) and \( S_g \) are sets of pure and mixed strategies of the government, respectively. We call the first strategy complete intervention and the second strategy non-intervention. In this situation, a strategy \( s_g = (\alpha, 1 - \alpha) \in S_g \) defines a mixed strategy between those two strategies, where \( \alpha \) is called an intervention rate and assumed to satisfy \( \alpha \in [0, 1] \). For the target population state \( s_p^* \in S_p \), we define the reallocation matrix \( Q \) as follows:

\[ Q = (1 - \alpha)I_m + \alpha X^*, \]  

(2)

where \( I \) is the \( l \)-dimensional unit matrix and \( X^* = [s_p^1 \ldots s_p^l]^T \). In this case, Eq. (1) is rewritten as follows:

\[ s_i^p = (1 - \alpha)s_i^p \left( r_i^p(s_p) - \bar{r}_p(s_p) \right) + \alpha \left( s_i^p - s_i^p \right) \bar{r}_p(s_p). \]  

(3)

When \( \alpha = 0 \) holds (non-intervention), no payoffs of players are collected and reallocated. Eq. (3) is rewritten as

\[ s_i^p = s_i^p \left( r_i^p(s_p) - \bar{r}_p(s_p) \right). \]  

(4)

On the other hand, when \( \alpha = 1 \) holds (complete intervention), the government collects all payoffs of all players and reallocates them depending on the target state \( s_p^* \). In this case, Eq. (3) is rewritten as

\[ s_i^p = (s_i^p - s_i^p) \bar{r}_p(s_p). \]  

(5)

Since \( \bar{r}_p(s_p) > 0 \), the target state \( s_i^p \) is a globally asymptotically stable equilibrium point of Eq. (3).

The following proposition about Eqs. (3) and (4) has been proved [7].

**Proposition 1** If the target state \( s_i^p \in S_p \) is an equilibrium point of Eq. (4), then it is an equilibrium point of Eq. (3) for any \( \alpha \in [0, 1] \). On the other hand, if \( s_i^p \) is not an equilibrium point of Eq. (4), then there does not exist \( \alpha \in [0, 1] \) such that it is not an equilibrium point of Eq. (3).

From Proposition 1, if \( s_i^p \) is not an equilibrium point of Eq. (3), then the government has to adopt the strategy complete intervention for leading the population state of \( P \) to the target state \( s_p^* \). Therefore, we consider the case that the target state \( s_p^* \) is an equilibrium point of Eq. (4) in this paper.

3. Dynamic Intervention Rate

Let \( r_i^g : S_p \times S_g \to \mathbb{R} \) be the payoff function for the government with the pure strategy \( g \in \Phi_g \) and \( \bar{r}_g(s_p, s_g) \) be the government’s current payoff, i.e., \( \bar{r}_g(s_p, s_g) = ar_g(s_p, s_g) + (1 - \alpha)r_i^g(s_p, s_g) \). In this paper, we suppose that the government increases the intervention rate \( \alpha \) in proportion to differences between the payoffs of the complete intervention strategy \( r_i^g \) and the current payoffs \( \bar{r}_g \) which the government earns. Such a rule of changing the intervention rate can also be modeled by replicator dynamics. Since the government has two strategies complete intervention and non-intervention, the replicator dynamics is formulated as follows:

\[ \dot{\alpha} = \alpha(1 - \alpha) \left( r_i(s_p, s_g) - \bar{r}_i(s_p, s_g) \right). \]  

(6)

Using payoff matrices \( A, B, \) and \( C \), we define the players’ and the government’s payoff as \( r_i(s_p, s_g) = e_i^T A r_p + e_i^T B s_p + e_i^T C s_g \), respectively, where \( e_i^T \) is the \( i \)-dimensional unit vector such that the \( i \)th element equals 1.

The matrices \( B \) and \( C \) are the payoffs which depend on the current population state of \( P \) and the current intervention rate, respectively. We consider that the matrix \( B \) is the government’s benefit depending on the current population state of \( P \) and the matrix \( C \) is a cost of the government’s intervention depending on the current intervention rate. Assuming that the non-intervention strategy makes no benefits and costs, we define the matrices \( A, B, \) and \( C \) as follows:

\[
A = \begin{bmatrix} a_{11} & \cdots & a_{1m_p} \\ \vdots & \ddots & \vdots \\ a_{m_p,1} & \cdots & a_{m_p,m_p} \end{bmatrix}, \\
B = \begin{bmatrix} b_1 & \cdots & b_{m_p} \\ 0 & \cdots & 0 \end{bmatrix}, \\
C = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix},
\]

where \( a_{ij} \geq 0 \) for all \( i \) and \( j \in \Phi_p, b_{ij} \geq 0 \) for all \( i \in \Phi_p, \) and \( c_1 < 0 \).

4. Two Strategy Game: Boundary Target Point

Suppose that players of the population \( P \) have two strategies and the target point is \( s_p^* = (1, 0)^T \). Since \( s_p^1 + s_p^2 = 1 \), we have

\[ s_i^p = \left( 1 - s_i^p \right) \left( d_1 s_i^p \right)^2 + d_2 s_i^p + \alpha a_{22}, \]  

(7)

\[ \dot{\alpha} = -c_1 \alpha(1 - \alpha) \left( \beta_1 - \beta_2 \right) s_i^p + \beta_2 - \alpha, \]  

(8)

where

\[ d_1 = a_{11} - a_{12} - a_{12} + a_{22}, \]  

(9)

\[ d_2 = (a_{12} - a_{22}) + (a_{21} - a_{22}), \]  

(10)

\[ \beta_1 = -\frac{b_1}{c_1}, \quad \beta_2 = -\frac{b_2}{c_1}. \]  

(11)

Since \( \beta_1 \) and \( \beta_2 \) are the ratios of the government’s benefits \( b_1 \) and \( b_2 \) to the intervention cost \( c_1 \), we consider them as cost-efficiencies of the government’s interventions to players’ strategies 1 and 2, respectively.
On the other hand, we have \( \dot{\alpha} = 0 \) if \( \alpha = 0 \) or \( \alpha = 1 \), or \( |\beta_1 - \beta_2|s_p + \beta_2 - \alpha = 0 \). Thus, \( s_p = 0 \) holds on the curve

\[
l_1 : \alpha = \left(1 - \frac{(a_{11} - a_{22})s_p + a_{12}}{(a_{21} - a_{22})s_p + a_{22}}\right)s_p,
\]

and \( \dot{\alpha} = 0 \) holds on the line

\[
l_2 : \alpha = (\beta_1 - \beta_2)s_p + \beta_2.
\]

\( s_p > 0 \) (resp. \( < 0 \)) holds above (resp. below) the curve \( l_1 \). \( \dot{\alpha} < 0 \) (resp. \( > 0 \)) holds above (resp. below) the line \( l_2 \). The curve \( l_1 \) depends only on players’ payoff matrix \( A \) while the line \( l_2 \) depends on the government’s payoff matrices \( B \) and \( C \). Figure 2 shows typical patterns of equilibrium points and schematics of vector fields.

The coordinates of the points \( V, T_1, T_0, T_\ast, \) and \( T_\ast \) are \((0, \beta_2), (1, 1), (1, 0), (1, \frac{a_{11} - a_{21}}{a_{21} - a_{22}}), \) and \((1, \beta_1)\), respectively. Moreover, the \( s_p \)-coordinates of the point \( W, W_\ast, \) and \( W_\ast \) are solutions of

\[
w_1 \left(s_p^2\right) + w_2s_p + a_{22}\beta_2 = 0,
\]

where

\[
w_1 = (a_{21} - a_{22})\beta_1 - \beta_2 + (a_{11} - a_{12} - a_{21} + a_{22}),
\]

\[
w_2 = a_{23}\beta_1 + (a_{21} - 2a_{22})\beta_2 + (a_{11} - a_{22}).
\]

Their \( \alpha \)-coordinates are given by Eq. (13).

Figure 2(a) shows the case \( \beta_1 > 1 \). Under the condition \( \frac{a_{11} - a_{21}}{a_{21} - a_{22}} < \beta_1 < 1 \), Figs. (2b), (c), and (d) show the cases \( (w_2)^2 < 4\alpha\beta_2 w_1 \), \( (w_2)^2 = 4\alpha\beta_2 w_1 \), and \( (w_2)^2 > 4\alpha\beta_2 w_1 \), respectively. If the condition \( \beta_1 < \frac{a_{11} - a_{21}}{a_{21} - a_{22}} \) holds, then we have Fig. 2(e).

As shown in Fig. 2, stability conditions of points \( T_1, T_\ast, T_0, \) and \( W_\ast \) are given as follows:

- \( T_1 \) is asymptotically stable if \( \beta_1 > 1 \) (Fig. 2(a));
- \( T_\ast \) is asymptotically stable if \( \frac{a_{11} - a_{21}}{a_{21} - a_{22}} < \beta_1 < 1 \) (Figs. 2(b)–(d));
- \( W \) is stable if \( \beta_1 < \frac{a_{11} - a_{21}}{a_{21} - a_{22}} \) (Fig. 2(e)); and
- \( W_\ast \) is stable if \( \frac{a_{11} - a_{21}}{a_{21} - a_{22}} < \beta_1 < 1 \) and \( (w_2)^2 > 4\alpha\beta_2 w_1 \) (Fig. 2(d)).

The target state \( s_p^* = (1, 0)^T \) corresponds to the boundary edge \( T_1T_0 \) in Fig. 2. Therefore, the achievement of the target state requires that the point \( T_\ast \) or \( T_1 \) is asymptotically stable.

If the stability condition of the point \( T_1 \) holds, then the government keeps increasing the intervention rate \( \alpha \) to the complete intervention strategy \( \alpha = 1 \) since there is no reason for the government to reduce \( \alpha \). Although the target state is achieved, the original game structure is completely lost.

On the other hand, if the stability condition of \( T_\ast \) holds, then the target state is achieved and the intervention rate \( \alpha \) keeps the intermediate level. Moreover, if the condition \( (w_2)^2 < 4\alpha\beta_2 w_1 \) also holds, then \( T_\ast \) is globally asymptotically stable (Fig. 2(b)). To achieve the target state inde-
In the case that $b_1 = 1.2$ and $b_2 = 0.03$, the point $T_{b_2}$ is above $T_1$, which is a globally asymptotically stable equilibrium point (see Figs. (iv) and 2(a)).

6. Conclusions

In this paper, we have defined the government’s payoff as a sum of benefits which depend on the current state of the population and a cost of the government’s intervention. Moreover, we have proposed a model which describes the evolution of its payoff reallocation strategy, and have investigated stability of its equilibrium points in the case that players have two strategies and the target state is on the boundary.

References


